INTRODUCTION TO DIFFERENTIAL TOPOLOGY

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Topological manifolds (I'll do a minicourse on topology on Monday if anyone wants a refresher).

Intuitive definition. A topological manifold is a space that, near any point, looks like Euclidean space.

We don't allow (line) + (sphere) + (plane), so let's be more precise:

Improvement. A topological manifold is a space which, near any point, looks like $\mathbb{R}^n$ for some $n \geq 0$.

What does "near" mean?

Improvement. A space M is a topological manifold if, for any $p \in M$, there is an open set $U \subseteq M$ containing p that looks like $\mathbb{R}^n$ for some $n \geq 0$.

What does "looks like $\mathbb{R}^n$" mean? Homeomorphic to $\mathbb{R}^n$? Mabye homeomorphic to an open subset of $\mathbb{R}^n$? Or should we only allow open balls in $\mathbb{R}^n$? In fact, these are all equivalent, and each version is useful in different situations. Let's use the second one for now.

Improvement. A space M is a topological manifold if, for any $p \in M$, there is an open set $U \subseteq M$ containing p that is homeomorphic to some open subset $V \subseteq \mathbb{R}^n$.

This is the definition of "locally Euclidean". Given open subsets $U \subseteq M$, $V \subseteq \mathbb{R}^n$, and a homeomorphism $\phi : U \to V$, the map ϕ is called a *chart* on U. The set U is called a *coordinate neighborhood* of p. Note that *not necessarily every* open subset of M has a chart.

For technical reasons, we also don't want to allow things like the line with two origins:

$$\{(x,y) \in \mathbb{R}^2 \mid y \in \{0,1\}\}/\sim \ \text{ where } (x,i) \sim (y,j) \text{ if } \begin{cases} x, y \neq 0 \text{ and } x = y \\ x, y = 0 \text{ and } i = j \end{cases}$$

or an uncountable disjoint union of copies of $\mathbb{R}$, e.g. $\bigsqcup_{\mathbb{R}} \mathbb{R}$, so we add two condititions to rule them out: we require a topological manifold to be Hausdorff and second-countable. (The line with two origins is not Hausdorff because the two origins cannot be separated by disjoint open sets, and the uncountable disjoint union of copies of $\mathbb{R}$ is not second-countable because any base for the topology would have to have at least one element for each copy of $\mathbb{R}$, hence an uncountable number of elements).

Proper Definition. A space M is a topological manifold if it is locally Euclidean, Hausdorff, and second-countable.

Examples. Euclidean space itself $\mathbb{R}^n$, the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$, a curve in $\mathbb{R}^n$, the Mobius strip, **any open subset of a manifold**, products of manifolds (so $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$)

Questions. Is $[0,1]$ a manifold? What about $\{*\}$, $\varnothing$, or $\mathbb{Q}$?

$\mathbb{R}^n$ is nice, and topological manifolds are modeled on it, so they are also nice. But still... they're kind of boring! So far we just have a definition and examples, and nothing to do with them. What's something exciting we can do with $\mathbb{R}^n$? A powerful tool for studying functions? Calculus!

Note that topology isn't enough to do calculus - a circle and a triangle are homeomorphic, but we can all agree that they aren't "differentiably" the same (whatever that might mean). We only know how to do calculus in $\mathbb{R}^n$, so if we want to do calculus in a manifold, we need to **transfer** it to the manifold. How can we do that?

Here is a hint: what might it mean for a function $f : M \to \mathbb{R}$, or a function $g : \mathbb{R} \to M$, to be differentiable?

We compose with a chart (or its inverse, since it is a homeomorphism), to get a map between Euclidean spaces! But wait - depending on what charts we use, this definition might not be consistent.

For example, suppose our manifold is $M = \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and we want to know if the function $f : M \to \mathbb{R}$ defined by $f(x, y) = x$ is differentiable. Let $U = \{(x, y) \in \mathbb{S}^1 \mid x, y \geq 0\}$ be an open quarter circle, and consider the two charts $\phi, \psi : U \to (0, 1)$ defined by $\phi(x, y) = y$ and

$$\psi(x, y) = \begin{cases} \frac{x+1}{2} & \text{if } x \geq \frac{1}{2}, \\ \frac{3x}{2} & \text{if } x \leq \frac{1}{2}. \end{cases}$$

Then $(f \circ \phi^{-1}) : (0, 1) \to \mathbb{R}$ is the map $(f \circ \phi^{-1})(t) = \sqrt{1 - t^2}$, which is differentiable, but the map

$$(f \circ \psi^{-1})(t) = \begin{cases} 2t - 1 & \text{if } t \geq \frac{3}{4} \\ \frac{2t}{3} & \text{if } t \leq \frac{3}{4} \end{cases}$$

fails to be differentiable at $t = \frac{3}{4}$. So, how can we fix this?

We want to define a smooth manifold to be a topological manifold with a consistent way of deciding which functions to or from it are differentiable. Clearly, in order to do this, we need to only allow charts that agree somehow. This motivates the following definitions:

Transition map. Suppose $U, W \subseteq M$ are non-disjoint open subsets ($U \cap W \neq \varnothing$) having charts $\phi : U \to V$ and $\psi : W \to Y$. The composition $(\psi \circ \phi^{-1}) : \phi(U \cap W) \to \psi(U \cap W)$ is the transition map from ϕ to ψ.

Smooth compatibility. Two charts $\phi : U \to V$ and $\psi : W \to Y$ are smoothly compatible if either $U \cap W = \varnothing$ or the transition map $\psi \circ \phi^{-1}$ is a C^∞-**diffeomorphism** (note that $\phi(U \cap W)$ and $\psi(U \cap W)$ are open subsets of $\mathbb{R}^n$, so we can talk about differentiability of maps between them).

Smooth atlas. A smooth atlas $\mathcal{A}$ on M is a collection of charts whose domains cover M, and which are all smoothly compatible with each other.

Naive Definition. A topological manifold M with a choice of smooth atlas is a smooth manifold.

Sounds good, right? But wait a second - do we really want to call

$$\mathbb{R}, \mathcal{A} = \{\mathrm{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}\} \qquad \mathbb{R}, \mathcal{B} = \{\mathrm{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}, \mathrm{id}_{(0,1)} : (0, 1) \to (0, 1)\}$$

"different" manifolds? No, that'd be silly. Instead, we define

Maximal smooth atlas. A smooth atlas $\mathcal{A}$ on M is a maximal smooth atlas if any chart that is smoothly compatible with $\mathcal{A}$ is already in $\mathcal{A}$. Alternatively, $\mathcal{A}$ is not contained in any strictly larger smooth atlas.

Given any smooth atlas $\mathcal{A}$, there is a unique maximal smooth atlas containing it:

$$\mathcal{B} = \{\text{charts } \phi : U \to V \text{ of } M \text{ that are smoothly compatible with every element of } \mathcal{A}\}$$

Proper definition. A smooth manifold is a topological manifold M together with a choice of maximal smooth atlas $\mathcal{A}$, also called a "smooth structure". Usually the choice of $\mathcal{A}$ is implied and we just write M.

When we speak of a chart of a smooth manifold M, we are implicitly taking the chart to belong to the atlas of M.

Examples. All my previous examples of topological manifolds are also examples of smooth manifolds - take the charts I mentioned before, check that they are smoothly compatible, and form the maximal smooth atlas containing those charts. However, there exist topological manifolds for which there does not exist any smooth structure. Also, there exist so-called "exotic 7-spheres", which are smooth structures for $\mathbb{S}^7$ that are not diffeomorphic to each other. There exist uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$, but for any $n \neq 4$, there don't exist any exotic smooth structures - in other words, if a manifold is homeomorphic to $\mathbb{R}^n$ for $n \neq 4$, it is diffeomorphic as well. Given any two n-manifolds, where $n \in \{1, 2, 3\}$, if they are homeomorphic, then they are diffeomorphic. It's an open question whether there exist exotic structures on the 4-sphere.

Smooth functions to $\mathbb{R}$. Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is smooth at $p \in M$ if there exists a chart $\phi : U \to V$ for which $p \in U$ such that $(f \circ \phi^{-1}) : V \to \mathbb{R}$ is a smooth function (as defined for Euclidean spaces). The function f is smooth is it is smooth at p for every $p \in M$.

Smooth maps. Let M and N be smooth manifolds. A map $f : M \to N$ is smooth if for every $p \in M$, there exist charts $\phi : U \to V$ and $\psi : W \to Y$ of M and N respectively, such that $p \in U$ and $f(p) \in W$, $f(U) \subseteq W$, and $(\psi \circ f \circ \phi^{-1}) : V \to Y$ is a smooth function between open sets of $\mathbb{R}^n$.

Smoothness is local. Let M and N be smooth manifolds, and let $f : M \to N$ be a function. Suppose that for every $p \in M$ there is an open $U \subseteq M$ containing p for which $f|_U : U \to N$ is smooth. Thus, there is a chart $\phi : A \to B$ of U for which $p \in A$ and a chart $\psi : C \to D$ of N for which $f(p) \in C$, with $f(A) \subseteq C$ and $(\psi \circ f \circ \phi^{-1}) : B \to D$ is smooth. But recall the definition of the smooth structure on U: the restriction of all charts of M to U. The restriction of a chart of M to an open set of M is still a chart of M, so in fact ϕ is a chart of M, and the definition of being a smooth map from M to N is satisfied. Conversely, it is clear that if f is smooth, then its restriction to any open subset is smooth.

Smooth maps are well-defined. Suppose that $f : M \to N$ is smooth - for any $p \in M$, **there exist** charts $\phi : U \to V$ and $\psi : W \to Y$ for which the composition with f is smooth. But is it true that given **any** charts $\sigma : A \to B$ and $\rho : C \to D$, the composition $\rho \circ f \circ \sigma^{-1} : B \to D$ is smooth? The answer is yes, because we only allowed charts in our atlas that were all smoothly compatible with each other (indeed, pretty much the entire motivation for defining smooth manifolds the way we did was to get a consistent way of saying when a map is smooth, so it'd better work). Therefore, the transition maps $\phi \circ \sigma^{-1}$ and $\rho \circ \psi^{-1}$ are smooth, and so the composition

$$(\rho \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \sigma^{-1}) = \rho \circ f \circ \sigma^{-1}$$

is smooth. Thus, it doesn't matter which charts we look at f with, f will consistently be either smooth or not smooth.

Smooth implies continuous. Suppose that $f : M \to N$ is smooth. Because, after applying charts, it is (locally) a smooth map between open sets of Euclidean space, it is (locally) continuous after applying charts, and because charts are homeomorphisms, f is locally continuous, hence continuous.

Compositions of smooth maps are smooth. Let $f : M \to N$ and $g : N \to P$ be smooth maps of manifolds. Applying the definition of smoothness and taking our chart on M to have a domain whose

image under f is contained in the domain of our chart on N, this follows immediately from the fact that the composition of smooth maps between Euclidean spaces is continuous.

Note that a smooth bijection between manifolds is **not** necessarily a diffeomorphism. The standard example of this phenomenon is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. It is bijective, and smooth at every point, but it's inverse $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^{1/3}$ is not differentiable at $x = 0$. This is similar to the situation with topological spaces: a continuous bijection is not necessarily a homeomorphism.

So, what is differential topology? It is the study of the properties of differentiable manifolds that are invariant under diffeomorphisms. Let's compare this with general topology, and with differential geometry.

A topologist sees a triangle, a circle, and an ellipse as equivalent - they are all homeomorphic.

A differential topologist sees a triangle as different, but the circle and ellipse are still the same - the circle and ellipse are diffeomorphic.

A differential geometer sees all three as different, because the ellipse has a different curvature than the circle - the ellipse is not isometric to the circle. So, differential geometry studies smooth manifolds that have some sort of extra information attached to them. There are in fact many different kinds of extra information one might look at - Riemannian, symplectic, metaplectic, etc.

As always, mathematics is amazingly interconnected, and studying more restrictive notions actually helps us understand the general case. For a minor example, the Poincare conjecture - a statement about differentiable manifolds - was proven using extensive study of differential geometry.