2.1.22. Let $X = \bigcup_{i\in I} U_i$ be an open cover of X. For each $i \in I$, let $\mathcal{F}_i$ be a sheaf on U_i, with ρ_i being the restriction maps for $\mathcal{F}_i$. For each $i, j \in I$, let $\phi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ be an isomorphism such that $\phi_{ii} = \text{id}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$.

We want to create a sheaf $\mathcal{F}$ on X, with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that for each $i, j \in I$, $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$, and show that $\mathcal{F}$ is unique up to isomorphism.

Definition of $\mathcal{F}(U)$. For any open $U \subseteq X$, we define

$$\mathcal{F}(U) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U \cap U_i) \mid \phi_{ij}(U \cap U_i \cap U_j)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i)) = {\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j) \text{ for all } i, j \in I\}$$

Note that the definition of $\mathcal{F}(U)$ is consistent with our assumptions about the ϕ_{ij} - we are requiring each $(s_i)_{i\in I} \in \mathcal{F}(U)$ to have the property that, for all $i, j \in I$,

$$\phi_{ij}(U \cap U_i \cap U_j)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i)) = {\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j)$$

and therefore, after applying ${\rho_j}^{U\cap U_i\cap U_j}_{U\cap U_i\cap U_j\cap U_k}$ to the above equation, requiring that each $(s_i)_{i\in I} \in \mathcal{F}(U)$ have the property that, for all $i, j \in I$,

$${\rho_j}^{U\cap U_i\cap U_j}_{U\cap U_i\cap U_j\cap U_k}(\phi_{ij}(U \cap U_i \cap U_j)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i))) = \phi_{ij}(U \cap U_i \cap U_j \cap U_k)({\rho_i}^{U\cap U_i\cap U_j}_{U\cap U_i\cap U_j\cap U_k}({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i))) =$$

$$\boxed{\phi_{ij}(U \cap U_i \cap U_j \cap U_k)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j\cap U_k}(s_i)) = {\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j\cap U_k}(s_j)} = {\rho_j}^{U\cap U_i\cap U_j}_{U\cap U_i\cap U_j\cap U_k}({\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j))$$

Thus, for any open $U \subseteq X$, $(s_i)_{i\in I} \in \mathcal{F}(U)$, and $i, j, k \in I$, we have that

$$\phi_{jk}(U\cap U_i\cap U_j\cap U_k)(\phi_{ij}(U\cap U_i\cap U_j\cap U_k)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j\cap U_k}(s_i))) = \phi_{jk}(U\cap U_i\cap U_j\cap U_k)({\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j\cap U_k}(s_j)) =$$

$${\rho_k}^{U\cap U_k}_{U\cap U_i\cap U_j\cap U_k}(s_k) = \phi_{ik}(U \cap U_i \cap U_j \cap U_k)({\rho_k}^{U\cap U_k}_{U\cap U_i\cap U_j\cap U_k}(s_k))$$

which agrees with our assumption that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$.

Definition of restriction maps. For any open $V \subseteq U \subseteq X$, we define $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ by

$$\rho^U_V((s_i)_{i\in I}) = ({\rho_i}^{U\cap U_i}_{V\cap U_i}(s_i))_{i\in I} \in \mathcal{F}(V)$$

Note that we have $({\rho_i}^{U\cap U_i}_{V\cap U_i}(s_i))_{i\in I} \in \mathcal{F}(V)$ because for each $i \in I$, we have that $s_i \in \mathcal{F}_i(U \cap U_i)$ and hence ${\rho_i}^{U\cap U_i}_{V\cap U_i}(s_i) \in \mathcal{F}_i(V \cap U_i)$, and because the ϕ_{ij} are morphisms, we have that

$$\phi_{ij}({\rho_i}^{V\cap U_i}_{V\cap U_i\cap U_j}({\rho_i}^{U\cap U_i}_{V\cap U_i}(s_i))) = \phi_{ij}({\rho_i}^{U\cap U_i}_{V\cap U_i\cap U_j}(s_i)) = {\rho_j}^{U\cap U_i}_{V\cap U_i\cap U_j}(\phi_{ij}(s_i)) =$$

$${\rho_j}^{U\cap U_i}_{V\cap U_i\cap U_j}(s_j) = {\rho_j}^{V\cap U_j}_{V\cap U_i\cap U_j}({\rho_j}^{U\cap U_j}_{V\cap U_j}(s_j))$$

Checking that they form a presheaf. For any open $U \subseteq X$ and $(s_i)_{i\in I} \in \mathcal{F}(U)$, we have that

$$\rho^U_U((s_i)_{i\in I}) = ({\rho_i}^{U\cap U_i}_{U\cap U_i}(s_i))_{i\in I} = (s_i)_{i\in I}$$

Thus, $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$. For any open $W \subseteq V \subseteq U \subseteq X$, we have that

$$\rho^V_W(\rho^U_V((s_i)_{i\in I})) = \rho^V_W(({\rho_i}^{U\cap U_i}_{V\cap U_i}(s_i))_{i\in I}) = ({\rho_i}^{V\cap U_i}_{W\cap U_i}({\rho_i}^{U\cap U_i}_{V\cap U_i}(s_i)))_{i\in I} = ({\rho_i}^{U\cap U_i}_{W\cap U_i}(s_i))_{i\in I} = \rho^U_W((s_i)_{i\in I})$$

Thus, $\rho^U_W = \rho^V_W \circ \rho^U_V$. Therefore $\mathcal{F}$, together with the restriction maps ρ, form a presheaf.

Checking axiom (3) for sheaves. Let $U \subseteq X$ be open, and let $U = \bigcup_{a\in A} V_a$ be an open cover. Suppose that $s = (s_i)_{i\in I} \in \mathcal{F}(U)$ has the property that, for all $a \in A$,

$$\rho^U_{V_a}(s) = ({\rho_i}^{U\cap U_i}_{V_a\cap U_i}(s_i))_{i\in I} = (0)_{i\in I} = 0 \in \mathcal{F}(V_a)$$

Note that for each $i \in I$, $U \cap U_i = \bigcup_{a\in A}(V_a \cap U_i)$ is an open cover and ${\rho_i}^{U\cap U_i}_{V_a\cap U_i}(s_i) = 0$ for all $a \in A$, so because $\mathcal{F}_i$ is a sheaf, we have that $s_i = 0 \in \mathcal{F}_i(U \cap U_i)$, and thus $s = (s_i)_{i\in I} = (0)_{i\in I} = 0 \in \mathcal{F}(U)$.

Checking the gluing axiom for sheaves. Let $U \subseteq X$ be open, and let $U = \bigcup_{a\in A} V_a$ be an open cover. Suppose that a collection of $s^a = (s^a_i)_{i\in I} \in \mathcal{F}(V_a)$, for $a \in A$, have the property that

$$\rho^{V_a}_{V_a\cap V_b}(s^a) = ({\rho_i}^{V_a\cap U_i}_{V_a\cap V_b\cap U_i}(s^a_i))_{i\in I} = ({\rho_i}^{V_b\cap U_i}_{V_a\cap V_b\cap U_i}(s^b_i))_{i\in I} = \rho^{V_b}_{V_a\cap V_b}(s^b) \text{ for all } a, b \in A$$

For each $i \in I$, we have that $U \cap U_i = \bigcup_{a\in A}(V_a \cap U_i)$ is an open cover for $U \cap U_i$. By the above equation, we have that for each $i \in I$ the collection of $s^a_i \in \mathcal{F}_i(V_a \cap U_i)$ have the property that

$${\rho_i}^{V_a\cap U_i}_{V_a\cap V_b\cap U_i}(s^a_i) = {\rho_i}^{V_b\cap U_i}_{V_a\cap V_b\cap U_i}(s^b_i)$$

so because $\mathcal{F}_i$ is a sheaf, there is a unique $s_i \in \mathcal{F}_i(U \cap U_i)$ such that ${\rho_i}^{U\cap U_i}_{V_a\cap U_i}(s_i) = s^a_i$ for all $a \in A$. Thus, $s = (s_i)_{i\in I}$ will be a gluing of the s^a if we can show that in fact $s = (s_i)_{i\in I} \in \mathcal{F}(U)$, i.e. for all $i, j \in I$,

$$\phi_{ij}(U \cap U_i \cap U_j)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i)) = {\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j)$$

Because $s^a = (s^a_i)_{i\in I} \in \mathcal{F}(V_a)$ for each $a \in A$, we have that for all $i, j \in I$,

$$\phi_{ij}(V_a \cap U_i \cap U_j)({\rho_i}^{V_a\cap U_i}_{V_a\cap U_i\cap U_j}(s^a_i)) = {\rho_j}^{V_a\cap U_j}_{V_a\cap U_i\cap U_j}(s^a_j)$$

Thus, we have that for all $i, j \in I$ and $a \in A$,

$${\rho_j}^{U\cap U_i\cap U_j}_{V_a\cap U_i\cap U_j}(\phi_{ij}(U \cap U_i \cap U_j)({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i))) = \phi_{ij}(V_a \cap U_i \cap U_j)({\rho_i}^{U\cap U_i\cap U_j}_{V_a\cap U_i\cap U_j}({\rho_i}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i))) =$$

$$\phi_{ij}(V_a \cap U_i \cap U_j)({\rho_i}^{U\cap U_i}_{V_a\cap U_i\cap U_j}(s_i)) = \phi_{ij}(V_a \cap U_i \cap U_j)({\rho_i}^{V_a\cap U_i}_{V_a\cap U_i\cap U_j}({\rho_i}^{U\cap U_i}_{V_a\cap U_i}(s_i))) =$$

$$\phi_{ij}(V_a \cap U_i \cap U_j)({\rho_i}^{V_a\cap U_i}_{V_a\cap U_i\cap U_j}(s^a_i)) = {\rho_j}^{V_a\cap U_j}_{V_a\cap U_i\cap U_j}(s^a_j)$$

Now let $t_{i,j} = {\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j)$. We have that

$${\rho_j}^{U\cap U_i\cap U_j}_{V_a\cap U_i\cap U_j}(t_{i,j}) = {\rho_j}^{U\cap U_i\cap U_j}_{V_a\cap U_i\cap U_j}({\rho_j}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j)) = {\rho_j}^{U\cap U_j}_{V_a\cap U_i\cap U_j}(s_j) = {\rho_j}^{V_a\cap U_j}_{V_a\cap U_i\cap U_j}({\rho_j}^{U\cap U_j}_{V_a\cap U_j}(s_j)) = {\rho_j}^{V_a\cap U_j}_{V_a\cap U_i\cap U_j}(s^a_j)$$

Thus, for all $i, j \in I$, we have that

$$\rho_{j}{}^{U\cap U_i\cap U_j}_{V_a\cap U_i\cap U_j}(\phi_{ij}(U\cap U_i\cap U_j)(\rho_{i}{}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i))) = \rho_{j}{}^{U\cap U_i\cap U_j}_{V_a\cap U_i\cap U_j}(t_{i,j}) \text{ for all } a \in A$$

Note that for each $i, j \in I$, we have that $U \cap U_i \cap U_j = \bigcup_{a\in A}(V_a \cap U_i \cap U_j)$ is an open cover of $U \cap U_i \cap U_j$. Because $\mathcal{F}_j$ is a sheaf, we have by the lemma below that for all $i, j \in I$,

$$\phi_{ij}(U\cap U_i\cap U_j)(\rho_{i}{}^{U\cap U_i}_{U\cap U_i\cap U_j}(s_i)) = t_{i,j} = \rho_{j}{}^{U\cap U_j}_{U\cap U_i\cap U_j}(s_j)$$

Thus, we have shown that $s = (s_i)_{i\in I} \in \mathcal{F}(U)$, and thus s is a gluing of the s_a.

Lemma. Let $\mathcal{H}$ be a sheaf on a topological space Y, with restriction maps $\hat{\rho}$. Let $E \subseteq Y$ be open, and let $E = \bigcup_{r\in R} E_r$ be an open cover. If $t_1, t_2 \in \mathcal{H}(E)$ have $\hat{\rho}^E_{E_r}(t_1) = \hat{\rho}^E_{E_r}(t_2)$ for all $r \in R$, then because the restriction maps are homomorphisms, we have

$$\hat{\rho}^E_{E_r}(t_1 - t_2) = 0 \in \mathcal{H}(E_r) \text{ for all } r \in R$$

and hence $t_1 - t_2 = 0 \in \mathcal{H}(E)$, i.e. $t_1 = t_2$. □

Thus, we have shown that $\mathcal{F}$ together with the restriction maps ρ form a sheaf.

Defining the isomorphisms ψ_h. Note that for any $h \in I$ and open $V \subseteq U_h$, we have that

$$\mathcal{F}|_{U_h}(V) = \mathcal{F}(V) = \{(s_i)_{i\in I} \in \prod_{i\in I}\mathcal{F}_i(V\cap U_i) \mid \phi_{ij}(V\cap U_i\cap U_j)(\rho_{i}{}^{V\cap U_i}_{V\cap U_i\cap U_j}(s_i)) = \rho_{j}{}^{V\cap U_j}_{V\cap U_i\cap U_j}(s_j) \text{ for all } i, j \in I\}$$

For any $h \in I$ and open $V \subseteq U_h$, define $\psi_h(V) : \mathcal{F}|_{U_h}(V) \to \mathcal{F}_h(V)$ by

$$\psi_h(V)((s_i)_{i\in I}) = s_h \in \mathcal{F}_h(V \cap U_h) = \mathcal{F}_h(V)$$

This defines a morphism $\psi_h : \mathcal{F}|_{U_h} \to \mathcal{F}_h$ because, for any open $W \subseteq V \subseteq U_h$, we have that

$$\psi_h(W)(\rho^V_W((s_i)_{i\in I})) = \psi_h(W)((\rho_{i}{}^{V\cap U_i}_{W\cap U_i}(s_i))_{i\in I}) = \rho_{h}{}^{V\cap U_h}_{W\cap U_h}(s_h) = \rho_{h}{}^V_W(s_h) = \rho_{h}{}^V_W(\psi_h(V)((s_i)_{i\in I}))$$

Checking that the ψ_h are isomorphisms. For any $h \in I$ and open $V \subseteq U_h$, define $\beta_h(V) : \mathcal{F}_h(V) \to \mathcal{F}|_{U_h}(V)$ by

$$\beta_h(V)(t) = (\phi_{hi}(V\cap U_i)(\rho_{h}{}^V_{V\cap U_i}(t)))_{i\in I} \in \mathcal{F}|_{U_h}(V) = \mathcal{F}(V)$$

Note that we do in fact have that $\beta_h(V)(t) \in \mathcal{F}(V)$, because

$$\phi_{ij}(V\cap U_i\cap U_j)(\rho_{i}{}^{V\cap U_i}_{V\cap U_i\cap U_j}(\phi_{hi}(V\cap U_i)(\rho_{h}{}^V_{V\cap U_i}(t)))) = \phi_{ij}(V\cap U_i\cap U_j)(\phi_{hi}(V\cap U_i\cap U_j)(\rho_{h}{}^{V\cap U_i}_{V\cap U_i\cap U_j}(\rho_{h}{}^V_{V\cap U_i}(t)))) =$$

$$\phi_{hj}(V\cap U_i\cap U_j)(\rho_{h}{}^V_{V\cap U_i\cap U_j}(t)) = \phi_{hj}(V\cap U_i\cap U_j)(\rho_{h}{}^{V\cap U_j}_{V\cap U_i\cap U_j}(\rho_{h}{}^V_{V\cap U_j}(t))) = \rho_{j}{}^{V\cap U_j}_{V\cap U_i\cap U_j}(\phi_{hj}(V\cap U_j)(\rho_{h}{}^V_{V\cap U_j}(t)))$$

Furthermore, this defines a morphism $\beta_h : \mathcal{F}_h \to \mathcal{F}|_{U_h}$ because, for any open $W \subseteq V \subseteq U_h$, we have that

$$\beta_h(W)(\rho_{h}{}^V_W(t)) = (\phi_{hi}(W\cap U_i)(\rho_{h}{}^W_{W\cap U_i}(\rho_{h}{}^V_W(t))))_{i\in I} = (\phi_{hi}(W\cap U_i)(\rho_{h}{}^V_{W\cap U_i}(t)))_{i\in I} =$$

$$(\phi_{hi}(W\cap U_i)(\rho_{h}{}^{V\cap U_i}_{W\cap U_i}(\rho_{h}{}^V_{V\cap U_i}(t))))_{i\in I} = (\rho_{i}{}^{V\cap U_i}_{W\cap U_i}(\phi_{hi}(V\cap U_i)(\rho_{h}{}^V_{V\cap U_i}(t))))_{i\in I} = \rho^V_W(\beta_h(V)(t))$$

Note that for any $h, i, j \in I$, any open $V \subseteq U_h$, and any $(s_i)_{i \in I} \in \mathcal{F}|_{U_h}(V) = \mathcal{F}(V)$, we have that

$$\phi_{ij}(V \cap U_i \cap U_j)(\rho_i{}^{V \cap U_i}_{V \cap U_i \cap U_j}(s_i)) = \rho_j{}^{V \cap U_j}_{V \cap U_i \cap U_j}(s_j)$$

and thus in particular

$$\phi_{hi}(V \cap U_i)(\rho_h{}^{V}_{V \cap U_i}(s_h)) = s_i$$

Thus, for any $h \in I$ and open $V \subseteq U_h$, and any $(s_i)_{i \in I} \in \mathcal{F}|_{U_h}(V) = \mathcal{F}(V)$, we have that

$$\beta_h(V)(\psi_h(V)((s_i)_{i \in I})) = \beta_h(V)(s_h) = (\phi_{hi}(V \cap U_i)(\rho_h{}^{V}_{V \cap U_i}(s_h)))_{i \in I} = (s_i)_{i \in I}$$

Furthermore, for any $h \in I$, any open $V \subseteq U_h$, and any $t \in \mathcal{F}_h(V)$, we have that

$$\psi_h(V)(\beta_h(V)(t)) = \psi_h(V)((\phi_{hi}(V \cap U_i)(\rho_h{}^{V}_{V \cap U_i}(t)))_{i \in I}) = \phi_{hh}(V \cap U_h)(\rho_h{}^{V}_{V \cap U_h}(t)) = \mathrm{id}_V(\rho_h{}^{V}_{V}(t)) = t$$

Thus we have shown that for any $h \in I$, $\psi_h : \mathcal{F}|_{U_h} \to \mathcal{F}_h$ and $\beta_h : \mathcal{F}_h \to \mathcal{F}|_{U_h}$ are morphisms and are inverse to each other. Thus, for any $h \in I$, ψ_h is an isomorphism.

Checking that the ψ_h satisfy the relation. For any $h, k \in I$ and any open $V \subseteq U_h \cap U_k$, we have that

$$\mathcal{F}|_{U_h \cap U_k}(V) = \mathcal{F}(V) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}_i(V \cap U_i) \mid \phi_{ij}(V \cap U_i \cap U_j)(\rho_i{}^{V \cap U_i}_{V \cap U_i \cap U_j}(s_i)) = \rho_j{}^{V \cap U_j}_{V \cap U_i \cap U_j}(s_j) \text{ for all } i, j \in I\}$$

In particular, for any $h, k \in I$, any open $V \subseteq U_h \cap U_k$, and any $(s_i)_{i \in I} \in \mathcal{F}(V)$, we have that

$$\phi_{hk}(V \cap U_h \cap U_k)(\rho_h{}^{V \cap U_h}_{V \cap U_h \cap U_k}(s_h)) = \phi_{hk}(V)(s_h) = s_k = \rho_k{}^{V \cap U_k}_{V \cap U_h \cap U_k}(s_k)$$

Thus, for any $h, k \in I$, any open $V \subseteq U_h \cap U_k$, and any $(s_i)_{i \in I} \in \mathcal{F}(V)$, we have that

$$\psi_k(V)((s_i)_{i \in I}) = s_k = \phi_{hk}(V)(s_h) = \phi_{hk}(V)(\psi_h(V)((s_i)_{i \in I}))$$

and therefore

$$\psi_k(V) = \phi_{hk}(V) \circ \psi_h(V)$$

Thus, for any $h, k \in I$, we have that $\psi_k|_{U_h \cap U_k} = \phi_{hk} \circ \psi_h|_{U_h \cap U_k}$.

Thus, $\mathcal{F}$ is a sheaf on X, with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ for each $i \in I$ such that $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

Uniqueness of $\mathcal{F}$ up to isomorphism. Suppose that $\mathcal{G}$ is a sheaf on X, with isomorphisms $\tau_i : \mathcal{G}|_{U_i} \to \mathcal{F}_i$ such that $\tau_j = \phi_{ij} \circ \tau_i$ on $U_i \cap U_j$. Let $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ and $\mathscr{H}om(\mathcal{G}, \mathcal{F})$ be the sheaves of local morphisms.

For each $i \in I$, let $f_i \in \mathscr{H}om(\mathcal{F}, \mathcal{G})(U_i)$ be $f_i = \tau_i^{-1} \circ \psi_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$, which is an isomorphism because ψ_i and τ_i are isomorphisms. Note that for any $i, j \in I$,

$$f_i|_{U_i \cap U_j} = \tau_i^{-1}|_{U_i \cap U_j} \circ \psi_i|_{U_i \cap U_j} = (\phi_{ji} \circ \tau_j|_{U_i \cap U_j})^{-1} \circ \phi_{ji} \circ \psi_j|_{U_i \cap U_j} =$$

$$\tau_j|_{U_i \cap U_j}^{-1} \circ \phi_{ji}^{-1} \circ \phi_{ji} \circ \psi_j|_{U_i \cap U_j} = \tau_j|_{U_i \cap U_j}^{-1} \circ \psi_j|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

Recall that $X = \bigcup_{i \in I} U_i$ is an open cover. The $f_i \in \mathscr{H}om(\mathcal{F}, \mathcal{G})(U_i)$ have the property that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, so because $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf (proven in Problem 2.1.15), there is a unique $f \in \mathscr{H}om(\mathcal{F}, \mathcal{G})(X)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

For each $i \in I$, let $g_i \in \mathscr{H}om(\mathcal{G}, \mathcal{F})(U_i)$ be $g_i = \psi_i^{-1} \circ \tau_i : \mathcal{G}|_{U_i} \to \mathcal{F}|_{U_i}$. By the same reasoning, there is a unique $g \in \mathscr{H}om(\mathcal{G}, \mathcal{F})(X)$ such that $g|_{U_i} = g_i$ for all $i \in I$.

Thus $g \circ f \in \mathscr{H}om(\mathcal{F}, \mathcal{F})$ and $f \circ g \in \mathscr{H}om(\mathcal{G}, \mathcal{G})$. For all $i \in I$, we have

$$(g \circ f)|_{U_i} = g_i \circ f_i = \mathrm{id}_{\mathcal{F}|_{U_i}} = \mathrm{id}_{\mathcal{F}}|_{U_i} \in \mathscr{H}om(\mathcal{F}, \mathcal{F})(U_i)$$

$$(f \circ g)|_{U_i} = f_i \circ g_i = \mathrm{id}_{\mathcal{G}|_{U_i}} = \mathrm{id}_{\mathcal{G}}|_{U_i} \in \mathscr{H}om(\mathcal{G}, \mathcal{G})(U_i)$$

Because $\mathscr{H}om(\mathcal{F}, \mathcal{F})$ and $\mathscr{H}om(\mathcal{G}, \mathcal{G})$ are sheaves, we therefore have that $g \circ f = \mathrm{id}_{\mathcal{F}}$ and $f \circ g = \mathrm{id}|_{\mathcal{G}}$. Thus $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.