Math 19620 – Section 50 – Final Exam

December 10, 2015

Information and Directions

- This exam will last 120 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Make sure to show all details of your work.

Name: ____

Problem	Points	Score
1	20	
2	20	
3	20	
4	15	
5	10	
6	10	
7	5	

Problem 1 (20 points)

Consider the linear transformation $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_3 - 2x_4 + 8x_5 \\ x_1 + 2x_3 + 7x_5 \\ 3x_1 - 3x_4 + 9x_5 \end{bmatrix}$$

Part 1 (5 points)

Find the solutions to $f(\vec{x}) = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$.

Part 2 (10 points)

Find a basis of the kernel of f.

Part 3 (5 points)

Find a basis of the image of f.

Problem 2 (20 points)

Consider this 3×3 matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Part 1 (10 points)

Is A invertible? If so, find its inverse. If not, explain how you know it is not invertible.

Part 2 (10 points)

Is A diagonalizable? If so, find a basis of $\mathbb{R}^3$ in which A becomes diagonal, and find what diagonal matrix A becomes in this basis. If not, explain how you know it is not diagonalizable.

Problem 3 (20 points)

Let $V = \operatorname{im}(A)$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 3 & 4 & 5 \\ 3 & 3 & 3 \end{bmatrix}$$

Part 1 (10 points)

Find an orthonormal basis of V.

Part 2 (5 points)

What is the dimension of $V^\perp$? Explain your answer.

Part 3 (5 points)

Find a basis of $V^\perp$.

Problem 4 (15 points)

Consider the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Part 1 (10 points)

Find the least-squares solutions of $A\vec{x} = \vec{b}$.

Part 2 (5 points)

Which vector in $\mathrm{im}(A)$ is the closest to $\vec{b}$?

Problem 5 (10 points)

Find a value of α where the parallelogram in $\mathbb{R}^2$ formed by $D_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $D_{2\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has area 1.

$$\boxed{\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \qquad \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \qquad \sin^2(\alpha) + \cos^2(\alpha) = 1}$$

Problem 6 (10 points)

Let A be the 2×2 matrix that represents scaling $\mathbb{R}^2$ by a factor of 5.

Let B be the 2×2 matrix that represents orthogonal projection of $\mathbb{R}^2$ onto the line $L = \{\left[\begin{smallmatrix} r \\ r \end{smallmatrix}\right] : r \in \mathbb{R}\}$.

Let $C = AB$.

Find the eigenvalues of C. For each eigenvalue, find a basis of the corresponding eigenspace.

(One approach is to simply compute what C is and do the necessary calculations to find the answer. However, I would also accept an answer using geometric intuition. Make sure to explain yourself as fully as possible if you choose the latter approach.)

Problem 7 (5 points)

Mark the following statements as true or false. **Read and think very carefully.**

The kernel of a 4×6 matrix A is a subspace of $\mathbb{R}^4$.	T	F
If A is a 3×3 matrix, then $\det(2A) = 8\det(A)$.	T	F
If A is a 2×5 matrix, then the columns of A must be linearly dependent.	T	F
The rank of a 3×7 matrix A can be equal to 4.	T	F
The rotation matrix D_{43° is an orthogonal matrix.	T	F