CATEGORY THEORY

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"*Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows.*"

- Saunders Mac Lane, one of the two creators of category theory

Categories

Category theory is a means of capturing entire mathematical theories in single objects, such that their common patterns may be studied alike and their relationships described. A category is essentially a directed graph with a composition law for edges; for many categories, the vertices would represent various mathematical objects, and the directed edges would represent the structure-preserving maps between them—for instance, appropriate homomorphisms, continuous maps, linear maps . . .

In category theory, then, we make no reference to the actual elements of sets; we instead characterize certain features of each theory through various universal properties that they satisfy, characterized by various existence and uniqueness conditions for diagrams of arrows. Such characterizations translate much more naturally to less familiar parts of mathematics.

A **category** $\mathcal{C}$ consists of

- a set $\mathrm{Ob}\,\mathcal{C}$ of **objects**;
- for all $x, y \in \mathrm{Ob}\,\mathcal{C}$, a set $\mathrm{Hom}(x, y)$ of **morphisms**;
- a **composition law** assigning to any pair of morphisms $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$ a **composite** $(g \circ f) \in \mathrm{Hom}(a, c)$,

such that

- for all $x \in \mathrm{Ob}\,\mathcal{C}$ there exists a two-sided identity morphism $id_x \in \mathrm{Hom}(x, x)$;
- composition is associative: if $f \in \mathrm{Hom}(a, b)$, $g \in \mathrm{Hom}(b, c)$, $h \in \mathrm{Hom}(c, d)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

So every category is a graph, but categories have also the algebraic structure of composition; this cross between combinatorial and algebraic structure is powerful. We could explicitly write down some finite categories, but we'll be mostly interested in categories so infinite that set theory begins to look inadequate.

Examples:

Set:	$\mathrm{Ob}\,\mathrm{Set}$ is all sets,	$\mathrm{Hom}(a, b)$ all functions from a to b;
Grp:	groups,	group homomorphisms
Rng:	rings,	ring homomorphisms
Vect_F:	vector spaces over F,	F-linear maps
$R-$Mod:	modules over R,	R-linear maps
Top:	topological spaces,	continuous maps
Matr_F:	natural numbers,	$a \times b$ matrices over F
Set_*:	sets with a chosen element,	functions respecting that choice
Graph:	graphs,	graph homomorphisms.

Of course, the set of all sets does not exist in most conceptions of set theory; we can work around this in various ways, for instance letting Set be the category of "small" sets, where $\mathrm{Ob}\,\mathrm{Set}$ is a "large" set. Most category theorists feel that a new foundation of mathematics based around categories would be much more adequate to express these ideas than set theory, and so we will gloss over such matters.

We say that a diagram of objects and morphisms **commutes** if the compositions of morphisms along any two paths between two objects yield the same result.

Functors

What are the structure-preserving maps between categories? If categories are graphs with some algebraic structure of composition, then functors are graph homomorphisms that respect the composition law.

A **functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of:

- a function (written F) from $\mathrm{Ob}\,\mathcal{A}$ to $\mathrm{Ob}\,\mathcal{B}$, and
- for all $x, y \in \mathrm{Ob}\,\mathcal{A}$, a function from $\mathrm{Hom}(x, y)$ to $\mathrm{Hom}(Fx, Fy)$, also written F,

such that if $a, b, c \in \mathrm{Ob}\,\mathcal{A}$ and $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$ then $F(g \circ f) = F(g) \circ F(f)$, and $F(id_a) = id_{F(a)}$.

Examples:

- Free : Set $\to \mathrm{Vect}_F$. Any set a is sent to the vector space of all formal $F-$linear combinations of elements of a. Any function $f : a \to b$ extends to a linear map Free f : Free $a \to$ Free b. Similarly there is a free functor from Set to Grp, Rng, $R-$Mod, ...
- Forget : Grp $\to$ Set. Any group is taken to its underlying set and any group homomorphism to itself as a mere function: it "forgets" the group

structure. Similarly we have a forgetful functor to Set from Rng, Ab, Top, Vect_F, R−Mod, ...

- P : Set $\to$ Set. Any set is sent to its power set, and any function induces a function on subsets acting elementwise.
- (In)disc : Set $\to$ Top. A set is equipped with the (in)discrete topology, and functions automatically become continuous.
- $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, the identity functor.
- $(\ )^\times$: Rng $\to$ Grp. Any ring is taken to its group of units, and ring homomorphisms restrict to group homomorphisms.
- $[-,-]$: Grp $\to$ Grp. Any group is taken to its commutator subgroup, and group homomorphisms restrict.

Note that the center Z is not a functor Grp $\to$ Ab, as group homomorphisms do not preserve the center.

Any group may be encoded in a category as follows: we let our category have only a single object a, and let it have a morphism for each element of the group; we then define a composition law on $\mathrm{Hom}(a, a)$ by our group law. Then the identity morphism on a corresponds to the group identity, and all morphisms are invertible. Indeed, we see then that the notions of "one-object category with all morphisms invertible" and "group" coincide. This suggests two generalizations of groups:

Relaxing invertibility, we have **monoids**, one-object categories; these can easily be defined more classically by removing the inverse axiom of groups. However, retaining invertibility but relaxing the one-object condition, we obtain **groupoids**, which are then essentially groups with only partial operations; these have found uses in algebraic topology and other branches of mathematics.

Special types of morphisms

A **isomorphism** between $x, y \in \mathrm{Ob}\,\mathcal{A}$ is a morphism $f \in \mathrm{Hom}(x, y)$ for which there exists a $g \in \mathrm{Hom}(y, x)$ such that $g \circ f = id_x \in \mathrm{Hom}(x, x)$ and $f \circ g = id_y \in \mathrm{Hom}(y, y)$. This g is unique, and is called the **inverse** of f.

Remark. It is important to realize that this is the correct definition of isomorphism in any category. Often, an isomorphism might be introduced as a "bijective homomorphism" or some other such definition; *this is not the proper definition*. Firstly, if we are working in a category whose objects are not sets with structure, the concept of "bijective" simply makes no sense. But secondly, it is, in fact, a *theorem* about groups that a group homomorphism is an isomorphism if and only if it is bijective. There are many mathematical objects for which the corresponding statement is false: for example, topological spaces (a continuous bijection need not be a homeomorphism) and smooth manifolds (a smooth bijection need not be a diffeomorphism).

A **monomorphism** or **monic** is a morphism $f \in \mathrm{Hom}(x, y)$ in a category $\mathcal{A}$ such that $f \circ g = f \circ h \implies g = h$ for all $z \in \mathrm{Ob}\,\mathcal{A}$ and all $g, h \in \mathrm{Hom}(z, y)$. The monics in a category are usually what we expect: a group homomorphism is a monic if and only if it is injective. The corresponding statement is true for most "algebraic" objects: rings, fields, vector spaces, etc. We usually have to be more careful with "topological" objects: topological spaces, manifolds, etc.

However, there are counterexamples, even among "algebraic"-style objects: let Div be the category whose objects are divisible abelian groups, and whose morphisms are just group homomorphisms (an abelian group G is said to be divisible when, for any $g \in G$ and $n \in \mathbb{N}$, there exists an $h \in G$ such that $nh = g$). The group of rational numbers $\mathbb{Q}$ under addition, is an example of a divisible group; so is the quotient group $\mathbb{Q}/\mathbb{Z}$. The quotient homomorphism $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ that sends $a \in \mathbb{Q}$ to its equivalence class $[a]$ in $\mathbb{Q}/\mathbb{Z}$ (which is definitely **not** injective) is actually still a monic in the category Div.

A **epimorphism** or **epic** is a morphism $f \in \mathrm{Hom}(x, y)$ in a category $\mathcal{A}$ such that $g \circ f = h \circ f \implies g = h$ for all $z \in \mathrm{Ob}\,\mathcal{A}$ and all $g, h \in \mathrm{Hom}(y, z)$. The epics in a category are also often what we expect: for example, a group homomorphism is an epic if and only if it is surjective. But even in the category of rings, there are morphisms that are unexpectedly epic: for example, the inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is definitely **not** surjective, yet it is an epic in the category Ring.

A final note of caution: a morphism $f \in \mathrm{Hom}(x, y)$ that is both a monic and an epic is **not** necessarily an isomorphism. An example of this phenomenon occurs in the category Haus of Hausdorff topological spaces, with continuous maps; for any Hausdorff topological space X, the inclusion $i : D \to X$ of a dense subset $D \subset X$ is both monic and epic, but not an isomorphism.

Natural transformations

We've defined categories, and their structure-preserving maps, namely functors. But what are the structure-preserving maps between two functors?

A **natural transformation** $\eta : F \Rightarrow G$ from a functor $F : \mathcal{A} \to \mathcal{B}$ to a functor $G : \mathcal{A} \to \mathcal{B}$ consists of

- for all $x \in \mathrm{Ob}\,\mathcal{A}$, a morphism $\eta_x \in \mathrm{Hom}(Fx, Gx)$

such that, for every $x, y \in \mathrm{Ob}\,\mathcal{A}$ and every morphism $f \in \mathrm{Hom}(x, y)$, the following diagram commutes:

$$\begin{tikzcd}
Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, "\eta_y"'] & Gy
\end{tikzcd}$$

If every η_x is an isomorphism, then we say that η is a **natural equivalence** or a **natural isomorphism**.

There are many examples of natural transformations in nature:

- Let $\mathcal{A} = \mathcal{B} =$ Grp, and let $F = id_{\text{Grp}}$ (the identity functor from Grp to itself), and $G = (\ )^{ab}$ (the functor that takes a group to its abelianization, and a group homomorphism to the corresponding induced homomorphism). We can define a natural transformation $\eta : F \Rightarrow G$ by letting $\eta_H : FH \to GH$ (i.e., $\eta_H : H \to H^{ab}$) be the quotient homomorphism q_H that takes a group H to its abelianization $H^{ab} := H/[H,H]$:

  $$\begin{tikzcd}
  H \arrow[r, "q_H"] \arrow[d, "f"'] & H^{ab} \arrow[d, "f^{ab}"] \\
  K \arrow[r, "q_K"'] & K^{ab}
  \end{tikzcd}$$

  (admittedly, this is kind of an boring example).

- Let $\mathcal{A} =$ CRing, let $\mathcal{B} =$ Grp, let $F : \mathcal{A} \to \mathcal{B}$ be the functor that takes a ring R to the matrix group $GL_n(R)$, and let $G : \mathcal{A} \to \mathcal{B}$ be the functor that takes a ring R to the unit group $R^\times$. Then we can define a natural transformation $\eta : F \Rightarrow G$ by letting $\eta_R : FR \to GR$ (i.e., $\eta_R : GL_n(R) \to R^\times$) be the determinant homomorphism $\det_R$ that takes matrix in $GL_n(R)$ to its determinant (which will be in $R^\times$, by the definition of $GL_n(R)$):

  $$\begin{tikzcd}
  GL_n(R) \arrow[r, "\det_R"] \arrow[d, "GL_n(f)"'] & R^\times \arrow[d, "f^\times"] \\
  GL_n(S) \arrow[r, "\det_S"'] & S^\times
  \end{tikzcd}$$

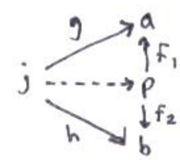
- Let $\mathcal{A} = \mathcal{B} = \text{Vect}_\mathcal{K}$, and let $F = id_{\text{Vect}_K}$, and $G = (\ )^{**}$ (the functor that takes a K-vector space V to its double dual $V^{**} = \text{Hom}(\text{Hom}(V,K),K)$). Then we can define a natural transformation $\eta : F \Rightarrow G$ by letting $\eta_V : FV \to GV$ (i.e., $\eta_V : V \to V^{**}$) be the linear map ev_V that takes an element $x \in V$ to the evaluation map $ev_x(f : V \to K) = f(x)$:

$$\begin{tikzcd}
V \arrow[r, "ev_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\
W \arrow[r, "ev_W"'] & W^{**}
\end{tikzcd}$$

Limits and colimits

We can characterize a number of important features of a category in terms of relationships between morphisms, when normally in a given theory they would be described in terms of elements.

Given a category $\mathcal{C}$, objects $a, b, p \in \mathrm{Ob}\,\mathcal{C}$, and morphisms $f_1 : p \to a$, $f_2 : p \to b$, we say (p, f_1, f_2) is a **product** of a and b if for all $j \in \mathrm{Ob}\,\mathcal{C}$ and all $g : j \to a$ and $h : j \to b$, there exists a unique $l : j \to p$ such that $g = f_1 \circ l$ and $h = f_2 \circ l$, i.e. the above diagram commutes.



For instance, if $a, b \in \mathrm{Ob}$ Set, then $(a \times b, \pi_a, \pi_b)$ is a product where π_a and π_b are the projections onto the two factors. Similarly in Top we have the product space, and in Vect_K we have the direct sum.

Claim. *Products are unique up to isomorphism.*

Proof. Suppose that (p_1, f_1, f_2) and (p_2, g_1, g_2) are two products of a and b. Then there exist unique morphisms $l_{12} : p_1 \to p_2$ and $l_{21} : p_2 \to p_1$ such that the diagram commutes. Define $l_{11} = l_{21} \circ l_{12}$, $l_{22} = l_{12} \circ l_{21}$. Then $f_1 l_{11} = f_1$, but the product property with p_1 and p_1 gives that there is a unique morphism with those commutativity conditions, and the identity id_{p_1} satisfies them. So $l_{11} = id_{p_1}$, and similarly $l_{22} = id_{p_2}$, so that l_{12} and l_{21} are mutual inverses, and hence are isomorphisms between p_1 and p_2. $\square$

Note that we have demonstrated a slightly stronger condition than claimed: the isomorphism between two products is unique within the commutativity conditions.

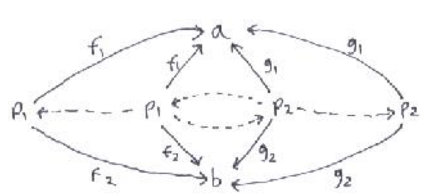
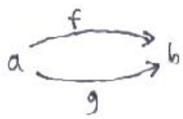
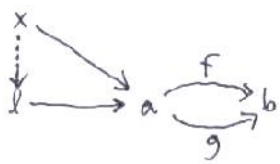
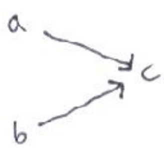


Figure 1: Uniqueness of products

More generally, given a subcategory $\mathcal{D}$ of $\mathcal{C}$, we define a **cone** to $\mathcal{D}$ to be an object x of $\mathcal{C}$ together with a morphism $f_y : x \to y$ for each object y of $\mathcal{D}$ such that the f_y commute with the morphisms of D, that is, for any morphism $k : a \to b$ in $\mathcal{D}$, $k \circ f_a = f_b$.

A **limit** of D is a cone (ℓ, f_y) such that for all cones (x, g_y) to $\mathcal{D}$, there exists a unique morphism $m : x \to \ell$ such that all morphisms commute, i.e. for all $y \in \operatorname{Ob}\mathcal{D}$, $g_y = f_y \circ m$.

Examples:

$\mathcal{D}$	cone diagram	name
$\emptyset$	$x \dashrightarrow \ell$	terminal object
		product
		equalizer
		pullback

Sometimes we consider infinite limits, for instance in the diagram:

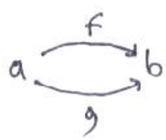
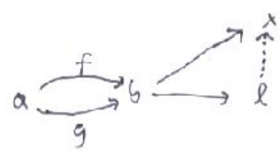
$$\ldots \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

The limit of this subcategory of Rng has limit $\mathbb{Z}_p$, the p-adic integers. We (loosely) write $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Similarly we define a **cocone** of a subcategory D as an object $x \in \mathrm{Ob}\,\mathcal{C}$ together with morphisms $f_y : y \to x$ for all $y \in \mathrm{Ob}\,\mathcal{D}$, such that the f_y commute with the morphisms of $\mathcal{D}$.

A **colimit** of $\mathcal{D}$ is a cocone (ℓ, f_y) such that for all cocones (x, g_y) there exists a unique morphism $m : \ell \to x$ such that all morphisms commute.

Examples:

$\mathcal{D}$	cocone diagram	name
$\emptyset$		initial object
		coproduct
		coequalizer

Theorem. *All limits and colimits are unique up to isomorphism, and this isomorphism is unique within the commutativity conditions of the two (co)cones.*

The proof is virtually identical to that of the product above.