Vector Bundles

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Review of tangent spaces to an abstract manifold

Let M be a manifold. The set of all smooth, real-valued functions

$$C^\infty(M) = \{\text{smooth } f : M \to \mathbb{R}\}$$

forms an $\mathbb{R}$-vector space, with operations

$$(f+g)(p) := f(p) + g(p) \qquad (r \cdot f)(p) = r \cdot f(p).$$

A **derivation at a point** $p \in M$ is an $\mathbb{R}$-linear function $X : C^\infty(M) \to \mathbb{R}$ satisfying

$$X(fg) = f(p)X(g) + g(p)X(f)$$

(this is a version of the product rule). The collection of all derivations at $p \in M$ form an n-dimensional $\mathbb{R}$-vector space, called T_pM, **the tangent space of** M **at** p, with operations

$$(X+Y)(f) := X(f) + Y(f) \qquad (r \cdot X)(f) := r \cdot X(f)$$

We call an $X \in T_pM$ a **tangent vector at** p. Intuitively, the tangent space is the collection of all possible "directions in which to take a directional derivative at p"; the $X \in T_pM$ are the resulting "directional derivative operators".

Let $\phi : U \to V$ be a chart of M whose domain U contains $p \in M$. For each $i = 1, \ldots, n$, define the coordinate functions $x^i : U \to \mathbb{R}$ by $x^i = \pi^i \circ \phi$, where $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is the projection on the ith coordinate. Then the tangent vectors $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p \in T_pM$ are defined by their action on $f \in C^\infty(M)$,

$$\left(\frac{\partial}{\partial x^i}\bigg|_p\right)(f) := \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(p))$$

For any chart ϕ, the corresponding coordinate tangent vectors $\frac{\partial}{\partial x^i}\big|_p$ form a basis for the tangent space T_pM (this was proven earlier in the lectures).

Let $\gamma : (a,b) \to M$ be a curve on M. The **tangent vector to** γ **at** $t_0 \in (a,b)$ is the tangent vector $\gamma'(t_0) \in T_{\gamma(t_0)}M$ which acts on functions $f \in C^\infty(M)$ by

$$(\gamma'(t_0))(f) := \frac{d}{dt}\bigg|_{t_0}(f \circ \gamma) = \frac{d(f\circ\gamma)}{dt}(t_0).$$

This can be thought of as taking the derivative of f "along" the curve γ.

We can see that this is a special case of the **pushforward**. Recall that, given any smooth map of manifolds $g : M \to N$ and a point $p \in M$, the pushforward $g_* : T_pM \to T_{g(p)}N$ is the linear map which takes a tangent vector $X \in T_pM$ to the tangent vector $g_*X \in T_{g(p)}N$ which acts on $f \in C^\infty(N)$ by

$$(g_*X)(f) := X(f \circ g) \in T_{g(p)}(N).$$

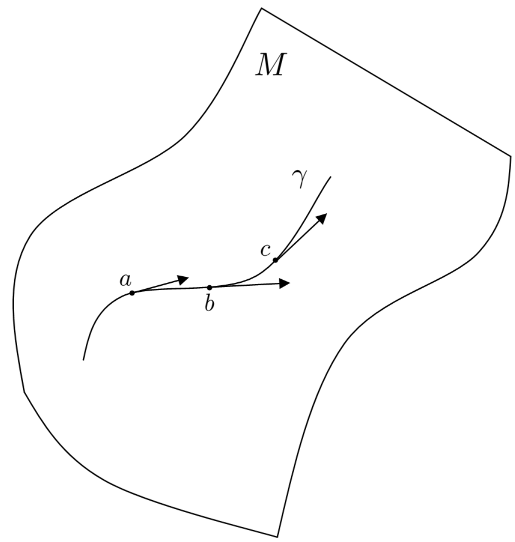
The tangent space $T_{t_0}(a,b)$ of (a,b) at t_0 is 1-dimensional, because (a,b) is a 1-dimensional manifold, and indeed, $T_{t_0}(a,b)$ is spanned by the tangent vector $\frac{d}{dt}|_{t_0}$ (this arises from the standard chart on (a,b), i.e. the identity map). Calculating the pushforward of the tangent vector $\frac{d}{dt}|_{t_0}$ by γ_*, we find

$$\left(\gamma_*\left(\frac{d}{dt}\bigg|_{t_0}\right)\right)(f) = \left(\frac{d(\ \ \circ\gamma)}{dt}\bigg|_{t_0}\right)(f) = \frac{d(f\circ\gamma)}{dt}(t_0) = (\gamma'(t_0))(f).$$

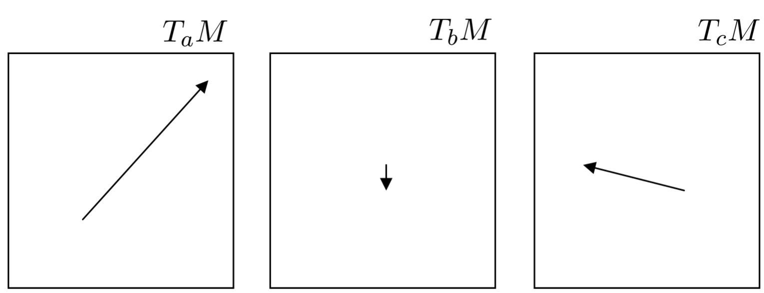
Motivation

It's kind of weird that tangent vectors at different points on a curve all live in completely unrelated spaces!

If a, b, and c are "nearby" points in the curve γ on the manifold M,



maybe the tangent vectors at a, b, and c look like this inside T_aM, T_bM, and T_cM, respectively:



Absurd! But how can we capture the idea that nearby tangent spaces should "cohere" or "be related"? What does it mean for the tangent vector to "smoothly vary" from point to point if the tangent spaces are all invisible to each other?

Of course, it'd be nice if all tangent vectors to M were elements of some big space. If we were working in a submanifold of $\mathbb{R}^n$, then we could try to compare the tangent vectors within $\mathbb{R}^n$; but we're working in an abstract manifold, so we have to do everything ourselves. How can we solve this problem?

Glue all of the T_pM's together, and voila, we have a big space that contains all of our tangent vectors!

The tangent bundle

We define the **tangent bundle** of M to be the set

$$TM := \coprod_{p \in M} T_pM$$

where $\coprod$ denotes the disjoint union. We will **not** be giving TM the disjoint union topology, though.

As a part of the definition of the disjoint union, each $X \in T_pM$ is now labeled with the point it came from; i.e., an element of TM looks like an ordered pair (p, X), where $X \in T_pM$. The set TM comes with a natural map $\pi : TM \to M$, defined by $\pi(p, X) = p$. Clearly, this means that $\pi^{-1}(p) = T_pM$.

TM **has a manifold structure**! (though really, what else did you expect?) It has dimension $2n$ when M has dimension n; intuitively, this is because we have n degrees of freedom from M, and then n more degrees of freedom from the tangent space at each point. We construct it as follows:

Given a chart $\phi : U \to V$ of M, with coordinate functions $x^1, \ldots, x^n$, define the map $\widetilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\phi}\left(p, \sum_{i=1}^{n} v_i \frac{\partial}{\partial x^i}\bigg|_p\right) = (x^1(p), \ldots, x^n(p), v_1, \ldots, v_n).$$

The image of $\widetilde{\phi}$ is $V \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. Also, $\widetilde{\phi}$ is bijective onto its image. This means we can simply "import" the topology from $V \times \mathbb{R}^n$, and put it on $\pi^{-1}(U)$; now, $\pi^{-1}(U)$ is homeomorphic to an open subset of $\mathbb{R}^{2n}$ via $\widetilde{\phi}$.

So, we want to simply take all of our charts ϕ of M, construct the corresponding maps $\widetilde{\phi}$ on TM, and declare those to be our charts on TM. But before we can do that, we have to check that they are smoothly compatible. So, let $\phi : U \to V$ and $\psi : W \to Y$ be two charts of M, having coordinate functions $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ respectively, and transition function $t = (\psi \circ \phi^{-1}) : \phi(U \cap W) \to \psi(U \cap W)$. Let $t^1, \ldots, t^n$ be the coordinate functions of t. Finally, let $\widetilde{\phi}$ and $\widetilde{\psi}$ be the corresponding soon-to-be-charts of TM.

Recall the statement of the chain rule using tangent vectors:

$$\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^{n} \frac{\partial t^j}{\partial x^i}(\phi(p)) \frac{\partial}{\partial y^j}\bigg|_p.$$

Let $\phi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n$. The transition map $(\widetilde{\psi} \circ \widetilde{\phi}^{-1}) : \phi(U \cap W) \times \mathbb{R}^n \to \psi(U \cap W) \times \mathbb{R}^n$ is given by

$$(\widetilde{\psi} \circ \widetilde{\phi}^{-1})(\underbrace{x^1(p), \ldots, x^n(p)}_{\phi(p)}, v_1, \ldots, v_n) =$$

$$\Big(\underbrace{t^1(\phi(p)), \ldots, t^n(\phi(p))}_{\psi(p)}, \sum_{j=1}^{n} \tfrac{\partial t^1}{\partial x^j}(\phi(p))v_j, \ldots, \sum_{j=1}^{n} \tfrac{\partial t^n}{\partial x^j}(\phi(p))v_j\Big)$$

The transition $\widetilde{\psi} \circ \widetilde{\phi}^{-1}$ applies t to the first n coordinates, and applies the matrix $\left(\frac{\partial t^i}{\partial x^j}(\phi(p))\right)_{ij}$ to the second n coordinates. The map t is smooth because t was the transition function of the charts ϕ and ψ on M, and linear transformations are always smooth, so together, this means that the transition $\widetilde{\psi} \circ \widetilde{\phi}^{-1}$ is smooth.

Thus, TM is indeed a smooth manifold of dimension $2n$.

But there is still the matter of "coherence" - we've made a big space where every T_pM is represented, but how are "nearby" T_pM's related?

The vector bundle structure on TM

Proposition. For any $p \in M$, there exists an open $U \subseteq M$ with $p \in U$ such that

- there is a **diffeomorphism** $\Phi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$,
- the following triangle commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi", "\sim"'] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

  (where $\pi_1 : U \times \mathbb{R}^n \to U$ is the projection on the first factor), and
- for all $q \in U$, the map Φ restricted to $\pi^{-1}(q) = T_qM$, i.e. the map $\Phi|_{T_qM} : T_qM \to \{q\} \times \mathbb{R}^n$, is a linear **isomorphism**.

Proof. For any $p \in M$, there is a chart $\phi : U \to V$ of M whose domain U contains p. Let $x^1, \ldots, x^n$ be the coordinate functions of ϕ. Then define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(q, \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_q\right) = (q, v^1, \ldots, v^n).$$

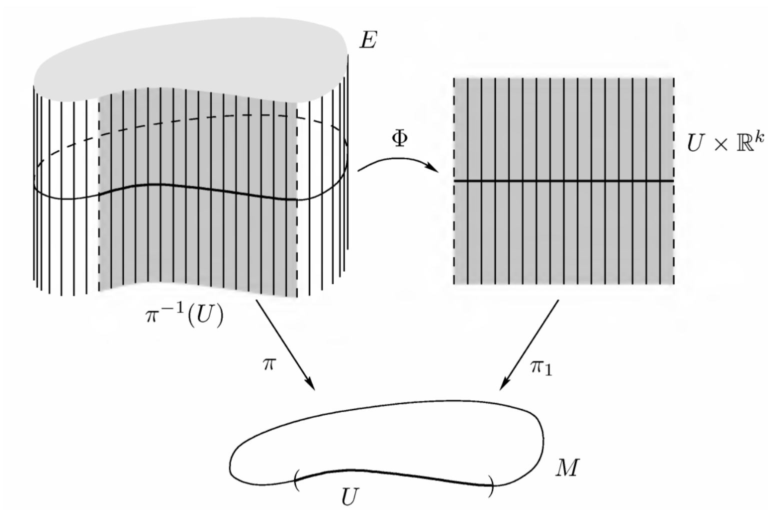
Clearly, we have that $\pi_1 \circ \Phi = \pi$. When we fix q (i.e. restrict Φ to a particular T_qM), this is also linear.

Let $\widetilde{\phi} : \pi^{-1}(U) \to V \times \mathbb{R}^n$ be the chart of TM corresponding to ϕ. Then the composition

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\phi \times \mathrm{id}_{\mathbb{R}^n}} V \times \mathbb{R}^n$$

equals $\widetilde{\phi}$, which is a diffeomorphism (simply look at the definition of $\widetilde{\phi}$). Also note that, because $\phi : U \to V$ is a diffeomorphism, so is $(\phi \times \mathrm{id}_{\mathbb{R}^n}) : U \times \mathbb{R}^n \to V \times \mathbb{R}^n$. Because $\Phi \circ (\phi \times \mathrm{id}_{\mathbb{R}^n})$ is a diffeomorphism and $\phi \times \mathrm{id}_{\mathbb{R}^n}$ is a diffeomorphism, we must have that Φ is itself a diffeomorphism. This, together with the fact that $\pi = \pi_1 \circ \Phi$, forces Φ to be bijective on fibers, i.e. $\Phi|_{T_qM} : T_qM \to \{q\} \times \mathbb{R}^n$ has to be a bijection for all $q \in U$. We've already verified these maps are linear, so they are in fact isomorphisms. $\square$

The above property is makes TM a **bundle**, in the proper sense. The diffeomorphisms ϕ are called **local trivializations**. Here is an illustration of a local trivialization (taken from Lee's *Introduction to Smooth Manifolds*, p.104):



Here are some examples of tangent bundles:

- $T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$, the cylinder.
- $T\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$, the product of the torus and the plane.
- $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, i.e. $2n$-dimensional Euclidean space.

Hmmm... maybe this isn't so interesting a concept after all?

Actually, if M is an n-manifold, then $TM \cong M \times \mathbb{R}^n$ **if and only if** M is the product of a Lie group and some number of copies of $\mathbb{S}^7$ (the examples I gave above were all Lie groups). When this is the case, we say that M is **parallelizable**. The only spheres that are Lie groups are $\mathbb{S}^1$ and $\mathbb{S}^3$. In general, it is not so easy to describe the tangent bundle to a manifold in such a nice way.

Vector bundles in general

The structure we found on TM with the local trivializations is just one example of a general concept, that of a **vector bundle**.

If $\pi : E \to M$ is a surjective smooth map of manifolds, and for any $p \in M$,

- $E_p := \pi^{-1}(p)$ is a k-dimensional vector space for all $p \in M$,
- there exists an open $U \subseteq M$ with $p \in U$ with a diffeomorphism $\Phi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^k$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi", "\sim"'] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

  commutes, and

- $\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism for all $q \in U$,

then we say that $\pi : E \to M$ is a **rank-k vector bundle over M**. The maps Φ are called **local trivializations**, and the open sets U they are defined over are called **trivializing neighborhoods**. The vector space E_p is called the **fiber** over p.

The easiest example of a vector bundle is the **trivial bundle** over M, where $E = M \times \mathbb{R}^k$ and $\pi : E \to M$ is just projection on the first factor. Such a bundle can take as a trivializing neighborhood all of M. (Also, do you see why the Φ's are called "trivializations" now?)

The first example of a non-trivial bundle is the Mobius bundle over $\mathbb{S}^1$. This bundle models the Mobius strip, hence the name. We let $E = [0,1] \times \mathbb{R}/\sim$, where

$$(a,b) \sim (c,d) \iff b = -d \text{ and } \begin{cases} a = 0, c = 1, \text{ or} \\ a = 1, c = 0 \end{cases}$$

This takes the infinite strip $[0,1]\times\mathbb{R}$, and glues together the edges in a way that gives it the characteristic half-twist of the Mobius strip. We'll denote the equivalence class of (x,y) under $\sim$ by $\langle x,y\rangle$. We let $M=\mathbb{S}^1$, define $\pi: E\to M$ by

$$\pi(\langle x,y\rangle)=e^{2\pi ix}$$

(which is easily checked to be well-defined).

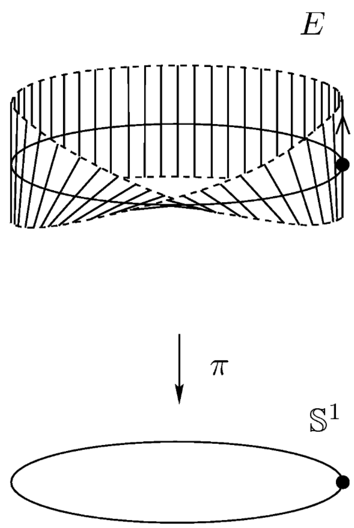
Let $U=\mathbb{S}^1-\{1\}$, so that $\pi^{-1}(U)=(0,1)\times\mathbb{R}/\sim$, which (after looking at the definition of $\sim$) is just $(0,1)\times\mathbb{R}$ itself. We can let the local trivialization Φ_1 over U be the identity map to $(0,1)\times\mathbb{R}$.

Let $V=\mathbb{S}^1-\{-1\}$, so that $\pi^{-1}(V)=\left([0,\frac{1}{2})\cup(\frac{1}{2},1]\right)\times\mathbb{R}/\sim$. We define the local trivialization over V to be the map

$$\Phi_2(\langle x,y\rangle)=\begin{cases}(x-\frac{1}{2},y) \text{ if } x>\frac{1}{2},\\ (x+\frac{1}{2},y) \text{ if } x<\frac{1}{2}\end{cases}$$

which is kind of a pain to check is smooth, but it is.

Here is a nice illustration of the Mobius bundle (taken from Lee's *Introduction to Smooth Manifolds*, p.106):



Operations on vector bundles

Basically, anything you can do with a vector space, you can do with a bundle, by "unstitching" the bundle, doing that operation to each fiber, and then "stitching" them all up again. For example, given bundles $\pi_1: E\to M$ and $\pi_2: F\to M$, we can create

- the **dual bundle** $\pi_1^*: E^*\to M$, for which $(E^*)_p:=(E_p)^*$,
- the **direct sum bundle** $(\pi_1\oplus\pi_2): E\oplus F\to M$, for which $(E\oplus F)_p:=E_p\oplus F_p$,
- the **tensor bundle** $(\pi_1\otimes\pi_2): E\otimes F\to M$, for which $(E\otimes F)_p:=E_p\otimes F_p$,
- the **alternating product bundle** $(\pi_1\wedge\pi_2): E\wedge F\to M$, for which $(E\wedge F)_p:=E_p\wedge F_p$,

and so on. Also, given a bundle $\pi: E\to N$ and a smooth map of manifolds $f: M\to N$, we can create the **pullback bundle** $f^*\pi: f^*E\to M$, defined by $(f^*E)_p:=E_{f(p)}$ (the fiber over $p\in M$ is just taken to be the fiber over $f(p)\in N$ in the original bundle).

Maps of vector bundles

The definition of a map of bundles is straightforward. Given two bundles $\pi_1 : E \to M$ and $\pi_2 : B \to N$, a **smooth bundle map** between them consists of a pair of smooth maps of manifolds, $F : E \to B$ and $f : M \to N$, such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

and such that the restriction to each fiber $F|_{E_p} : E_p \to B_{f(p)}$ is linear. The map F actually determines the map f, so we usually only refer to F as being the bundle map.

Vector fields as sections of the tangent bundle

Finally, remember that part of our motivation for defining the tangent bundle was that we had no notion of a "smoothly varying" choice of vector at each point, because all the tangent spaces were disconnected from each other. Now that we have the manifold structure on TM, we can go back and fix this with a precise notion of "smooth vector field on M":

For any sets X and Y and function $f : X \to Y$, a **section** of f is a function $g : Y \to X$ such that $(f \circ g) : Y \to Y$ is the identity map.

Given a manifold M, the natural projection map $\pi : TM \to M$ is smooth. A **vector field** is a smooth map of manifolds $\sigma : M \to TM$ that is a section of π, i.e. $\pi \circ \sigma = \mathrm{id}_M$. The condition of being a section is exactly what we need to ensure that we actually send $p \in M$ to a tangent vector at p (i.e. an element of T_pM); we don't want to be assigning tangent vectors from all over the manifold willy-nilly to whatever points we feel like!

In general, a section of a bundle has an analogous definition. Two VERY IMPORTANT! types of sections in differential topology and geometry are:

- **Differential forms**, which are sections of the bundle $\pi : \bigwedge^k(T^*M) \to M$.

  The symbol $\bigwedge^k$ denotes the kth alternating power operation, and T^*M (known as the **cotangent bundle**) is the dual bundle of the tangent bundle TM.

- **Riemannian metrics**, which are sections of the bundle $\pi : S^2(T^*M) \to M$ that at each fiber are positive-definite.

  The symbol S^2 denotes the symmetric square operation.