Playing Fast and Loose With Geometric Series

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Act 1

Given a function $f(x)$, let the symbol $\int f$ stand for $\int_0^x f(t)\,dt$.

Let's say we want to find the function $f(x)$ that satisfies

$$\int f = f - 1.$$

Rearranging, we have

$$1 = f - \int f.$$

Factoring,

$$1 = \left(1 - \int\right) f.$$

Multiplying both sides by the inverse,

$$f = \left(1 - \int\right)^{-1} 1.$$

Applying the geometric series,

$$\begin{aligned}
f &= \left(1 + \int + \int^2 + \int^3 + \cdots\right) 1 \\
&= 1 + \left(\int 1\right) + \left(\int^2 1\right) + \left(\int^3 1\right) + \cdots \\
&= 1 + x + \left(\int x\right) + \left(\int^2 x\right) + \cdots \\
&= 1 + x + \frac{x^2}{2} + \left(\int \frac{x^2}{2}\right) + \cdots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\
&= e^x
\end{aligned}$$

which can be easily checked to be the correct answer.

Act 2

Given a function $f(n)$, let the symbol $\sum f$ stand for $\sum_{k=0}^{n-1} f(k)$.

Let's say we want to find the function $f(n)$ that satisfies

$$\sum f = f - 1.$$

Rearranging, we have

$$1 = f - \sum f.$$

Factoring,

$$1 = \left(1 - \sum\right) f.$$

Multiplying both sides by the inverse,

$$f = \left(1 - \sum\right)^{-1} 1.$$

Applying the geometric series,

$$\begin{aligned}
f &= \left(1 + \sum + \sum^2 + \sum^3 + \cdots\right) 1 \\
&= 1 + \left(\sum 1\right) + \left(\sum^2 1\right) + \left(\sum^3 1\right) + \cdots \\
&= 1 + n + \left(\sum n\right) + \left(\sum^2 n\right) + \cdots \\
&= 1 + n + \frac{n(n-1)}{2} + \left(\sum \frac{n(n-1)}{2}\right) + \cdots \\
&= 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} + \cdots \\
&= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots \\
&= 2^n
\end{aligned}$$

which can be easily checked to be the correct answer.

Act 3

Let ϵ be really small. Given a function $f(x)$, let the symbol Sf stand for the function $f(x-\epsilon)$.

Define the "derivative" $D = \dfrac{1-S}{\epsilon}$, so that the symbol Df stands for the function $\dfrac{f(x)-f(x-\epsilon)}{\epsilon}$.

Let's say we want to find the function $D^{-1}f$.

Applying the geometric series,

$$\begin{aligned}
D^{-1}f &= \left(\frac{1-S}{\epsilon}\right)^{-1} f \\
&= \left(\frac{\epsilon}{1-S}\right) f \\
&= \epsilon\left(1+S+S^2+S^3+\cdots\right) f \\
&= \epsilon(f+Sf+S^2f+S^3f+\cdots) \\
&= \epsilon(f(x)+f(x-\epsilon)+f(x-2\epsilon)+f(x-3\epsilon)+\cdots)
\end{aligned}$$

which is the "antiderivative" of f, as it is reminiscent of the Riemann sum for

$$\int_{-\infty}^{x} f(t)\,dt.$$