THE ITERATION OF DIGIT-RELATED ARITHMETIC FUNCTIONS

Zev Chonoles[1]
Submitted: April 6, 2009

Abstract

We will answer the questions posed by Silverman in [1] about the sum-of-squares-of-digits function as special cases in our study of analogous functions for other bases and exponents, and propose multiple avenues for further research.

Notation and Preliminaries

In this paper, all objects will be elements of $\mathbb{N}$ unless stated otherwise, though we may explicitly state an object's membership in $\mathbb{N}$ for emphasis. For conciseness, given any set A and a map $f : A \to A$, and for an $a \in A$ and an $m \in \mathbb{N}$, we will write $f^m(a)$ to denote $f(f^{m-1}(a))$, if $m > 1$, and $f(a)$, if $m = 1$; it is also occasionally convenient to say $f^0(a) = a$. In other words, $f^m(a)$ denotes f applied to a, m times. We will use the simple fact that $f^n(f^m(a)) = f^{n+m}(a)$ without proof. We will also need some definitions:

Definition 1. A *cycle* of a map $f : A \to A$ is a finite, non-empty subset $C \subset A$ with $\forall\, c_1, c_2 \in C, f(c_1) \in C$ and $\exists\, m \in \mathbb{N} : f^m(c_1) = c_2$.

Definition 2. A *fixed point* of a map $f : A \to A$ is a cycle with one element. Equivalently, $\{a\}$ is a fixed point of f if $f(a) = a$.

Definition 3. An element $a \in A$ *reaches* a cycle C under the map $f : A \to A$ if $\exists\, m \in \mathbb{N} : f^m(a) \in C$.

Definition 4. If $a \in A$ reaches a cycle C under the map $f : A \to A$, then the *preperiod* of a is defined to be $\min\{m \in \mathbb{N} \cup \{0\} : f^m(a) \in C\}$.

Finally, recall the system of base representation for natural numbers; for a $b \in \mathbb{N}$, $b \geq 2$, the *base*, we can examine combinations $\sum_{i=0}^{k} a_i b^i$ of the powers of b, whose coefficients a_i are called *digits* when each $0 \leq a_i < b$. We will assume throughout this paper the elementary result that every $n \in \mathbb{N}$ has a unique representation in any base b, that is, $n = \sum_{i=0}^{k} a_i b^i$ for a unique collection of digits $0 \leq a_i < b$. In such a representation, $k = \lfloor \log_b(n) \rfloor$.

Introduction to Problem

In [1], the function F is given as $F(n) =$ the sum of the squares of the digits of n, where we are implicitly using the digits of n in base 10. In more precise terms,

Definition 5. $F : \mathbb{N} \to \mathbb{N}$ is the map taking $n = \sum_{i=0}^{k} a_i 10^i$ to $F(n) = \sum_{i=0}^{k} a_i^2$, where $k = \lfloor \log_{10}(n) \rfloor$ and each a_i is an integer $0 \leq a_i < 10$.

The sequence $F^0(n), F^1(n), F^2(n), \ldots$ is the subject of our investigation. One initial observation is that, because $F(1) = 1$, $\{1\}$ is a fixed point of F. The specific questions we are charged with are

- Are there infinitely many numbers n which reach $\{1\}$ under F?
- Are there infinitely many numbers n which don't reach $\{1\}$ under F?
- For those n which don't reach $\{1\}$ under F, what does the sequence $F^0(n), F^1(n), F^2(n), \ldots$ do?

Due to space considerations we will go straight to a more general formulation.

Statement of General Problem

Let $b, e \in \mathbb{N}$, with $b \geq 2$. We will generalize the above function F to consider bases other than 10, and exponents for the digits other than 2, as follows:

Definition 6. $F_{b,e}(n) : \mathbb{N} \to \mathbb{N}$ is the map taking $n = \sum_{i=0}^{k} a_i b^i$ to $\sum_{i=0}^{k} a_i^e$, where $k = \lfloor \log_b(n) \rfloor$ and each a_i is an integer $0 \leq a_i < b$.

[1] The author was supported by a grant from his father.

From this definition, we have that $F_{10,2} = F$. We also have that $\{1\}$ is a fixed point of $F_{b,e}$ for all b and e, because the representation of 1 base b is "1" for all bases b, and $1^e = 1$, so that $F_{b,e}(1) = 1$. Some of the questions about $F_{b,e}$ we will be approaching are generalizations of the ones posed in [1]; others are new. Given some b and e,

- Do all n reach a cycle under $F_{b,e}$?
- If n reaches a cycle under $F_{b,e}$, how long is its preperiod?
- For each cycle of $F_{b,e}$ - in particular, $\{1\}$ - are there infinitely many n that reach it?
- How many cycles of $F_{b,e}$ are there? What lengths can the cycles of $F_{b,e}$ have?
- How many fixed points of $F_{b,e}$ are there? How many digits do the fixed points of $F_{b,e}$ have?
- What else can be said about the action of $F_{b,e}$?

Results

We first introduce some basic results that rigorize our intuition about cycles.

Proposition 1. If $a \in A$ has $a \in C$ for some cycle C, then $f^m(a) \in C$ for all $m \in \mathbb{N}$.

Proof. We proceed by induction. By the definition of a cycle, if $a \in C$, then $f(a) = f^1(a) \in C$; this is the base case. Suppose that if $a \in C$, then $f^i(a) \in C$ for $1 \le i \le k$. Then apply the base case to $f^k(a) \in C$ to see that $f(f^k(a)) = f^{k+1}(a) \in C$. Thus, $f^m(a) \in C$ for all $m \in \mathbb{N}$. □

Proposition 2. If C_1 and C_2 are two distinct cycles of $f : A \to A$, then C_1 and C_2 are disjoint.

Proof. We prove the contrapositive. Suppose $a \in C_1$ and $a \in C_2$, but that $\exists\, c \in C_1$ which $c \notin C_2$. Because $c \in C_1$, we have $c = f^m(a)$ for some $m \in \mathbb{N}$ by the definition of cycle. But by Proposition 1, $f^m(a) \in C_2$ as well because $a \in C_2$; contradiction. Thus if two cycles C_1 and C_2 share one element a, then $C_1 = C_2$. □

Proposition 3. If $a \in A$ reaches a cycle C under $f : A \to A$, then a reaches no cycle other than C.

Proof. Suppose $a \in A$ reached both C_1 and C_2. Then $\exists\, m_1 \in \mathbb{N} : f^{m_1}(a) \in C_1$ and $\exists\, m_2 \in \mathbb{N} : f^{m_2}(a) \in C_2$. Without loss of generality, $m_1 \le m_2$. If $m_1 = m_2$, then $f^{m_2}(a) \in C_1$ and $f^{m_2}(a) \in C_2$, and so by Proposition 2, $C_1 = C_2$. If $m_1 < m_2$, then $f^{m_2 - m_1}(f^{m_1}(a)) = f^{m_2}(a) \in C_1$ by Proposition 1, so once again $f^{m_2}(a) \in C_1$ and $f^{m_2}(a) \in C_2$, and so by Proposition 2, $C_1 = C_2$. □

Proposition 4. For each $m \in \mathbb{N}$, $a \in A$ reaches a cycle C under $f : A \to A$ iff $f^m(a)$ reaches C.

Proof. We proceed by induction. If $a \in A$ reaches C, then $\exists\, n \in \mathbb{N} : f^n(a) \in C$. If $n > 1$, then $f^n(a) = f^{n-1}(f(a)) \in C$, and so $f(a)$ reaches C; if $n = 1$, then $f^1(a) = f(a) \in C$, and by definition of a cycle, $f(f(a)) \in C$, so that $f(a)$ reaches C. Thus, if a reaches C, then $f(a)$ reaches C. Conversely, if $f(a)$ reaches C, then $\exists\, n \in \mathbb{N} : f^n(f(a)) \in C$. Thus $f^{n+1}(a) \in C$, and thus a reaches C. Therefore, we have a reaches C iff $f(a)$ reaches C. Now, suppose a reaches C iff $f^i(a)$ reaches C, for each $1 \le i \le k$. Then apply the base case to $f^k(a)$ to see that a reaches C iff $f^k(a)$ reaches C iff $f(f^k(a)) = f^{k+1}(a)$ reaches C. Thus, for each $m \in \mathbb{N}$, a reaches C iff $f^m(a)$ reaches C. □

The ordering of $\mathbb{N}$ allows us to draw much stronger conclusions. The following is our key theorem.

Theorem 1. *Suppose a map $g : \mathbb{N} \to \mathbb{N}$ satisfies $\exists\, n_0 \in \mathbb{N} : \forall\, n \in \mathbb{N}, n > n_0,\ g(n) < n$. Then*

1. *Every n reaches exactly one cycle of g.*

2. *There are at most n_0 cycles of g.*

3. *Each cycle is at most M long, where $M = \max\{g(n) : n \le n_0\}$.*

4. *The preperiod of n is at most $M + \max\{0, n - n_0\}$.*

Proof. Choose any $n \in \mathbb{N}$. First, we will show that $\exists\, m \in \mathbb{N} \cup \{0\} : g^m(n) \le n_0$ (that is, either $n \le n_0$ or $\exists\, m \in \mathbb{N} : g^m(n) \le n_0$). If $n \le n_0$, we are done; so suppose $n > n_0$. Consider the numbers $n = g^0(n), g^1(n), g^2(n), \ldots, g^{n-n_0}(n)$. If all of these numbers were greater than n_0, then by hypothesis, for each $1 \le i \le n - n_0$, $g^i(n) < g^{i-1}(n)$, or equivalently, $g^i(n) \le g^{i-1}(n) - 1$. Combining these inequalities, we have $g^i(n) \le g^0(n) - i = n - i$. But if this were true for each $1 \le i \le n - n_0$, then $g^{n-n_0}(n) \le n - (n - n_0) = n_0$; contradiction. Thus, for any $n \in \mathbb{N}$, either $n \le n_0$ or $\exists\, m \in \mathbb{N} : g^m(n) \le n_0$, and we can combine these statements by saying that $g^m(n) \le n_0$ for some $0 \le m \le \max\{0, n - n_0\}$.

Now we will prove that if $n \le n_0$, then $g^m(n) \le M$ for all $m \in \mathbb{N}$. We proceed by induction. For any $n \le n_0$, $g(n) \le M$ by our choice of M. Suppose that for all $n \le n_0$, $g^i(n) \le M$ for all $1 \le i \le k$. If $g^k(n) \le n_0$, then $g(g^k(n)) = g^{k+1}(n) \le M$, once again by our choice of M; if $g^k(n) > n_0$, then $g^{k+1}(n) = g(g^k(n)) < g^k(n) \le M$ by our choice of n_0. Thus if $n \le n_0$, then $g^m(n) \le M$ for all $m \in \mathbb{N}$. But this means that, for any $n \le n_0$, there is at least one repeat $g^i(n) = g^j(n)$, $i < j$ among the numbers $g^1(n), g^2(n), \ldots, g^{M+1}(n)$ because each is an integer between 1 and M. Choose any repeat among these for which $j - i$ is minimal; it will necessarily have $g^i(n) \ne g^k(n)$ for $i < k < j$, by minimality. Then $C = \{g^i(n), g^{i+1}(n), \ldots, g^{j-1}(n)\}$ is a cycle with $j - i$ elements, because for $i \le k < j - 1$, $g(g^k(n)) = g^{k+1}(n) \in C$, and $g(g^{j-1}(n)) = g^j(n) = g^i(n) \in C$, so that for all $g^k(n) \in C$, $g^{k+1}(n) \in C$; and because for any $i \le i + a < j$ and any $i \le i + b < j$, $g^{j-i-a+b}(g^{i+a}(n)) = g^{j+b}(n) = g^b(g^j(n)) = g^b(g^i(n)) = g^{i+b}(n)$, so for any $g^{i+a}(n), g^{i+b}(n) \in C$, $\exists\, m \in \mathbb{N}$: $g^m(g^{i+a}(n)) = g^{i+b}(n)$. Thus C satisfies the definition of cycle. Clearly, n reaches C under g because $g^i(n) \in C$, and by Proposition 3, this is the only cycle n goes to. Thus, we have shown that every $n \le n_0$ reaches exactly one cycle. But we have also proven that every $n \in \mathbb{N}$ has some $0 \le m \le \max\{0, n - n_0\}$ for which $g^m(n) \le n_0$, and by Proposition 4, n reaches whatever cycle $g^m(n)$ does; thus every $n \in \mathbb{N}$ reaches exactly one cycle of g.

The other parts of the theorem follow easily from our construction above. Because for every $n \in \mathbb{N}$ there is an $x \le n_0$ which reaches the same cycle as n, the set of cycles which the numbers less than n_0 reach necessarily comprises all the cycles any $n \in \mathbb{N}$ can reach. The "worst case scenario" is if each $x \le n_0$ has a distinct cycle, in which case there would be n_0 cycles; if some of the numbers $\le n_0$ share a cycle, there are less than n_0 cycles. Similarly, because all cycles of g are necessarily subsets of $\{1, 2, \ldots, M\}$ (by our construction), the longest one could possibly be is M. Finally, our construction showed that it takes at most $0 \le m \le \max\{0, n - n_0\}$ applications of g to n to reach a number $\le n_0$; after that point, the "worst case scenario" would be that the only cycle is a single fixed point c, and that we would have to go through all $M - 1$ of the other numbers $\le M$ before we reached it. Thus, it takes at most $M + \max\{0, n - n_0\}$ applications of g to n to reach its cycle. □

Theorem 2. *For all $b, e \in \mathbb{N}$, $b \ge 2$, $\exists\, n_0 \in \mathbb{N} : \forall n \in \mathbb{N}, n > n_0$, $F_{b,e}(n) < n$. Thus, $F_{b,e}$ has all the properties described in Theorem 1.*

Proof. Fix some $b, e \in \mathbb{N}$, $b \ge 2$. Recall that for $n = \sum_{i=0}^{k} a_i b^i$, $F_{b,e}(n) = \sum_{i=0}^{k} a_i^e$, where $k = \lfloor \log_b(n) \rfloor$. We want to find an n_0 such that, for all $n > n_0$, $n = \sum_{i=0}^{k} a_i b^i > \sum_{i=0}^{k} a_i^e = F_{b,e}(n)$. Because each a_i has $0 \le a_i < b$, if we find an n_0 such that for all $n > n_0$, $n = \sum_{i=0}^{k} a_i b^i > \sum_{i=0}^{k} b^e$, that n_0 would certainly have $n > F_{b,e}(n)$ for all $n > n_0$. Because $k = \lfloor \log_b(n) \rfloor$, this is equivalent to $n > (\lfloor \log_b(n) \rfloor + 1) b^e$; we can drop the $\lfloor\ \rfloor$, because doing so only makes the condition for n_0 stronger; thus, we want a point n_0 for which $\frac{n}{\log_b(n)+1} > b^e$ for all $n > n_0$. Because the left side is monotonically increasing and unbounded and the right side is just a constant, there is obviously such an n_0. □

Definition 7. Define the function $n_0(b, e)$ to be the least $m \in \mathbb{N}$ for which $\frac{m}{\log_b(m)+1} > b^e$.

Remark. Note that $n_0(b, e)$ is not the least number m with the property that $n > F_{b,e}(n)$ for all $n > m$, because we replaced the digits with b's and so our answer is larger than necessary. There is also no elementary expression for $n_0(b, e)$ - an analytic solution requires the Lambert W-function - but Mathematica is able to provide a good numerical approximation.

Thus, to find all cycles of $F_{b,e}$, we simply have Mathematica compute the bound $n_0(b,e)$, and find which cycles the numbers less than $n_0(b,e)$ reach. I wrote a Mathematica program to do this[2], and it determined for $b = 10$, $e = 2$ that $n_0 = 356$ and $M = 166$. It then found the only cycles reached by numbers less than 356 were $\{1\}$ and $\{4, 16, 37, 58, 89, 145, 42, 20\}$ (certainly consistent with Theorem 1's upper bound of 356 cycles, and upper bound of cycles having 166 elements), and by Theorem 1, these are the only cycles of $F_{10,2}$, and every $n \in \mathbb{N}$ reaches one or the other. Also by Theorem 1, the preperiod of n under $F_{10,2}$ is at most $166 + \max\{0, n - 356\}$. Given any other choice of b and e, we can do these calculations, though because this is only an algorithm, there is no "formula" in terms of b and e.

We now present some results on fixed points[3] for the case of $e = 2$.

Proposition 5. If $b \in \mathbb{N}$, $b \geq 2$ is odd, then $F_{b,2}$ has at least 3 fixed points.

Proof. Besides the fixed point 1, which is fixed for any base b, if $b = 2r+1$ for some $r \in \mathbb{N}$, then $rb+(r+1)$ and $(r+1)b+(r+1)$ are fixed points because $r^2+(r+1)^2 = 2r^2+2r+1 = r(2r+1)+(r+1) = rb+(r+1)$ and $(r+1)^2 + (r+1)^2 = 2r^2 + 4r + 2 = (r+1)(2r+1) + (r+1) = (r+1)b + (r+1)$, respectively. □

We can actually vastly generalize such an analysis, via the following theorem:

Theorem 3. *For all $b \in \mathbb{N}$, $b \geq 2$, a fixed point n of $F_{b,2}$ has either that $n = 1$, or that n has 2 base-b digits. There are an odd number of fixed points of $F_{b,2}$. There are at most $b^2 - b + 1$ fixed points of $F_{b,2}$.*

Proof. First, we check this for $b = 2, 3, 4$. By following the approach outlined above, Mathematica determined that the only cycle of $F_{2,2}$ is $\{1\}$; that the only cycles of $F_{3,2}$ are $\{1\}, \{5\}, \{8\}, \{2, 4\}$; and that the only cycle of $F_{4,2}$ is $\{1\}$. Written in their respective bases, the fixed points of the above are 1 for $b = 2$; $1, 12, 22$ for $b = 3$; and 1 for $b = 4$. Thus, we can see that there are an odd number of fixed points of $F_{b,2}$ for $b = 2, 3, 4$, and that other than 1, each fixed point has 2 base-b digits. Now, we assume $b > 4$. Note that $n_0 = b^3$ works as a bound in Theorem 2, because $\frac{b^3}{3+1} > b^2$ (since $b > 4$). Thus, any fixed points of $F_{b,2}$ will have to be $\leq b^3$. b^3 itself is not a fixed point, because $F_{b,2}(b^3) = 1^2+0^2+0^2+0^2 = 1$; thus any fixed points of $F_{b,2}$ have to be $< b^3$, i.e. have at most 3 base-b digits. Suppose $a_2b^2 + a_1b + a_0$ were a fixed point, with $a_2 \neq 0$ and each $0 \leq a_i < b$. Then $a_2b^2+a_1b+a_0 = a_2^2+a_1^2+a_0^2$, so that $a_2(b^2-a_2)+a_1(b-a_1) = a_0(a_0-1)$. The term $a_2(b^2 - a_2) \geq (b^2 - 1)$ because the derivative of $a_2(b^2 - a_2)$ is positive on the interval $(0, b)$ (since $b > 4$), so that the minimum value of $a_2(b^2 - a_2)$ is given by the minimum value of a_2, which is 1. In addition, the term $a_1(b - a_1) \geq 0$ by a similar analysis (the difference being that a_1 could be 0). However, the term $a_0(a_0 - 1)$ is at most $(b - 1)(b - 2)$, because $a_0(a_0 - 1)$ reaches its maximum value for the maximum value of a_0, which is $b - 1$. Thus the left side of the equation is $\geq (b^2 - 1)$, and the right side is $\leq (b - 1)(b - 2)$; contradiction. Thus, no fixed points of $F_{b,2}$ have 3 base-b digits, so fixed points of $F_{b,2}$ can only have either 1 or 2 base-b digits. If $0 \leq a_0 < b$ is a fixed point, then $a_0 = F_{b,e}(a_0) = a_0^2$; thus the only fixed point with 1 base-b digit is 1, and thus all other fixed points of $F_{b,2}$ must have 2 base-b digits.

Now, suppose $a_1b + a_0$ is a fixed point of $F_{b,2}$ with $a_1 \neq 0$ and $0 \leq a_0, a_1 < b$; then $(b - a_1)b + a_0$ is also a fixed point, because $F_{b,2}((b - a_1)b + a_0) = (b - a_1)^2 + a_0^2 = b^2 - 2a_1b + a_1^2 + a_0^2 = b^2 - 2a_1b + (a_1b + a_0) = (b - a_1)b + a_0$. This suffices to show that there are an even number of fixed points with 2 base-b digits if we can prove that there are no fixed points with $a_1 = b - a_1$, i.e. $b = 2a_1$. If $a_1(2a_1) + a_0$ were a fixed point, then $a_1(2a_1)+a_0 = a_1^2+a_0^2$, so that $a_1^2 = a_0(a_0-1)$, but a_0 and a_0-1 cannot both be squares; contradiction. Thus, we can pair off fixed points having 2 base-b digits (pairing each $a_1b + a_0$ with $(b - a_1)b + a_0$) and combined with the fixed point 1, we have shown there are an odd number of fixed points of $F_{b,2}$. Finally, if every number with 2 base-b digits, and 1, were fixed points, we would have $(b-1)b+1 = b^2-b+1$ fixed points (because there are $(b - 1)b$ numbers with 2 base-b digits); thus there are at most that many. □

[2]The code I wrote to investigate this project can be found here: http://zatomics.googlepages.com/projectcode.nb

[3]For notational convenience, we will identify the fixed point $\{n\}$ with its element, n.

Theorem 4. *Given any $b, e \in \mathbb{N}$, $b \geq 2$, for all cycles C of $F_{b,e}$, $\exists$ infinitely many $m \in \mathbb{N} : m$ reaches C.*

Proof. We can actually prove something much stronger: for all $n, b, e \in \mathbb{N}$, $b \geq 2$, $\exists$ infinitely many $m \in \mathbb{N} : F_{b,e}(m) = n$ (thus, choosing any $c \in C$, there are infinitely many $m \in \mathbb{N} : F_{b,e}(m) = c$ and so infinitely many $m \in \mathbb{N}$ which reach C in a single application of $F_{b,e}$). We simply construct infinitely many. Let $h(b, n, x) = \sum_{i=0}^{n-1} b^{x+i}$. Written in base b, this number is a string of n 1's followed by x 0's. Thus, we have $F_{b,e}(h(b, n, x)) = \underbrace{1^e + \cdots + 1^e}_{n} + \underbrace{0^e + \cdots 0^e}_{x} = n$ for all $x \geq 0$. ☐

In particular, infinitely many numbers reach both $\{1\}$ and $\{4, 16, 37, 58, 89, 145, 42, 20\}$ under $F_{10,2}$. Using a similar style of clever reasoning, we can prove

Proposition 6. For all $n, e \in \mathbb{N}$, $\exists$ infinitely many $b \in \mathbb{N}, b \geq 2 : n$ reaches $\{1\}$ under $F_{b,e}$.

Proof. We construct infinitely many. Let $b(n, e, x) = n^{e^x}$. We have that $n^{e^i} < b(n, e, x)$ for all $0 \leq i < x$, so $F_{b(n,e,x),e}(n) = n^e$ because n is represented by a single digit in base $b(n, e, x)$ (namely, itself); similarly, $F^2_{b(n,e,x),e}(n) = F_{b(n,e,x),e}(n^e) = n^{e^2}$ because n^e is represented by a single digit; we continue until we get $F^x_{b(n,e,x),e}(n) = n^{e^x} = b(n, e, x)$, which is the first point at which we get a number whose base $b(n, e, x)$ representation isn't just itself considered as a digit - instead, the base $b(n, e, x)$ representation of $F^x_{b(n,e,x),e}(n) = b(n, e, x)$ is "10", so that $F^{x+1}_{b(n,e,x),e}(n) = 1^e + 0^e = 1$. Thus, for any $n, e \in \mathbb{N}$, n reaches $\{1\}$ under $F_{b(n,e,x),e}$ for all $x \geq 0$. ☐

We have answered all the questions we stated at the beginning; here are some more ideas.

Suggested Topics For Further Exploration

- *Asymptotic density of n which reach a given cycle of $F_{b,e}$?* I ran a Mathematica evaluation of the numbers less than 1,000,000 under $F_{10,2}$, which determined that approximately 13.3% went to $\{1\}$.
- *Mixed-radix representations? Zeckendorf (Fibonacci) representation?* These representation systems can likely be tackled by the analysis in this work. In particular, I hypothesize that in a mixed-radix system where b_i is the least radix used and b_j is the greatest, all we need to do to prove an analogous version of Theorem 2 is to use $\log_{b_i}(n)$ as an upper bound of the number of digits and b_j as an upper bound on the size of each digit.
- *Balanced base $2b+1$ representations?* This works similarly to base $2b+1$, but it allows digits a_i from $-b \leq a_i \leq b$ instead of from $0 \leq a_i \leq 2b$. Thus, if e is odd, it may very well happen that the sum of the eth powers of the digits is negative, in which case we would need to expand the scope of our function to all of $\mathbb{Z}$.
- *Negative exponents e?* For example, $e = -1$ would yield the sum of the reciprocals of the digits. For negative e, we would need to work in $\mathbb{Q} \cup \{\infty\}$, because we would need to have multiplicative inverses of the digits, and if the sum of their eth powers yielded a non-terminating base-b representation (e.g. 3 maps to $0.\overline{1}$ under $F_{10,-2}$), the next step would be an infinite sum.
- *Other functions of the digits, besides power sums?* It would appear from our work that the analytic properties of the function, such as the existence of a bound like the one described in Theorem 1, are very important. Thus, it seems unlikely that applying number theoretic functions (φ, σ, etc.) to the digits would yield much insight, while polynomial functions seem more promising. In particular, I wonder if there might be a distinct difference in the action of symmetric polynomials vs. non-symmetric polynomials of the digits.
- *Any application to Benford's Law?* This concerns the leading digits in naturally occuring data.

References

[1] Joseph H. Silverman, *MATH 1560 Research Project*, Homework Assignments **1** (Mar 2009), no. 1, 1–2.