Math 13100 – Section 58 – Midterm 3

March 4, 2016

Information and Directions

- This exam will last 50 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Partial credit is given for showing your calculations and explaining your thoughts.

Name: SOLUTIONS

Problem	Points	Score
1	30	
2	40	
3	30	

Problem 1 (30 points)

Part 1 (10 points)

Let x be a real number, and let f be a function. Give the precise definition (using a limit) of the derivative of f at x:

$$f'(x) = \lim_{\boxed{h \to 0}} \boxed{\frac{f(x+h) - f(x)}{h}}$$

Part 2 (10 points)

Using only the definition you wrote above, and your experience with computing limits, compute the derivative $f'(x)$, where $f(x) = x^2$. Don't use the derivative rules from Section 2.3.

(No need to do any ϵ-δ stuff, just compute the limit. Please show your work though.)

(Of course, you are welcome to use the derivative rules we learned to *check* your answer.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

(Power Rule confirms that $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$)

Part 3 (5 points)

Remember that $f(x) = x^2$ was defined in Part 2 of this problem, and $f'(x)$ was computed.
Find the equation of the tangent line to the graph of $y = f(x)$ at 1.

The tangent line to $y=f(x)$ at 1 is characterized by these two facts:

- It goes through the point $(1, f(1))$, which since $f(x) = x^2$, is equal to $(1, 1^2)$, i.e. $(1, 1)$
- Its slope is equal to $f'(1)$, which since $f'(x) = 2x$, is equal to 2.

By the point-slope formula, the equation of the tangent line at 1 is $y - 1 = 2(x-1)$, or equivalently, $y = 2x - 1$

Part 4 (5 points)

Remember that $f(x) = x^2$ was defined in Part 2 of this problem, and $f'(x)$ was computed.
For which real numbers c is it true that the tangent line to $y = f(x)$ at c goes through the point $(2,3)$?
(Hint: keeping c as an unknown, find the equation for the tangent line at c. Plug in $x = 2$ and $y = 3$ and deduce the possible values for c.)

The tangent line to $y=f(x)$ at c is characterized by these two facts:

- It goes through the point $(c, f(c))$, which since $f(x)=x^2$, is equal to (c, c^2)
- Its slope is equal to $f'(c)$, which since $f'(x) = 2x$, is equal to $2c$.

By the point-slope formula, the equation of the tangent line at c is

$y - c^2 = 2c(x-c)$, or equivalently, $y = 2cx - c^2$.

The point $(2,3)$ is on the line $y = 2cx - c^2$ if, and only if, plugging in $x=2$ and $y=3$ makes a true equation. Thus, we are looking for the values of c for which

$$y = 2cx - c^2$$

$$3 = (2c)\cdot 2 - c^2$$

$$3 = 4c - c^2$$

$$c^2 - 4c + 3 = 0$$

$$(c-3)(c-1) = 0$$

so the solutions are $c=1$ and $c=3$.

By the way, the corresponding tangent lines are

$$y = 2cx - c^2$$

$(c=1)$ $\quad y = 2x - 1$ $\qquad\qquad$ $(c=3)$ $\quad y = 6x - 9$

and we can check that the point $(2,3)$ is on both of these lines since $3 = 2(2) - 1$ ✓ and $3 = 6(2) - 9$ ✓

The image on the next page shows

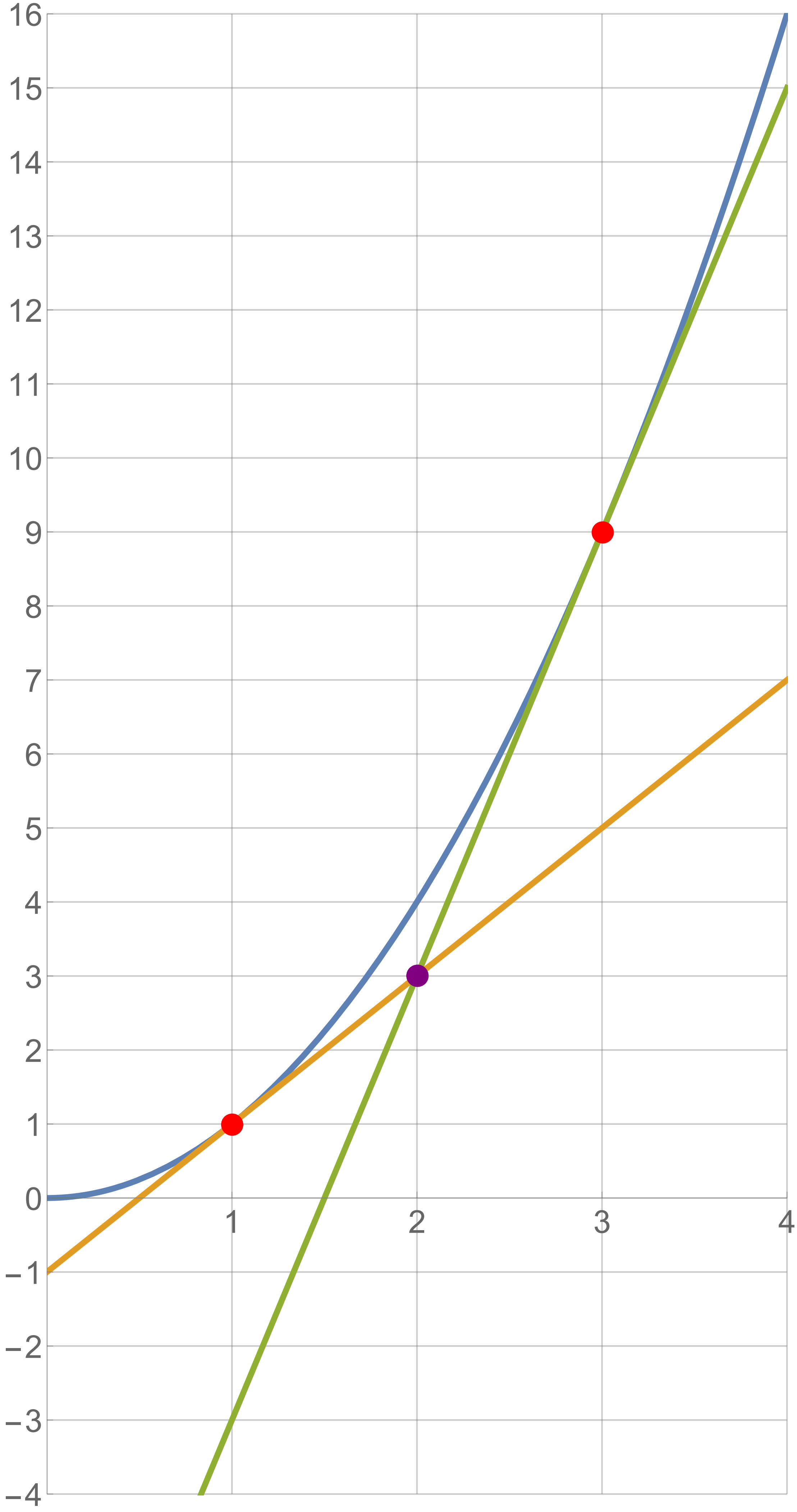
a graph of $y = f(x)$, i.e., $y = x^2$ (blue)

the tangent line to $y = f(x)$ at 1 (yellow)

the tangent line to $y = f(x)$ at 3 (green)

the points $(1,1)$ and $(3,9)$, where the tangent lines touch the graph of $y = x^2$ (red)

the point $(2,3)$, which we wanted the the tangent lines to go through (purple)


16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
1
2
3
4

Problem 2 (40 points)

Part 1 (10 points)

Find the derivative of the function $f(x) = 3x^5 + 7x - 8$.

$$f'(x) = \frac{d}{dx}(3x^5 + 7x - 8)$$

$$= \frac{d}{dx}(3x^5) + \frac{d}{dx}(7x) + \frac{d}{dx}(-8)$$

$$= 3\frac{d}{dx}(x^5) + 7\frac{d}{dx}(x) + (-8)\frac{d}{dx}(1)$$

power rule

$$= 3(5x^4) + 7(1) + (-8)(0)$$

$$= 15x^4 + 7$$

Part 2 (10 points)

Find the derivative of the function $f(x) = (2x^2 + 5x)(3x^2 - 1)$.

If $g(x) = (2x^2+5x)$ and $h(x) = 3x^2-1$, then

$g'(x) = 4x+5$ and $h'(x) = 6x$

$$f'(x) = (gh)'(x) = g'(x)h(x) + g(x)h'(x)$$

product rule

$$= (4x+5)(3x^2-1) + (2x^2+5x)(6x)$$

$$= (12x^3 + 15x^2 - 4x - 5) + (12x^3 + 30x^2)$$

$$= 24x^3 + 45x^2 - 4x - 5$$

Part 3 (10 points)

Find the derivative of the function $f(x) = \dfrac{x^2 - 1}{5x + 3}$.

If $g(x) = x^2 - 1$ and $h(x) = 5x + 3$, then

$g'(x) = 2x$ and $h'(x) = 5$

$$f'(x) = \left(\frac{g}{h}\right)'(x) = \frac{h(x)\,g'(x) - g(x)\,h'(x)}{h^2(x)}$$ quotient rule

$$= \frac{(5x+3)(2x) - (x^2-1)(5)}{(5x+3)^2}$$

$$= \frac{(10x^2 + 6x) - (5x^2 - 5)}{25x^2 + 30x + 9}$$

$$= \frac{5x^2 + 6x + 5}{25x^2 + 30x + 9}$$

Part 4 (10 points)

Find the derivative of the function $f(x) = (x^2 + 1)^{17}$.

If $g(x) = x^{17}$ and $h(x) = x^2 + 1$, then

$g'(x) = 17x^{16}$ and $h'(x) = 2x$

$$f'(x) = (g \circ h)'(x) = g'(h(x)) \cdot h'(x)$$ chain rule

$$= 17(x^2+1)^{16} \cdot 2x$$

$$= 34x(x^2+1)^{16}$$

Problem 3 (30 points)

Suppose that at time $x = 0$, a ball is thrown straight up in the air with a velocity of 64 ft/sec. Taking into account gravity, and the fact that the ball started at 80 ft above the ground, the height of the ball x seconds after being thrown is equal to

$$f(x) = -16x^2 + 64x + 80$$

Part 1 (10 points)

At what time does the ball hit the ground?

The ball hits the ground when its height $f(x)$ is equal to 0.

$$f(x) = -16x^2 + 64x + 80 = 0$$

$$-16(x^2 - 4x - 5) = 0$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

Since the ball was thrown at time $x = 0$ seconds, the answer $x = -1$ seconds doesn't make sense. Therefore the ball hits the ground at time $x = 5$ seconds.

Part 2 (10 points)

What is the velocity of the ball at the time that it hits the ground?

If $v(x) =$ velocity of the ball at time x, then $v(x) = f'(x)$

$$v(x) = f'(x) = \frac{d}{dx}(-16x^2 + 64x + 80) = -32x + 64$$

So the velocity when the ball hits the ground is the velocity at $x = 5$ seconds, using Part 1 of the problem, and the answer is

$$v(5) = -32(5) + 64 = -160 + 64 = -96 \text{ ft/sec}$$

Part 3 (5 points)

What direction is the ball moving in at time $x = 2.5$?

Circle one (though please still show your work or thought process):

up   not moving   down

These options correspond to:

$$v(2.5) > 0 \qquad v(2.5) = 0 \qquad v(2.5) < 0$$

Since positive velocity corresponds to upward movement, similarly negative velocity corresponds to downward movement, and zero velocity corresponds to not moving.

$$v(2.5) = -32(2.5) + 64 = -80 + 64 = -16 \text{ ft/sec}$$

which is negative, so the ball is moving down at $x = 2.5$ seconds.

Part 4 (5 points)

What is the maximum height the ball reaches?

(Hint: remember, the ball reaches its maximum height at the exact moment it stops going up and starts going down. What does that mean about the velocity at that moment?)

The ball reaches its maximum height when $v(x) = 0$ (as the hint alludes to, it is the moment when $v(x)$ stops being positive and starts being negative).

$$v(x) = -32x + 64 = 0$$

$$-32x = -64$$

$$x = 2 \text{ seconds}$$

Thus, the ball reaches its maximum height 2 seconds after being thrown.

Its height at that time is simply $f(2)$, which is

$$f(2) = -16(2)^2 + 64(2) + 80 = 144 \text{ ft}$$