Splitting Fields of Generalized Rikuna Polynomials

SMALL REU - Algebraic Number Theory

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Our Picture

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$$\begin{array}{ccc} \mathcal{O}_K & \subset & K \\ | & & | \\ \mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

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$$-2 \cdot 3 = -6 = (\sqrt{-6})^2$$

Because -2, 3, and $\sqrt{-6}$ are irreducible in $\mathbb{Z}[\sqrt{-6}]$, these are two distinct factorizations of -6.

Therefore, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-6}]$ is not a UFD.

Class Group

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Define an equivalence relation $\sim$ on non-zero ideals of $\mathcal{O}_K$ by:

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Remark

The class number measures the failure of unique factorization in $\mathcal{O}_K$; the larger h_K is, the further $\mathcal{O}_K$ is from being a UFD.

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Galois Theory

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Suppose the polynomial $f \in \mathbb{Q}[x]$ has roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Then the **splitting field** of f over $\mathbb{Q}$ is $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

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For any algebraic integer $\alpha \in \mathbb{C}$, we say $\beta \in \mathbb{C}$ is an **algebraic conjugate** of α if there is some irreducible $f \in \mathbb{Q}[x]$ having both α and β as roots.

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Theorem

A number field K is Galois if and only if K is the splitting field of some $f \in \mathbb{Q}[x]$.

Example

$L = \mathbb{Q}(\sqrt[3]{2})$ is not Galois. Because $\sqrt[3]{2}$, $\zeta\sqrt[3]{2}$, and $\zeta^2\sqrt[3]{2} \in \mathbb{C}$ are the roots of $f = x^3 - 2$, they are algebraic conjugates, but $\zeta\sqrt[3]{2}$ and $\zeta^2\sqrt[3]{2}$ are complex, while $L \subset \mathbb{R}$.

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$F = \mathbb{Q}(\sqrt[3]{2}, \zeta)$ is Galois because it is the splitting field of $x^3 - 2$. Note that F contains L.

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If K is Galois, then $|\mathrm{Gal}(K/\mathbb{Q})| = [K : \mathbb{Q}]$.

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Some Concluding Remarks About Function Fields

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However, looking at the class number of function fields is still very hard.

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- These fields were explicity constructed using the properties of the *Shanks polynomials*.

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- They were used by Washington to find infinitely many cubic fields with class number divisible by n.

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There are infinitely many function fields of any degree m over $\mathbb{F}_q(T)$ with class number indivisible by ℓ, for **any odd prime** ℓ. [SMALL Algebraic Number Theory 2008]

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There are infinitely many quadratic function fields over $\mathbb{F}_q(T)$ with class number indivisible by 3. [Ichimura 1999]

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- These fields were explicity constructed using the properties of the *Shanks polynomials*.

There are infinitely many function fields of any degree m over $\mathbb{F}_q(T)$ with class number indivisible by ℓ, for **any odd prime** ℓ. [SMALL Algebraic Number Theory 2008]

- These fields were also explicitly constructed, but it required more than just the Shanks polynomials.

The Rikuna Polynomials

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Definition

Define the polynomials $p, q \in K[x]$ to be

$$p = \frac{\zeta_\ell^{-1}(x - \zeta_\ell)^\ell - \zeta_\ell(x - \zeta_\ell^{-1})^\ell}{\zeta_\ell^{-1} - \zeta_\ell}, \quad q = \frac{(x - \zeta_\ell)^\ell - (x - \zeta_\ell^{-1})^\ell}{\zeta_\ell^{-1} - \zeta_\ell}.$$

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Remark

When $\ell = 3$, the Rikuna polynomial reduces to the Shanks polynomial for $u = T$.

Generalizing Rikuna Polynomials Using Iterations

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Definition

The m-th **generalized Rikuna polynomial** is defined to be

$$r_m = p_m - Tq_m \in K(T)[x].$$

This was our main object of study.

Define K_m to be the splitting field of r_m over $K(T)$.

Splitting Fields of Generalized Rikuna Polynomials

Define K_m to be the splitting field of r_m over $K(T)$.

This gives a tower of fields, each containing $K(T)$.

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K_m \arrow[d, dash] \\
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One thing to study about such towers is the Galois group $\mathrm{Gal}(K_m/K(T))$.

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The Roots of Generalized Rikuna Polynomials

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Theorem

Define $\alpha(T) = \frac{\zeta_\ell - T}{\zeta_\ell^{-1} - T}$*. For all* $m \geq 1$*, the roots of* r_m *are*

$$\theta_c^{(m)} = \frac{\zeta_\ell - \zeta_{\ell m}^c \sqrt[\ell^m]{\alpha(T)}}{1 - \zeta_\ell \zeta_{\ell m}^c \sqrt[\ell^m]{\alpha(T)}}, \qquad \text{for } 0 \leq c \leq \ell^m - 1.$$

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Galois theory tells us how to find $\mathrm{Gal}(K_m/K(T))$ once we know $\mathrm{Gal}(L_m/K(T))$.

Describing $\mathrm{Gal}(L_m/K(T))$

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$$\rho_m : \begin{array}{ccc} \zeta_{\ell m} & \mapsto & \zeta_{\ell m}^{(\ell-1)^{\ell^{v-1}}} \\ \sqrt[\ell m]{\alpha(T)} & \mapsto & \dfrac{1}{\sqrt[\ell m]{\alpha(T)}} \end{array}, \quad \gamma_m : \begin{array}{ccc} \zeta_{\ell m} & \mapsto & \zeta_{\ell m} \\ \sqrt[\ell m]{\alpha(T)} & \mapsto & \zeta_{\ell m} \sqrt[\ell m]{\alpha(T)} \end{array},$$

where $v = \min\{b, m\}$ *and* b *depends only on* K.

Solving for $\mathrm{Gal}(K_m/K(T))$

Having a description of $\mathrm{Gal}(L_m/K(T))$, we find $\mathrm{Gal}(K_m/K(T))$ by restricting automorphisms of L_m to automorphisms of K_m.

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Theorem (SMALL 2010)

For all $m \geq 1$,

$$\mathrm{Gal}(K_m/K(T)) \simeq \mathbb{Z}/\ell^m\mathbb{Z} \rtimes_{\phi_m} \mathbb{Z}/\ell^{m-v}\mathbb{Z},$$

where $\rtimes_{\phi_m}$ is a semi-direct product.

Ramification of Primes of $K(T)$

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Proposition (Cullinan, 2010)

Let $\omega = \zeta_\ell + \zeta_\ell^{-1}$. The discriminant of r_m is given by

$$\operatorname{disc}(r_m) = \pm \ell^{m(\ell^m)} \omega^{(\ell^m - 2)(\ell^m - 1)} (T^2 - \omega T + 1)^{\ell^m - 1}.$$

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- The prime at infinity

Ramification of the Finite Prime

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 & K(T)(\zeta_{\ell^m}) \\
K(T) \arrow[ur, dash, "2\ell^{m-v}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathfrak{P}_1^{\ell^m}\mathfrak{P}_2^{\ell^m} \arrow[dd, dash] \\
\mathcal{P}^{\ell^m} \arrow[ur, dash] \arrow[dd, dash] & \\
 & \mathfrak{p}_1\mathfrak{p}_2 \\
P \arrow[ur, dash] &
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$$P = T^2 - \omega T + 1, \quad \mathfrak{p}_1 = T - \zeta_\ell, \quad \mathfrak{p}_2 = T - \zeta_\ell^{-1}$$

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  Factor the irreducible polynomial from $K(T)(\zeta_{\ell^m})$ to L_m in $K((\frac{1}{T}))(\zeta_{\ell^m})$, the completion of $K(T)(\zeta_{\ell^m})$ with the valuation of the prime of infinity.

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Riemann-Hurwitz Formula and Genus

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Theorem (Riemann-Hurwitz Formula)

For a finite, separable, geometric extension L/K of function fields, we have:

$$2g_L - 2 \geq [L:K](2g_K - 2) + \sum_{\mathfrak{P}} (e(\mathfrak{P}|P) - 1) \deg_L \mathfrak{P}$$

where the sum is over all primes $\mathfrak{P}$ of L which are ramified in L/K. The inequality is an equality if and only if all ramified primes are tamely ramified.

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Theorem (SMALL 2010)

For all $m \geq 1$, K_m and L_m have genus 0.

Main Theorem

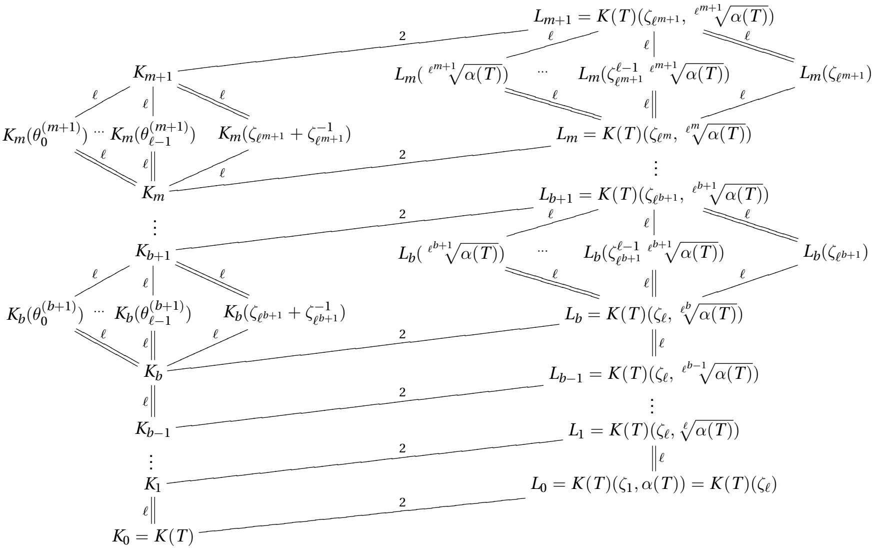
Theorem (SMALL 2010)

Let ℓ be any odd prime and ζ_ℓ be a ℓ-th root of unity. Let K be any perfect field with $\zeta_\ell + \zeta_\ell^{-1} \in K$ and $\zeta_\ell \notin K$.

We can construct explicitly an infinite tower of function fields $K(T) = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots$ such that

- *For all $m \geq 0$, K_{m+1}/K_m is an ℓ-extension.*
- *Exactly one prime of $K(T)$ ramifies in the tower.*
- *For all $m \geq 0$, $h_{K_m} = 1$.*

Towers of K_m and L_m



Ongoing Research, Further Questions

- For any odd integer $\ell \geq 3$,

$$\mathrm{Gal}(K_m/K(T)) \simeq \mathbb{Z}/\ell^m\mathbb{Z} \rtimes \mathbb{Z}/(\ell^m/b_m)\mathbb{Z},$$

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THANK YOU!