# CARTIER DIVISORS

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Abstract. All schemes will be assumed separated and of finite type over a fixed field $\Bbbk$.

## 1. Line bundles, rational sections, and Weil divisors

Let $X$ be an integral scheme. Recall that a *Weil divisor* on $X$ is a formal linear combination of codimension-one subvarieties of $X$. In other words, it is what we have called a cycle of pure codimension one. If we let $n$ be the dimension of $X$, then the group of Weil divisors is precisely $Z_{n-1}(X)$.

Now, let $\mathscr{L}$ be a line bundle on $X$. As you may recall from Vakil's notes, a rational section of $\mathscr{L}$ is a section of $\mathscr{L}$ that is defined over a dense open subset of $X$. (We'll need a more involved definition when we stop assuming $X$ is integral.) If $s$ is a nonzero rational section of $\mathscr{L}$, and $V$ is a codimension-one subvariety of $X$, then we can define

$$\operatorname{ord}_V(s),$$

the "order of $s$ along $V$," much as we did for rational functions.[1] In this way, a rational section $s$ of $\mathscr{L}$ gives rise to a Weil divisor

$$[\operatorname{div}(s)] = \sum_V \operatorname{ord}_V(s)[V],$$

where $V$ ranges over codimension-one subvarieties of $X$. Intuitively, we think of $[\operatorname{div}(s)]$ as "zeros of $s$" − "poles of $s$".

If $X$ is locally factorial, then every Weil divisor can be obtained as the divisor of a rational section of some line bundle. Moreover, we can reconstruct $\mathscr{L}$ from the Weil divisor (by a process I prefer not to go into at this time) and we can further reconstruct the rational section $s$ up to multiplication by a "unit," i.e., a global section of $\mathscr{O}_X^*$. Two Weil divisors are linearly equivalent if and only if they give rise to rational sections of the same line bundle.

A final note is that for rational sections $s$ of $\mathscr{L}$ and $t$ of $\mathscr{M}$,

$$[\operatorname{div}(s \otimes t)] = [\operatorname{div}(s)] + [\operatorname{div}(t)], \tag{1}$$

where $s \otimes t$ is a rational section of $\mathscr{L} \otimes \mathscr{M}$. In particular,

$$-[\operatorname{div}(s)] = [\operatorname{div}(s^{-1})], \tag{2}$$

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[1]Since $\mathscr{L}$ is locally free of rank one, there exists an isomorphism $\sigma\colon \mathscr{L}_{X,V} \to \mathscr{O}_{X,V}$. We set $\operatorname{ord}_V(s) = \operatorname{ord}_V(\sigma(s))$. If $\sigma_1, \sigma_2$ are two such isomorphisms, then $\sigma_1\sigma_2^{-1}$ is multiplication by a unit of the stalk $\mathscr{O}_{X,V}$, and thus does not change the order.[2] So, this is well-defined.

[2]The order is determined by its behavior on elements $r \in \mathscr{O}_{X,V}$. For such $r$, it is defined as the length of $\mathscr{O}_{X,V}/(r)$. Since multiplication by a unit respects the ideal $(r)$, it respects the order.

where $s^{-1}$ is the section of $\mathscr{L}^\vee$ such that $s^{-1}(s) = 1 \in \mathscr{O}_X$. (This $s^{-1}$ exists since $s$ is not identically zero and $X$ is integral.)

## 2. Cartier divisors

If $X$ is not integral or locally factorial, then the nice correspondence between Weil divisors and rational sections of line bundles need not hold. So, to get a nicely behaved theory of divisors on these more general schemes, we apply the "French trick of turning a theorem into a definition." We are going to *define* a Cartier divisor to be a pair $(s, \mathscr{L})$, where $s$ is a rational section of the line bundle $\mathscr{L}$. This definition will be subject to certain restrictions (generalizations of the notion that the rational section $s$ cannot be identically zero) and an equivalence relation telling us when two rational sections represent the same divisor. We will denote the Cartier divisor associated to a pair $(s, \mathscr{L})$ by

$$\mathrm{div}(s);$$

intuitively, we will think of it as "zeros of $s$" − "poles of $s$", as before.

2.1. **What is a rational section of $\mathscr{L}$?** First, since we are no longer requiring $X$ to be integral, we need to embark on the more involved definition of rational sections promised earlier.

Recall that, when $X$ was integral, we defined a rational section of $\mathscr{L}$ to be a section of $\mathscr{L}$ over a dense open subset $U \subset X$. One issue here that we brushed over is the following: If $U' \subset U$ are both open dense subsets of $X$, and $s$ is a section over $U$, we'd like to be able to say that $s$ and $s|U'$ represent the same rational section. In order for this to make sense, we need to know that we can reconstruct $s$ from $s|U'$—in other words, that for any section of $\mathscr{L}$ over $U'$, there is at most one way of extending it to a section over $U$. For integral schemes, this is not really a problem: as it turns out, a section of $\mathscr{L}$ over any nonempty open set is uniquely specified by its value in the stalk at the generic point.

If $X$ is not necessarily integral, this becomes a bit more of a problem, primarily because $X$ does not have just a single generic point. Instead, it may have a number of "associated points." These correspond to the irreducible components and the embedded components. As it turns out, to specify a rational section uniquely, we really need to know its behavior at the stalks of *all* the associated points. (If you want to try to work out the algebra here, use the fact that the set of zero divisors of a Noetherian ring is precisely the union of its associated primes.) In order for this to happen, the open set over which it is defined must include all of the associated points. This gives rise to the following definition:

**Definition 1.** A *rational section* of a line bundle $\mathscr{L}$ is, up to equivalence, a pair $(s, U)$, where

- $U$ is an open subset of $X$ that includes all the associated points of $X$,
- $s$ is a section of $\mathscr{L}$ over $U$, and
- $(s, U) \sim (s', U')$ if $s$ and $s'$ agree over $U \cap U'$.

The requirement that $U$ include all the associated points of $X$ amounts to saying that $s$ pulls back to a well-defined rational section of every component (irreducible or embedded) of $X$. Loosely speaking, none of the poles of $s$ can include a component of $X$.

2.2. **Invertible rational sections.** We're going to want to make the Cartier divisors of $X$ into a group, imitating the identities (1) and (2) for Weil divisors. In particular, this means that if $(s, \mathscr{L})$ represents a Cartier divisor, its inverse will need to be given by a section $s^{-1}$ of $\mathscr{L}^\vee$ such that the natural map

$$\begin{aligned} \mathscr{L} \otimes \mathscr{L}^\vee \to \mathscr{O}_X &\qquad \text{takes} \\ s \otimes s^{-1} \mapsto 1. \end{aligned}$$

Thus, such a section $s^{-1}$ must exist.

**Definition 2.** We will call a rational section $s$ of a line bundle $\mathscr{L}$ *invertible* if there exists a rational section $s^{-1}$ of $\mathscr{L}^\vee$ such that the natural pairing

$$\begin{aligned} \mathscr{L} \otimes \mathscr{L}^\vee \to \mathscr{O}_X &\qquad \text{takes} \\ s \otimes s^{-1} \mapsto 1. \end{aligned}$$

Perhaps this goes without saying, but the section we have called $s^{-1}$ is unique (if it exists) and will be denoted $s^{-1}$.

The most obvious rational section that is *not* invertible is the zero section; and if $X$ is integral, this is the only non-invertible section. More generally, for $s$ to be a rational section, it must pull back to a rational section on each associated component of $X$; and it turns out that $s$ is invertible if and only if none of these pullbacks is the zero section.

Here's an intuition for this: For $s$ to be a rational section, it must be defined (rationally) over every associated component. For $s$ to be invertible, $s^{-1}$ must be defined over every associated component. And $s^{-1}$ is defined over an associated component $Z$ iff $s$ does not vanish identically on $Z$.

2.3. **When do two sections represent the same divisor?** This particular definition is incredibly natural—no special considerations required.

**Definition 3.** A *Cartier divisor* on $X$ is represented by a pair $(s, \mathscr{L})$, where $\mathscr{L}$ is a line bundle on $X$ and $s$ is an invertible rational section of $\mathscr{L}$. Two such pairs $(s, \mathscr{L})$ and $(t, \mathscr{M})$ represent the same Cartier divisor if there exists an isomorphism of sheaves

$$\begin{aligned} \mathscr{L} \xrightarrow{\sim} \mathscr{M} &\qquad \text{taking} \\ s \longmapsto t. \end{aligned}$$

The Cartier divisor represented by $(s, \mathscr{L})$ will be denoted

$$\operatorname{div}(s),$$

and should be thought of as "zeros of $s$" − "poles of $s$".

An obvious question here is the following: If $s$ and $s'$ are both rational sections of $\mathscr{L}$, when do they represent the same divisor? To answer this question, consider that

$$\begin{aligned} \operatorname{Hom}(\mathscr{L}, \mathscr{L}) &= \Gamma(X, \mathcal{H}om(\mathscr{L}, \mathscr{L})) \\ &= \Gamma(X, \mathscr{L}^\vee \otimes \mathscr{L}) \\ &= \Gamma(X, \mathscr{O}_X). \end{aligned}$$

Likewise, the isomorphisms $\mathscr{L} \to \mathscr{L}$ correspond precisely to global sections of $\mathscr{O}_X^*$. Thus, if $s$ and $s'$ are two invertible rational sections of $\mathscr{L}$, then $\operatorname{div}(s) = \operatorname{div}(s')$ iff $s$ and $s'$ differ by a (globally defined) unit of the structure sheaf.

2.4. **The group structure.** We define a group structure on the collection of Cartier divisors by setting

$$\operatorname{div}(s) + \operatorname{div}(s') = \operatorname{div}(s \otimes s'),$$

where $s$ is an invertible rational section of $\mathscr{L}$, $s'$ is an invertible rational section of $\mathscr{L}'$, and $s \otimes s'$ is consequently an invertible rational section of $\mathscr{L} \otimes \mathscr{L}'$. It is left as an exercise to show that this operation is well-defined and does in fact induce a group structure.

Intuitively, if we want this to correspond to the Weil divisor addition, the identity ought to be the "empty divisor." A rational section $(s, \mathscr{L})$ ought to represent the empty divisor if it has no zeros and no poles; i.e., if it is nowhere zero and is globally defined. If such a section $s$ exists, then we can use it to produce an isomorphism

$$\begin{aligned} \mathscr{O}_X &\longrightarrow \mathscr{L} \\ 1 &\longmapsto s. \end{aligned}$$

In other words, such a section $s$ exists iff $\mathscr{L}$ is (globally) trivial and, under an appropriate trivialization, we can take $s = 1$. Furthermore,

$$1 \otimes t = t$$

for any rational section $t$ of any line bundle, and so $\operatorname{div}(1)$ is, in fact, the group's zero element.

Unsurprisingly, we have inverses given by

$$-\operatorname{div}(s) = \operatorname{div}(s^{-1}),$$

since this was the whole point of requiring $s^{-1}$ to exist in the first place.

The group of Cartier divisors on $X$ is denoted $\operatorname{Div}(X)$.

2.5. **Some notation.** To more closely echo the notation for Weil divisors, we will often denote a Cartier divisor by a single capital letter, e.g., $D$. If $D = \operatorname{div}(s)$ is a Cartier divisor, where $s$ is a rational section of $\mathscr{L}$, we will write

- $\mathscr{O}_X(D)$ for the line bundle associated to $D$, i.e., $\mathscr{O})X(D) = \mathscr{L}$.
- $s_D$ for the associated rational section $s$ of $\mathscr{O}_X(D)$.
- $|D|$ for the support of $D$, which we are about to define.

**Definition 4.** Let $D$ be a Cartier divisor, with associated line bundle $\mathscr{O}_X(D)$ and rational section $s_D$. Let $U$ be the maximal open subset of $X$ over which $s_D$ is defined and nonzero. The *support* of $D$, denoted $|D|$, is defined to be the closed set $X \smallsetminus U$. In other words, the support of $D$ consists of the union of the poles and the zeros of $D$.

2.6. **Linear equivalence.**

**Definition 5.** A Cartier divisor $D = \operatorname{div}(s)$ is *principal* if $s$ is a rational function, i.e., a rational section of the structure sheaf $\mathscr{O}_X$. In other words, $D$ is principal if $\mathscr{O}_X(D)$ is (isomorphic to) the trivial line bundle $\mathscr{O}_X$.

We say that two Cartier divisors $D, D'$ are *linearly equivalent* if their difference is a principal divisor.

**Claim 6.** Two Cartier divisors $D$, $D'$ are linearly equivalent if and only if their line bundles $\mathscr{O}_X(D)$, $\mathscr{O}_X(D')$ are isomorphic.

*Proof.* $\Longrightarrow$**:** Assume that $D$ and $D'$ are linearly equivalent. Thus, there exists a rational function $f$ such that

$$D + \operatorname{div}(f) = D'.$$

Writing $D = \operatorname{div}(s_D)$ and $D' = \operatorname{div}(s_{D'})$, we see that

$$\begin{aligned} \operatorname{div}(s_D) + \operatorname{div}(f) &= \operatorname{div}(s_{D'}) \\ \operatorname{div}(s_D \otimes f) &= \operatorname{div}(s_{D'}). \end{aligned}$$

By the definition of what it means for a two sections to represent the same Cartier divisor, this implies that there is an isomorphism of line bundles

$$\begin{aligned} \mathscr{O}_X(D) \otimes \mathscr{O}_X &\longrightarrow \mathscr{O}_X(D') \qquad \text{taking} \\ s_D \otimes f &\longmapsto s_{D'}. \end{aligned}$$

A fortiori, this implies that $\mathscr{O}_X(D')$ is isomorphic to $\mathscr{O}_X(D) \otimes \mathscr{O}_X = \mathscr{O}_X(D)$.

$\Longleftarrow$**:** Assume that $\mathscr{O}_X(D)$ and $\mathscr{O}_X(D')$ are isomorphic. Then $s_D \otimes s_{D'}^{-1}$ is a rational section of

$$\mathscr{O}_X(D) \otimes \mathscr{O}_X(D')^\vee \cong \mathscr{O}_X.$$

Thus, there is a rational function $f$ (the image of $s_D \otimes s_{D'}^\vee$ under this isomorphism) such that

$$\begin{aligned} \operatorname{div}(s_D \otimes s_{D'}^{-1}) &= \operatorname{div}(f) \\ \operatorname{div}(s_D) - \operatorname{div}(s_{D'}) &= \operatorname{div}(f) \\ D - D' &= \operatorname{div}(f). \end{aligned}$$

□

The *divisor class group* of a scheme $X$, sometimes denoted $\mathrm{Cl}(X)$, is given by Cartier divisors modulo linear equivalence. From what we've done so far, it should be obvious that there is a natural injective group homomorphism

$$\begin{aligned} \mathrm{Cl}(X) &\hookrightarrow \mathrm{Pic}(X) \\ [D] &\longmapsto [\mathscr{O}_X(D)]. \end{aligned}$$

(Really, there is a homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ such that two divisors have the same image iff they are linearly equivalent.)

The line bundles that can be represented by a Cartier divisor are precisely those line bundles that admit a rational section. If $X$ is sufficiently nice (e.g., integral; I think quasiprojective also works), then every line bundle admits a rational section, and so

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$$

is an isomorphism.

2.7. **The more usual definition: local data.** In most books (including Fulton's), you will see a definition of Cartier divisor more like the following:

**Definition 7.** Abstract version: A Cartier divisor is a global section of the quotient sheaf $\mathscr{K}^*/\mathscr{O}_X^*$, where $\mathscr{K}$ is the sheaf of rational functions on $X$. (In other words, the sections of $\mathscr{K}$ over an open set $U$ are precisely the rational sections of $\mathscr{O}_U$.)

Concrete version: A Cartier divisor is given by the "local data" consisting of a set of pairs $(U_\alpha, f_\alpha)$, where the $U_\alpha$ form an open cover of $X$ and $f_\alpha$ is an invertible rational section of $\mathscr{O}_U$. Additionally, on overlaps $U_\alpha \cap U_\beta$, the rational section $f_\alpha f_\beta^{-1}$ is required to be a regular section of $\mathscr{O}_X^*$ over $U_\alpha \cap U_\beta$.

It is an exercise best done in private to see that these two definitions are equivalent. The second, concrete definition relates more intuitively to the notion of Weil divisor: the $f_\alpha$ are the "local equations" of the divisor $D$. Over $U_\alpha$, the zeros and poles of $D$ are precisely the zeros and poles of $f_\alpha$. In many contexts, it makes sense to say that "a Cartier divisor is a Weil divisor that is locally principal," i.e., that is defined locally as the divisor of a single rational function.

If you have a Cartier divisor in our sense, represented by $(s, \mathscr{L})$, you can get a Cartier divisor in the usual sense by taking a cover $\{U_\alpha\}_\alpha$ such that over each $U_\alpha$, $\mathscr{L}$ admits a trivialization

$$\sigma_\alpha \colon \mathscr{L}|U_\alpha \xrightarrow{\sim} \mathscr{O}_X|U_\alpha.$$

Then set $f_\alpha = \sigma_\alpha(s|U_\alpha)$.

This can also be reversed. If you have a Cartier divisor in the usual sense, represented by local data $(U_\alpha, f_\alpha)$, we know by definition that $f_\alpha^{-1} f_\beta$ is a section of $\mathscr{O}_X^*$ over $U_\alpha \cap U_\beta$. It is easy to verify the cocycle condition, showing that these $f_\alpha^{-1} f_\beta$ provide transition maps that allow us to glue together a line bundle $\mathscr{L}$. This gluing process gives us natural isomorphisms $\mathscr{O}_X|U_\alpha \to \mathscr{L}|U_\alpha$. Moreover, unless I got them backwards (which is quite possible), the transition functions are precisely chosen so that $f_\alpha$ and $f_\beta$ map to the same rational section of $\mathscr{L}$ over $U_\alpha \cap U_\beta$. Thus, the $f_\alpha$ glue together to a rational section $s$ of $\mathscr{L}$, and $(s, \mathscr{L})$ gives a Cartier divisor in our sense. By construction, over each $U_\alpha$, $s$ and $f_\alpha$ have the same zeros and poles. (In some sense, they're even the same section.)

2.8. **Associated Weil divisor.** For this subsection, suppose again that $X$ is integral. As discussed previously, for any rational section $s$ of a line bundle $\mathscr{L}$, and any codimension-one subvariety $V$ of $X$, we have a definition for

$$\operatorname{ord}_V(s).$$

Moreover, this definition gives the same answer if we replace $s$ by $\sigma s$, for any isomorphism $\sigma \colon \mathscr{L} \to \mathscr{L}$. Thus,

$$\operatorname{ord}_V(\operatorname{div}(s)) := \operatorname{ord}_V(s)$$

is well-defined on Cartier divisors. Moreover, the map

$$D \mapsto \sum_V \operatorname{ord}_V(D)[V],$$

where $V$ ranges over codimension-one subvarieties, gives a well-defined group homomorphism from $\operatorname{Div}(X)$ to the group of Weil divisors.