- $\exists x_n \in E : x_n \to x \Rightarrow x \in \overline{E} \Leftrightarrow$ for every open U, $x \in U$, we have $U \cap E \neq \emptyset$
- $x \in \overline{E}$ is a **limit point** of E when for every open U, $x \in U$, we have $(U - \{x\}) \cap E \neq \emptyset$
- D **dense** $\subseteq X$ when $\overline{D} = X$
- $A^O = \emptyset \Leftrightarrow A^C$ dense $\subseteq X$
- D dense in $E \subseteq X \Leftrightarrow \forall x \in E$ and open $U \subseteq X$, $x \in U$, we have $D \cap U \neq \emptyset$
- X **separable** when $\exists$ countable dense $D \subseteq X$
- $f : X \to Y$ **continuous** when $f^{-1}(U)$ is open in X for every open $U \subseteq Y$
- $f : X \to Y$ **homeomorphism** when f is a bijection and f, f^{-1} are continuous
- X **metric space** when $\exists d : X \times X \to \mathbb{R}_{\geq 0}$ such that $d(x,y) = d(y,x)$, $d(x,y) + d(y,z) \geq d(x,z)$, $d(x,y) \geq 0\, (= 0 \Leftrightarrow x = y)$
- X **(real) inner product space** when X an $\mathbb{R}$-vector space, $\exists$ bilinear $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R} : \langle x, y \rangle = \langle y, x \rangle$, $\langle x, x \rangle \geq 0\, (= 0 \Leftrightarrow x = 0)$
- X inner product space, then $|\langle x, y \rangle| \leq \|x\| \|y\|$ for any $x, y \in X$ (**Cauchy-Schwarz inequality**)
- X, Y metric spaces, $f : X \to Y$ **uniformly continuous** when $\forall \epsilon > 0$, $\exists \delta > 0 : \rho(f(x), f(y)) < \epsilon$ for all $x, y \in X$ with $d(x,y) < \delta$
- $\mathcal{T}$ a topology, then $\mathcal{B} \subseteq \mathcal{T}$ is a **base** when $\forall U \in \mathcal{T}$, we have $U = \bigcup_{G \in \mathcal{B} : G \subseteq U} G$
- $\mathcal{B} \subseteq \mathcal{T}$ a base $\Leftrightarrow \bigcup_{U \in \mathcal{B}} U = X$ and $\forall \{U_1, \ldots, U_n\} \subseteq \mathcal{B}$ and $x \in \bigcap U_i$, $\exists V \in \mathcal{B} : x \in V \subseteq \bigcap U_i$
- S **subbase** for $\mathcal{T}$ when $\mathcal{T}$ is the weakest topology containing S
- **Product topology** on $Y = \prod X_i$ is defined by base $\{U_1 \times \cdots \times U_n : U_i \text{ open} \subseteq X_i\}$, i.e. $\pi_i : Y \to X_i$ continuous
- x_n **converges to** x (i.e., $x_n \to x$) when $\forall$ open $U \subseteq X$, $x \in U$, $\exists N \in \mathbb{N} : \forall n > N, x_n \in U$
- X **Hausdorff** when $\forall x, y \in X$, $x \neq y$, $\exists U, V$ open $\subseteq X : x \in U$, $y \in V$, $U \cap V = \emptyset$
- X Hausdorff, then $(x_n \to x \text{ and } x_n \to y) \Rightarrow x = y$
- Y Hausdorff, $f, g : X \to Y$ continuous, then $\{x \in X : f(x) = g(x)\}$ is closed
- X metric space, $E \subseteq X$, then $x \in \overline{E} \Leftrightarrow \exists \{x_n\} \subseteq E : x_n \to x$
- $f : X \to Y$ continuous $\Rightarrow (x_n \to x \Rightarrow f(x_n) \to f(x))$ - X metric space, then they are equivalent
- $\{x_n\}$ **Cauchy** when $\forall \epsilon > 0$, $\exists N \in \mathbb{N} : \forall m, n > N$, $d(x_m, x_n) < \epsilon$
- convergent $\Rightarrow$ Cauchy $\Rightarrow$ bounded - for monotonic sequences, all three are equivalent
- If a Cauchy sequence has a convergent subsequence, the sequence converges, and to the same limit
- X **complete** when every Cauchy sequence converges
- X complete, $E \subseteq X$, then E closed $\Leftrightarrow E$ complete
- X complete, then $\{f_n : X \to \mathbb{R}\}$ pointwise bounded $\Rightarrow \exists$ open $V \neq \emptyset, M \in \mathbb{R} : |f_n(x)| \leq M$ for all $x \in V$ and all n
- X complete, $f : X \to X$ has $\exists \alpha < 1 : d(f(x), f(y)) \leq \alpha d(x,y)$, then $\exists! x \in X : f(x) = x$ and $\forall x_0 \in X$, $x_n = f^n(x_0)$, $x_n \to x$
- X complete, then G_n dense, open $\subseteq X \Rightarrow \bigcap G_n$ dense in X (**Baire category theorem**)
- $E \subseteq X$ **nowhere dense** if $(\overline{E^C})^O = \emptyset$
- $E \subseteq X$ **first category** if $E = \bigcup A_n$ where A_n are all nowhere dense (example: $\mathbb{Q}$)
- $E \subseteq X$ **second category** if E is not first category (example: $\mathbb{R} - \mathbb{Q}$)
- $E \subseteq X$ $\mathbf{F}_\sigma$ if $E = \bigcup C_n$ where C_n are all closed (example: $\mathbb{Q}$)
- $E \subseteq X$ $\mathbf{G}_\delta$ if $E = \bigcap U_n$ where U_n are all open (example: $\mathbb{R} - \mathbb{Q}$)
- F_n each first category, G_n each second category, then $\bigcup F_n$ first category, $\bigcap G_n$ second category
- $f : X \to \mathbb{R}$ a function, then $C = \{x \in X : f \text{ continuous at } x\}$ is G_δ $(U_n = \{x \in X : \exists \text{ open } V_{\frac{1}{n}, x} \subset X : x \in V_{\frac{1}{n}, x}, f(x) \in B_{\frac{1}{n}}(f(x))\})$
- Any closed set in $\mathbb{R}^n$ is G_δ
- Counterexample: $A = ([0,1] \cap \mathbb{Q}) \cup ([2,3] - \mathbb{Q})$ is neither F_σ nor G_δ
- X **compact** when any open cover $X = \bigcup U_\alpha$ has a finite subcover $X = \bigcup_{i=1}^n U_i$
- X compact, K_n closed $\subseteq X$ with $K_n \neq \emptyset$ and $K_{n+1} \subseteq K_n$ for all n, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$
- $(X, \mathcal{T})$ Hausdorff, $(X, \mathcal{F})$ compact, $\mathcal{T} \subseteq \mathcal{F}$, then $\mathcal{T} = \mathcal{F}$
- X compact, $E \subseteq X$, then E closed $\Rightarrow E$ compact
- X Hausdorff, $E \subseteq X$, then E compact $\Rightarrow E$ closed
- $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow K$ is closed and bounded
- X compact, $f : X \to Y$ continuous, then $f(X)$ compact
- X compact, Y Hausdorff, $f : X \to Y$ continuous bijection, then f homeomorphism
- X compact metric space, Y metric space, $f : X \to Y$ continuous, then f uniformly continuous
- X metric space, $E \subseteq X$ **bounded** if $\sup\{d(x,y) : x, y \in E\} < \infty$
- X metric space, $E \subseteq X$ **totally bounded** if $\forall \epsilon > 0$, $\exists \{x_1, \ldots, x_n\} \subseteq E : E \subseteq \bigcup B(x_k, \epsilon)$
- E totally bounded $\Rightarrow E$ bounded, $\overline{E}$ totally bounded, any $F \subseteq E$ is totally bounded - for $\mathbb{R}^n$, totally bounded $\Leftrightarrow$ bounded
- X metric space, then X compact $\Leftrightarrow X$ complete and totally bounded
- X **sequentially compact** when every sequence in X has a convergent subsequence
- X metric space, then X compact $\Leftrightarrow X$ sequentially compact
- X compact metric space, Y metric space, then Y complete $\Rightarrow C(X,Y)$ complete in uniform metric $d(f,g) = \sup\{\rho(f(x), g(x))\}$
- Y metric space, $\mathcal{F} \subseteq C(X,Y)$ **equicontinuous** at $x \in X$ when $\forall \epsilon > 0, \exists$ open $U \subseteq X$, $x \in U : \forall y \in U, f \in \mathcal{F}, \rho(f(x), f(y)) < \epsilon$
- X compact, Y compact metric space, $\mathcal{F} \subseteq C(X,Y)$, then $\mathcal{F}$ totally bounded $\Leftrightarrow \mathcal{F}$ equicontinuous
- X compact, Y compact metric space, $\mathcal{F} \subseteq C(X,Y)$ equicontinuous $\Rightarrow$ every $\{f_n\} \subseteq \mathcal{F}$ has uniformly convergent subsequence
- $X \subset \mathbb{R}$ compact, $\{f_n\} \subseteq C(X, \mathbb{R})$ and $f_n \to f$ uniformly, then $\{f_n\}$ equicontinuous
- X **connected** when $\nexists$ open $U, V \subseteq X$: $U \cap V = \emptyset$, $U \cup V = X$
- X connected $\Leftrightarrow \nexists\, U$ open, closed $\subseteq X$: $U \neq \emptyset, X$
- $E \subseteq \mathbb{R}$ is connected $\Leftrightarrow E$ is an interval
- X connected, $f : X \to Y$ continuous, then $f(X)$ connected
- If E_α connected for all α and $\bigcap E_\alpha \neq \emptyset$, then $\bigcup E_\alpha$ connected
- E connected $\Rightarrow \overline{E}$ connected
- For each $x \in X$, $C_x = \bigcup_{\text{connected } E \subseteq X : x \in E} E$, then the C_x are connected and closed, and partition X

$\boxed{\overline{A} \neq \overline{A^O},\ A^O \neq (\overline{A})^O}$

$\boxed{(E^O)^C = \overline{(E^C)}}$

- The polynomial functions are dense in $C([a,b], \mathbb{R})$ with the uniform topology (**Stone-Weierstrass**)

- $\mathcal{A} \subseteq P(X)$ **algebra** when $\emptyset \in \mathcal{A}$ and $\forall A, B \in \mathcal{A}$, $A^C, A \cup B \in \mathcal{A}$
- $\mu : \mathcal{A} \to \mathbb{R}^*_{\geq 0}$ **additive** when $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$
- For $x \in X$, **point mass** at x is $\delta_x : P(X) \to \mathbb{R}$ with $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise

- $\mathcal{A} \subseteq P(X)$ **σ-algebra** when $\mathcal{A}$ algebra and $\forall \{A_k\} \subseteq A, \bigcup A_k \in \mathcal{A}$
- $\mathcal{B}$ **Borel algebra** is σ-algebra generated by closed subsets of X
- $\mathcal{A}$ algebra, $\mu : \mathcal{A} \to \mathbb{R}^*_{\geq 0}$ **countably additive** if $\mu(\bigcup A_k) = \sum \mu(A_k)$ for countably many disjoint $A_k \in \mathcal{A}$ and $\bigcup A_k \in \mathcal{A}$
- μ additive, then μ countably additive $\Leftrightarrow$ $(A_n \in \mathcal{A},\ A_n \subseteq A_{n+1}$ for all n, $\bigcup A_n \in \mathcal{A} \Rightarrow \mu(\bigcup A_n) = \lim_{n\to\infty} \mu(A_n))$
- $\mathcal{A}$ σ-algebra, $\mu : \mathcal{A} \to \mathbb{R}^*_{\geq 0}$ **measure** when μ is countably additive
- $\mathcal{A}$ σ-algebra, μ, ν measures on $\mathcal{A}$, then so are $\mu + \nu$, $t\mu$ (for $t \in \mathbb{R}^+$), and μ_A (where $\mu_A(E) = \mu(A \cap E)$)
- $\mathcal{A}$ σ-algebra, $\mathcal{M}$ directed set of measures on $\mathcal{A}$ ($\mu \leq \nu$ when $\forall A \in \mathcal{A}, \mu(A) \leq \nu(A)$), then $\sup \mathcal{M}$ is a measure on $\mathcal{A}$
- $\mu^* : P(X) \to \mathbb{R}^*_{\geq 0}$ **outer measure** when $\mu^*(\emptyset) = 0$, $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$, and $\forall \{A_n\} \subseteq P(X), \mu^*(\bigcup A_n) \leq \sum \mu^*(A_n)$
- μ^* outer measure on X, $E \subseteq X$ **μ^*-measurable** when $\forall A \subseteq X$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C)$ (we always have $\leq$)
- $\mu^*(E) = 0 \Rightarrow E$ measurable
- μ^* outer measure on X, $\mathcal{M} = \{\mu^*\text{-measurable sets}\}$, then $\mu^*_{\mathcal{M}}$ a measure on X
- m^* **Lebesgue outer measure** when $m^*(E) = \inf\{\sum \ell(I_n) : E \subseteq \bigcup I_n\}$
- If $A \subseteq \mathbb{R}^n$ is Lebesgue measurable, $\exists$ Borel $F, G : F \subseteq A \subseteq G$ and $m(G - A) = m(A - F) = 0$
- Lebesgue outer measure is translation invariant
- $A, B \subseteq \mathbb{R}$ measurable, $A \subseteq E \subseteq B$, then E measurable
- If $m(E) > 0$, then E contains a non-measurable set
- COUNTEREXAMPLE: $A = \{\text{representatives for } \sim\} \subset [0, 1)$ where $x \sim y$ when $x - y \in \mathbb{Q}$ is not Lebesgue measurable
- COUNTEREXAMPLE: Fat Cantor set is nowhere dense but has positive Lebesgue measure (take $[0, 1]$, at nth step subtract width $\frac{1}{2^{2n}}$ interval from each of the 2^{n-1} remaining intervals)
- COUNTEREXAMPLE: Cantor set C is perfect and bounded, hence compact and uncountable, but $m(C) = 0$ and C is nowhere dense (because it is closed and contains no interval) - C is also totally disconnected
- μ measure on $\mathcal{A}$, then $\mu(\bigcup E_n) \leq \sum \mu(E_n)$ for any $E_n \in \mathcal{A}$ (not necessarily disjoint)
- μ measure on $\mathcal{A}$ and $\mu(E_1) < \infty$ and $E_{n+1} \subseteq E_n$ for all n, then $\mu(\bigcap E_n) = \lim_{n\to\infty} \mu(E_n)$
- $\liminf E_n = \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty E_m$ (all but finitely many), $\limsup E_n = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty E_m$ (infinitely many)
- μ measure on $\mathcal{A}$, $E_n \in \mathcal{A}$, then $\mu(\liminf E_n) \leq \liminf \mu(E_n)$
- μ measure on $\mathcal{A}$, $E_n \in \mathcal{A}$, $\sum \mu(E_n) < \infty$, then $\mu(\limsup E_n) = 0$
- $f : X \to Y$ continuous, then $B \subseteq Y$ Borel $\Rightarrow f^{-1}(B) \subseteq X$ Borel

$\{x : \sup f_n(x) > t\} = \bigcup\{x : f_n(x) > t\}$

$\{x : \inf f_n(x) < t\} = \bigcup\{x : f_n(x) < t\}$

- $f : X \to \overline{\mathbb{R}}$ **$\mathcal{A}$-measurable** when $f^{-1}(\infty), f^{-1}(-\infty), f^{-1}(U) \in \mathcal{A}$ for all open $U \in \mathbb{R}$
- f measurable $\Leftrightarrow \{x : f(x)(>, \geq, \leq, <)t\} \in \mathcal{A} \Leftrightarrow f^{-1}(B) \in \mathcal{A}$ whenever B Borel or $B = \{\infty\}, \{-\infty\}$
- f_n, f, g measurable $\Rightarrow |f|, f^2, f + g, fg, \sup f_n, \inf f_n, \limsup f_n, \liminf f_n$ all measurable
- f_n, f measurable, $f_n \to f$ **converges in measure** when, $\forall \epsilon > 0$, $\lim \mu(\{x : |f(x) - f_n(x)| \geq \epsilon\}) = 0$
- f **simple** if image is finite
- $f \geq 0$ measurable, then $\exists$ $f_n \geq 0$ simple measurable : $f_n \uparrow$ and $\lim f_n = f$ pointwise (use $f_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbf{1}_{A_{n,k}} + n\mathbf{1}_{B_n}$ where $A_{n,k} = \{x : (k-1)2^{-n} < f(x) \leq k2^{-n}\}$ and $B_n = \{x : f(x) > n\}$)
- f measurable, then $\exists$ f_n simple measurable such that $f_n \to f$ pointwise - when f is bounded, $f_n \to f$ uniformly
- $f = \sum c_k \mathbf{1}_{A_k} \geq 0$ simple measurable, **integral** of f is $\int f = \sum c_k \mu(A_k)$ (note $0 \leq \int f d\mu \leq \infty$)
- $\int f d\mu \leq \int g d\mu$ if $f \leq g$, $\int (f + g) d\mu = \int f d\mu + \int g d\mu$, $\int t f d\mu = t \int f d\mu$ for $t \geq 0$
- $f \geq 0$ measurable, then **integral** of f is $\int f d\mu = \sup\{\int g d\mu : 0 \leq g \leq f, g \text{ simple measurable}\}$
- $f \geq 0$, then $\int f d\mu = 0 \Leftrightarrow f = 0$ a.e.
- For any $f \geq 0$ measurable, $\nu(E) = \int_E f d\mu$ is a measure on $\mathcal{A}$
- $f_n \geq 0$ measurable with $f_n \uparrow$, then $\int \lim f_n d\mu = \lim \int f_n d\mu$ (**monotone convergence**)
  $f = \lim f_n = \sup f_n$ is measurable because sup's are, so $\int f d\mu$ exists. $f_n \leq f_{n+1}$, so $\int f_n d\mu \leq \int f_{n+1} d\mu$, so $\lim \int f_n d\mu$ exists (though could be ∞). $f_n \leq f$, so $\lim \int f_n d\mu \leq \int f d\mu$. Let g simple measurable have $0 \leq g \leq f$, and fix $0 < \epsilon < 1$, and let $A_n = \{x \in X : f_n(x) \geq (1 - \epsilon) g(x)\}$ - then $A_n \subseteq A_{n+1}$ for all n because $f_n \uparrow$, and $\bigcup A_n = X$ because $\lim f_n = \sup f_n \geq g$. Becuase $\int f_n d\mu \geq \int_{A_n} f_n d\mu \geq (1 - \epsilon) \int_{A_n} g d\mu$ and $\nu(E) = \int_E g d\mu$ is a measure, $\int_{A_n} g d\mu \to \int g d\mu$, so that $\lim \int f_n d\mu \geq (1 - \epsilon) \int g d\mu$ for all $\epsilon > 0$, so $\lim \int f_n d\mu \geq \int g d\mu$, thus $\lim \int f_n d\mu \geq \int f d\mu$.
- $f_n \geq 0$, then $\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$ (**Fatou's lemma**)
- COUNTEREXAMPLE: $f_n = n\mathbf{1}_{(0, \frac{1}{n})}$ has $\liminf f_n = 0$ and hence $\int \liminf f_n d\mu = 0$, but $\liminf \int f_n d\mu = \liminf 1 = 1$
- f measurable, **integral** of f is $\int f = \int f^+ d\mu - \int f^- d\mu$ - f **integrable** (or **summable**) if $\int |f| d\mu < \infty$
- f_n measurable, $f_n \to f$, and $|f_n| \leq g$ where g integrable, then f integrable, and $\int f d\mu = \lim \int f_n d\mu$ (**dominated convergence**)
- $\mu(\{x : f(x) > t\}) \leq \frac{1}{t} \int f d\mu$ (**Chebyshev's inequality**)
- μ a measure on X with $\mu(X) = 1$, $f : X \to J$ integrable for some interval J, ϕ convex on J, then $\phi(\int f d\mu) \leq \int \phi(f) d\mu$ (**Jensen's inequality**)
- $1 < p < \infty$, $q = \frac{p}{p-1}$, $f, g \geq 0$ measurable, then $\int fg d\mu \leq (\int f^p d\mu)^{\frac{1}{p}} (\int g^q d\mu)^{\frac{1}{q}}$ - if $f \in L^p$ and $g \in L^q$, $fg \in L^1$ and $|\int fg| \leq \|f\|_p \|g\|_q$ (**Holder's inequality**)
- f bounded, then f measurable $\Leftrightarrow f \in L^1$
- $1 \leq p < \infty$, $f_n \in L^p$, $f_n \to f$ in L^p, then $\lim \int |f_n|^p = \int |f|^p$
- $1 \leq p < \infty$, $f_n, f \in L^p$, $f_n \to f$ a.e., then $f_n \to f$ in $L^p \Leftrightarrow \|f_n\|_p \to \|f\|_p$
- Always: $f_n \to f$ in $L^p \Rightarrow$ in measure - if $\mu(X) < \infty$, a.e. $\Rightarrow$ in measure - if $|f_n| \leq g \in L^p$, $L^p \Leftrightarrow$ in measure and a.e. $\Rightarrow$ in measure
- COUNTEREXAMPLE: $f_1 = \mathbf{1}_{[0,1]}$, $f_2 = \mathbf{1}_{[0, \frac{1}{2}]}$, $f_3 = \mathbf{1}_{[\frac{1}{2}, 1]}$, etc. $f_n \to 0$ in measure, L^p but $f_n \not\to f$ pointwise a.e.
- COUNTEREXAMPLE: $f_n = n\mathbf{1}_{[\frac{1}{n}, \frac{2}{n}]}$, then $f_n \to 0$ pointwise a.e., $f_n \to 0$ in measure, but $\|f_n\|_p = 1$, so cannot converge to 0 in L^p
- COUNTEREXAMPLE: $f_n = \mathbf{1}_{[n, n+1]}$, then $f_n \to 0$ everywhere, but $f_n \not\to 0$ in measure
- $f \in L^1([a, b])$ and $\forall x \in [a, b], \int_a^x f dt = 0$, then $f = 0$ a.e.
- $\sum \int |f_n| < \infty \Rightarrow \sum f_n$ converges, and $\int \sum f_n = \sum \int f_n$
- $\int |f|^p < \infty \Rightarrow F(t) \leq Ct^{-p}$, $f \in L^1 \Rightarrow nF(n) \to 0$, $\int |f| < \infty \Rightarrow \sum F(n) < \infty$ - for $\mu(X) < \infty$, they are equivalent
- f measurable, $E \subseteq \mathbb{R}^n$ measurable, then $\int_E |f|^p dm = \int_0^\infty p t^{p-1} m(\{x \in E : |f(x)| > t\}) dt$
- $T : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation, $A \subseteq \mathbb{R}^n$ Lebesgue measurable, then $T(A)$ is as well and $m(T(A)) = |\det(T)| m(A)$
- T invertible, $f : \mathbb{R}^n \to \mathbb{R}$ measurable, and either $f \geq 0$ or f integrable, $\int f dm = |\det(T)| \int f \circ T dm$
- U, V open $\subseteq \mathbb{R}^n$, $\phi : U \to V$ homeomorphism, $\phi, \phi^{-1} \in C^1$, then for any $f \geq 0$ measurable on V, $\int_V f dm = \int_U (f \circ \phi) |J_\phi| dm$
- $f \geq 0$ Borel on $\mathbb{R}^n$, $F(x) = \int_{\mathbb{R}^\ell} f(x, \cdot) dm$, $G(y) = \int_{\mathbb{R}^k} f(\cdot, y) dm$, then F, G Borel and $\int_{\mathbb{R}^n} f dm = \int_{\mathbb{R}^k} F dm = \int_{\mathbb{R}^\ell} G dm$
- f Borel on $\mathbb{R}^n$ and integrable, then $\int_{\mathbb{R}^n} f dm = \int_{\mathbb{R}^k} \int_{\mathbb{R}^\ell} f(x, y) dy dx = \int_{\mathbb{R}^\ell} \int_{\mathbb{R}^k} f(x, y) dx dy$ (**Fubini-Tonelli theorem**)
- COUNTEREXAMPLE: μ Lebesgue, ν counting, $A = \{(x, y) : x = y\} \subset [0, 1]^2$, $\iint \mathbf{1}_A d\mu d\nu = 0$ but $\iint \mathbf{1}_A d\nu d\mu = 1$

Graph of a measurable function has measure 0