1. PRELIMINARIES ON DIFFERENTIABLE FUNCTIONS

Definition 1.1. Let U be an open subset of $\mathbb{R}^m$. A function $f : U \to \mathbb{R}^n$ is differentiable at $x \in U$ if there is a linear transformation $T : |R^n \to \mathbb{R}^m$ such that: for all $\epsilon > 0$ there is some $\delta > 0$ with the following property:

$$v \in \mathbb{R}^n, \|v\| < \delta \implies x + v \in U \text{ and } \|f(x+v) - f(x) - Tv\| \leq \epsilon\|v\|$$

There is only one linear transformation T with the above property and it is denoted by $f'(x)$.

Theorem 1.2. ***chain rule***: *Let U, V, W be open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. Let $f : U \to V$ and $g : V \to W$ be functions that are differentiable at $x \in U$ and $y \in V$ respectively, Assume furthermore that $f(x) = y$. Then*

1. *the composite $g \circ f$ is differentiable at x and*
2. *$(g \circ f)'(x) = g'(f(x) \circ f'(x)$.*

Proof. left to the reader. See [1] or [14] □

Definition 1.3. $f : U \to \mathbb{R}$ is a function defined on an open subset $U \subset \mathbb{R}^n$. f is C^0 if f is continuous.

f is C^1 if all its partial derivatives $\partial_i f$ are defined and continuous on U. More generally, f is C^k (where $k \in \mathbb{N}$) if all its partial derivatives $\partial_i f$ are defined and are C^{k-1}.

f is C^∞ if f is C^k for every non-negative integer k.

The collection of C^k functions on U will be denoted by $C^k(U)$.

Proposition 1.4.
1. *If $f \in C^k(U)$ then $f \in C^{k-1}(U)$.*
2. *If $f, g \in C^k(U)$ then $f + g$ and fg are also C^k. All constant functions are C^k.*

Proof. Part 2 follows from part (1) and the Liebniz rule. □

Lemma 1.5. *$U \subset \mathbb{R}^n$ is open. Let $f : U \to \mathbb{R}$ be C^1. Then f is differentiable (see the first definition) at every point $x \in U$. Its derivative $f'(x) : \mathbb{R}^n \to \mathbb{R}$ is given by*

$$f'(x)(a_1, a_2, ..., a_n) = a_1\partial_1 f(x) + a_2\partial_2 f(x) + ... + a_n\partial_n f(x) \, \forall (a_1, ..., a_n) \in \mathbb{R}^n$$

Proof. An iterated use of the mean-value theorem. See [1] or [14]. □

Definition 1.6. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. Given $f : U \to V$ define $f_i : U \to \mathbb{R}$ for $i = 1, 2, ..., m$ by $f(x) = (f_1(x), ..., f_m(x))$ for every $x \in U$. If all the f_i are C^k then f is said to be C^k as well.

Theorem 1.7. *Let U, V, W be open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. If $f : U \to V$ and $g : V \to W$ are both C^k then so is $g \circ f$.*

Proof. Induction on k. The case $k = 0$ is OK.

Assume $k > 0$. In view of the lemma, one sees that "f is C^k" is equivalent to the reformulation "f is differentiable at all $x \in U$ and $x \mapsto f'(x)$ is a C^{k-1} function, denoted by f', from U to the space of $m \times n$ matrices."

The chain rule now tells us that $g \circ f$ is differentiable, and also that $(g \circ f)' = (g' \circ f).f'$ where . denotes matrix multiplication. Now both g' and f are C^{k-1}. By the induction hypothesis, $g' \circ f$ is C^{k-1}. In addition, f' is also C^{k-1}. It follows that the matrix product $g' \circ f.f'$ is also C^{k-1}. We have shown that $(g \circ f)'$ is C^{k-1}. In view of the equivalent reformulation, the theorem follows.

□

Proposition 1.8. *Let $f : U \times V \to \mathbb{R}$ be C^k where $k > 0$ and $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^n$ are both open subsets. Assume furthermore that $0 \in U$. Then there is a C^{k-1} function $g : U \times V \to \mathbb{R}$ such that*

$$f(x, y) = f(0, y) + xg(x, y) \text{ for all } x \in U, y \in V \tag{1}$$

Proof. The above equation in fact defines a C^k function g on the domain $(U \setminus \{0\}) \times V$. We have $a > 0$ such that the open interval $(-a, a)$ is contained in U. Now let $0 < |x| < a, y \in V$ and let $\gamma(t) = (tx, y)$ for $0 \leq t \leq 1$. An application of the fundamental theorem of calculus to the function $f \circ \gamma$ shows that

$$g(x, y) = \int_0^1 \partial_1 f(tx, y) dt \text{ whenever } 0 < |x| < a, y \in V \tag{2}$$

By assumption, $\partial_1 f$ is C^{k-1} on $U \times V$. It follows easily that the expression on the right in equation (2) defines a C^{k-1} function on $U \times V$, once again denoted by g by abuse of notation. The desired equation (1) holds on $(U \setminus \{0\}) \times V$ and continuity shows that it holds on $U \times V$ as well. □

Corollary 1.9. *Let $a, b > 0$, let $X = (-a, a)^r \times (-b, b)^s$ and let $f : X \to \mathbb{R}$ be a C^∞ function that vanishes on $0 \times (-b, b)^s$. (here $0 = (0, 0, ..., 0) \in (-a, a)^r$). Then there are C^∞ functions $g_1, ...g_r$ defined on X such that*

$$f = x_1 g_1 + x_2 g_2 + ... + x_r g_r$$

2. Inverse Function Theorem

Theorem 2.1. ***Inverse function theorem*** *(a) Let U be an open subset of $\mathbb{R}^n$, let $p \in U$ and let $f : U \to R^n$ be a C^1 function such that the linear transformation $f'(p) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then $f|_V$ is a one-to-one open map for some nbhd V of p in U. Thus $f(V)$ is open and there is a homeomorphism $h : V \to f(V)$ such that $f(x) = h(x)$ for all $x \in V$. (b) If the above f is C^k for some $k \geq 1$ then $h^{-1} : f(V) \to V$ is also C^k.*

Part (b) of the theorem is deduced from part (a) in exactly the same manner in all the sources (that I've seen) and is omitted. We concentrate on part (a). The proof by the Contraction Principle in [11, 13, 14] is valid even for Banach spaces. The proof of part (a) in [1] is interesting and different.

Problem 2.2. Deduce part (a) of the inverse function theorem from 'one-variable-calculus' (Intermediate Value thm. and Mean-Value thm. precisely) when $f(x_1, ..., x_n) = (g(x_1, ..., x_n), x_2, x_3, ..., x_n)$ under the assumption that $\partial_1 g(p) \neq 0$.

Problem 2.3. Assume thet the f in the statement of IFT is given by $f = (f_1, ..., f_n)$ where the $f_i : U \to \mathbb{R}$ are C^1. Prove that there is a permutation σ such that if $g_i = f_{\sigma(i)}$ for all i, then F_i defined by

$$F_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, g_i, g_{i+1}, ..., g_n)$$

satisfies the hypothesis of the inverse function theorem at p for all $i = 1, 2, ...n$.

Problem 2.4. Deduce the inverse function thm. from the previous two problems.

Problem 2.5. Assume that $f : U \to \mathbb{R}^n$ is continuous everywhere, and differentiable at a given point $p \in U$. Assume that the linear transformation $f'(p)$ is invertible. Prove that there is a nbhd V of p in U such that $f(V)$ contains a neighbourhood of $f(p)$.

Hint: prove for all sufficiently small and positive r, that
(i) $\|x - p\| = r$ implies $f(x) \neq f(p)$
(ii) the map from the sphere of radius r to $\mathbb{R}^n \setminus \{f(p)\}$ given by $x \mapsto f(x)$ is homotopic to $x \mapsto f(p) + f'(p)(x - p)$.

Now deduce that $\{f(x) : \|x - p\| \leq r\}$ contains the connected component of $f(p)$ in the complement of the compact set $\{f(x) : \|x - p\| = r\}$.

3. THE DEFINITION OF A C^∞ MANIFOLD

It is a good idea to read the definition of a presheaf (also a sheaf) on a topological space. Even though the little that is required for the moment has been spelt out below.

Notation 3.1. Recall that $C^0(Y)$ denotes the collection of continuous functions $Y \to \mathbb{R}$. Let $U \subset V \subset Y$ both be open subsets of Y. If $f \in C^0(V)$ then $f|_U \in C^0(U)$. Denote $f \mapsto f|_U$ by $Res(U, V) : C^0(V) \to C^0(U)$.

Let $f : Y \to Z$ be continuous. For every open $V \subset Z$ and for every $g \in C^0(V)$ we obtain $f^*g \in C^0(f^{-1}V)$ giiven by $(f^*g)x = g(f(x))$ for all $x \in f^{-1}V$.

Definition 3.2. Let Y be a topological space. A *subpresheaf* (or a presheaf of subsets) R of C^0_Y consists of

(1) the data: a subset $R(U) \subset C^0(U)$ for every open subset U of Y
subject to the condition:
(2) $Res(U, V)R(V) \subset R(U)$ whenever $U \subset V$ are both open subsets of Y.

R is a *subsheaf* of C^0_Y if in addition
(3) $f \in C^0(V), V = \cup\{V_i : i \in I\}, \forall i \in I\, Res(V_i, V)f \in R(V_i) \implies f \in R(V)$.

A *sheaf of* $\mathbb{R}$*-subalgebras* of C^0_Y is a subsheaf R of C^0_Y such that $R(U)$ is a $\mathbb{R}$-subalgebra of $C^0(U)$ for every open $U \subset Y$.

Definition 3.3. A *ringed space*[1] is a pair (Y, R) where Y is a topological space and R is a sheaf of $\mathbb{R}$-subalgebras of C^0_Y.

If (Y, R) is a ringed space and if $U \subset Y$ is open, then we obtain *the ringed space* (U, R_U) by setting $R_U(V) = R(V)$ for all open $V \subset U$.

A morphism of ringed spaces $f : (Y, R) \to (Y', R')$ is simply a continuous map $f : Y \to Y'$ with the property: U' open in Y' and $g \in R'(U')$ implies $f^*g \in R(f^{-1}(U'))$. See notation above for f^*g.

In addition, if f is a homeomorphsim and if its inverse $f^{-1} : Y \to Y'$ gives rise to a morphism $(Y', R') \to (Y, R)$ of ringed spaces then f is said to be an *isomorphism* of ringed spaces.

[1]the term 'ringed space' in mathematical literature covers a more general situation than encountered in these notes

Example 3.4. Let $0 \leq k \leq \infty$ and let U be open in $\mathbb{R}^n$. We have the sheaf of $\mathbb{R}$-subalgebras C^k_U given by $C^k_U(V) = C^k(V)$ for all open subsets $V \subset U$. See Prop.1.4(2).

Definition 3.5. A C^k-manifold is a ringed space (M, R) with the property that M is covered by its open subsets U for which (U, R_U) is isomorphic to (Ω, C^k_Ω) where Ω is open in $\mathbb{R}^n$ for some n.

For a C^k-manifold (M, R) it is traditional to denote R by C^k_M.

Given C^k manifolds (M, C^k_M) and (N, C^k_N), a map $f : M \to N$ is C^k if it is a morphism $(M, C^k_M) \to (N, C^k_N)$ of ringed spaces. In other words,

(1) $f : M \to N$ is continuous and
(2) For every open $V \subset N$ and for every $g \in C^k_N(V)$, the continuous function f^*g defined on $f^{-1}V$ belongs to $C^k_M(f^{-1}V)$.

Problem 3.6. Let U and V be open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. They are C^k manifolds in a natural manner. Show that the two definitions of "$f : U \to V$ is C^k" given in 3.5 and 1.6 are equivalent to each other.

Problem 3.7. Let U be an open subset of C^k manifold M. Let $f_1, f_2, ..., f_r$ be real-valued functions defined on U. Denote by $f : U \to \mathbb{R}^r$ the map $x \mapsto (f_1(x), ..., f_r(x))$. Prove that $f : M \to \mathbb{R}^r$ is C^k in the sense of definition 3.5 if and only if $f_i \in C^k_M(U)$ for all $i = 1, 2, ..., r$.

Problem 3.8. $f : M \to N$ is a local homeomorphism and (N, C^k_N) is a C^k manifold. Show that M acquires the structure of a C^k manifold for which $f : M \to N$ is C^k in the sense of 3.5.

Problem 3.9. In the previous problem, is there only one C^k structure on M for which f is C^k?

Problem 3.10. Define real analytic manifolds and complex analytic manifolds (the latter are referred to as complex manifolds). State and prove the analog of the previous problem in this context.

Problem 3.11. It is well known that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \exp(-1/t)$ for $t > 0$ and
$f(t) = 0$ otherwise
is C^∞. Using this function alone (and 1.4,1.7, 3.5) show that if $p \in U \subset M$ where M is a C^k manifold and U is open in M there is a C^k function $\phi : M \to \mathbb{R}$ which (a) vanishes on the complement of U and (b) satisfies $\phi(p) = 1$.

Problem 3.12. Let M and N be C^k manifolds. Let $f : M \to N$ be any function. Show that f is C^k if and only if

$$g : N \to \mathbb{R} \text{ is } C^k \implies g \circ f \text{ is } C^k$$

4. Tangent-spaces and cotangent-spaces

There are plenty of books on Differential geometry that contain the basic concepts and definitions: tangent bundles, one-parameter groups, the Frobenius theorem on sub-bundles of the tangent bundle:[7, 3, 10, 13, 18] are some of them.

The definition given of the tangent space of a C^∞ manifold in most sources e.g. [7, 13, 18] is the same. In Eucliden space, given a vector v we may compute the directional derivative $D_v f(p)$ of a function f defined on a neighbourhood U of $p \in \mathbb{R}^n$. Now $E(f) = D_v f(p)$ is a linear functional that satisfes the Liebniz rule:

$$E(fg) = f(p)E(g) + g(p)E(f)$$

The domain of the linear functional is the $\mathbb{R}$-algebra of germs(see [13, 18] of C^∞ functions at p is denoted by $C^\infty_{\mathbb{R}^n,p}$.

It is a simple exercise to deduce from Corollary 1.9 that every linear functional that satisfies the Liebniz rule is of the type $f \mapsto D_v f(p)$ for a unique $v \in \mathbb{R}^n$.

Definition 4.1. The tangent-space T_pM of a C^∞ manifold M at p is the collection of linear functionals $E : C^\infty_{M,p} \to \mathbb{R}$ that satisfies the Leibniz rule. By the above remarks, we see that if $\dim(M) = n$ then T_pM is a vector space of rank n.

If $f : (M,p) \to (N,q)$ is C^∞ then we get a $\mathbb{R}$-algebra homomorphism $f^* : C^\infty_{N,q} \to C^\infty_{M,p}$. If $E \in T_pM$ then the composite

$$C^\infty_{N,q} \xrightarrow{f^*} C^\infty_{M,p} \xrightarrow{E} \mathbb{R}$$

also satisfies the Leibniz rule. The corresponding element of T_qN is denoted by $f'(p)E$. This gives rise to the linear transformation $f'(p) : T_pM \to T_qN$.

Definition 4.2.

5. DIFFERENTIABLE SUBMANIFOLDS

What does it mean for an arbitrary subset A of a C^∞ manifold M to be a C^∞ submanifold?

For this purpose, we define a function $f : A \to \mathbb{R}$ to be admissible if every point $p \in A$ has a neighborhood U_p in M and a C^∞ function $f_p : U_p \to \mathbb{R}$ that agrees with f on the intersection $A \cap U_p$. The collection of admissible functions $f : A \to \mathbb{R}$ will be denoted by $R(A)$. This is clearly a $\mathbb{R}$-subalgebra of $C^0(A)$.

It is clear that if B is any subset of A and if $f \in R(A)$, then $f|_B \in R(B)$. The only subsets B we are interested in, however, are the open subsets (in the relative topology) of A. The $B \mapsto R(B)$ (for such B) defines a sheaf of $\mathbb{R}$-subalgebras of C^0_A.

Definition 5.1. *A is a C^∞ submanifold of M* if the ringed space(A, R) is a C^∞ manifold.

Given a point $p \in A$ we will say that A *is smooth at p* if there is a neighbourhood U of p in A such that the ringed space (U, R_U) is a C^∞ manifold (see prop 5.6).

Thus a subset A is a C^∞ submanifold iff A is smooth at every point $p \in A$.

Example 5.2. Let U_1 and U_2 be open in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively. Let $y_0 \in U_2$. Then $U_1 \times \{y_0\}$ is a C^∞ submanifold of $U_1 \times U_2$.

One frequently obtains C^∞ submanifolds through the two lemma below, both of which are simple consequences of the Inverse fUnction Thm.

Lemma 5.3. ***Parametric form***. *Let U and V be open subsets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and let $f : (U,q) \to (V,p)$ be a C^∞ function. Assume that the linear transformation $f'(q) : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one. Then there is*

a neighbourhood U_1 of q in U,
a neighbourhood U_2 of zero in $\mathbb{R}^{n-m}$,
a neighbourhood Ω of p in V, and
a diffeomorphism $\phi : U_1 \times U_2 \to \Omega$
such that $\phi(x, 0) = f(x)$ for all $x \in U_1$.

In view of example 5.2, it follows that $f(W)$ is a C^∞ submanifold of V for all W open in U_1–and in particular this is the case for W = all sufficiently small nbhds of q in U.

Lemma 5.4. ***Implicit form.*** *If U, V are open in $\mathbb{R}^n, \mathbb{R}^{n-m}$ respectively, if $f : (U, p) \to (V, q)$ is C^∞ and if $f'(p) : T_p\mathbb{R}^n \to T_q\mathbb{R}^{n-m}$ is surjective, then*

there are nbhds Ω of p in U, Ω_2 of q in V and Ω_1 of 0 in $\mathbb{R}^m$, and a diffeomorphism $\phi : \Omega \to \Omega_1 \times \Omega_2$ such that $f(x) = p_2\phi(x)$ for all $x \in \Omega$. Here $p_2 : \Omega_1 \times \Omega_2 \to \Omega_2$ is the projection to the second factor.

In view of example 5.2, we see that $f^{-1}(q)$ is smooth at p. Furthermore, $T_pf^{-1}(q)$ is the kernel of $f'(p)$.

Remark 5.5. In view of the fact that $f'(p) : T_pM \to T_qN$ has been defined for every C^∞ map $f : (M, p) \to (N, q)$ we may re-state the second statement of the above lemmas as follows:

(1) If $f'(p)$ is one-to-one, then $f(W)$ is a C^∞ submanifold of N for all sufficiently small nbhds W of p in M.
(2) If $f'(p)$ is onto, then $f^{-1}(q)$ is smooth at p. Furthermore the tangent-space of $f^{-1}(q)$ at p is the kernel of the linear transformation $f'(p)$.

Proposition 5.6. *Given $p \in A \subset M$ where M is a C^∞ manifold, then A is smooth at p if and only if there is a diffeomeorphism $U_1 \times U_2 \to W$ (where W is a nbhd of p in M, and U_1, U_2 are open in $\mathbb{R}^m, \mathbb{R}^{n-m}$ respectively) such that $W \cap \phi^{-1}A$ is simply $U_1 \times \{c\}$ for some point $c \in U_2$*

Proof. $\Leftarrow$ has already been remarked in example 5.2.

Now for the converse. The question being local at p, we may assume that M is an open subset of $\mathbb{R}^n$. Because A is smooth at p there is an isomorphism of ringed spaces

$$g : (M_1 \cap A, R_{M_1 \cap A}) \to (U, C^\infty_U)$$

where M_1 is a nbhd of p in M and U is an open subset of $\mathbb{R}^m$. Put $q = g(p)$. Denote the inverse of g by f. The composite

$$U \xrightarrow{f} M_1 \cap A \hookrightarrow M_1 \hookrightarrow \mathbb{R}^n$$

is then of the type $(f_1, ..., f_n)$. Denote the i-the co-ordinate function on $\mathbb{R}^n$ by $p_i : \mathbb{R}^n \to \mathbb{R}$. Because the p_i are C^∞, the $p_i|_{M_1 \cap A}$ belong to $R(M_1 \cap A)$. And because f is a morphism of ringed spaces these pull back to C^∞ functions of U. In other words, all the f_i are C^∞ functions on U.

Note next that g is given by $(g_1, ..., g_m)$ where the $g_i \in R(M_1 \cap A)$. It follows that there is a nbhd of M_2 of p in M_1 and C^∞ functions $\widetilde{g}_i$ defined on M_2 such that $\widetilde{g}_i(x) = g_i(x)$ for all $x \in M_2 \cap A$. Putting $\widetilde{g} = (\widetilde{g}_1, ..., \widetilde{g}_m)$ we obtain a C^∞ map $\widetilde{g} : M_2 \to \mathbb{R}^m$ such that $\widetilde{g}(x) = g(x)$ for all $x \in M_2 \cap A$.

Summarising, we have open subsets $U \subset \mathbb{R}^m$ and $M_2 \subset \mathbb{R}^n$ and C^∞ maps $f : U \to \mathbb{R}^n$ and $\widetilde{g} : M_2 \to \mathbb{R}^m$. Furthermore $\widetilde{g}(f(x)) = x$ for all $x \in f^{-1}(M_2 \cap A)$. The chain rule shows that $f'(q)$ is one-to-one. The claim now follows from lemma 5.3. □

Problem 5.7. A subset of a topological space is locally closed if it is the intersection of an open subset and a closed subset. Prove that a submanifold A of a manifold M is a locally closed subset. (No differentiability assumptions anywhere).

Problem 5.8. let A' denote the transpose of a matrix A. Let $X(resp.Y)$ be the space of $(k \times n)(resp.symmetric(k \times k))$ matrices with entries in $\mathbb{R}$. Define $F : X \to Y$ by $F(A) = A'A$ for all $A \in X$. Deduce from the implicit function theorem that the Stiefel variety $V_k(\mathbb{R}^n) = F^{-1}id_{k\times k}$ is a C^∞ submanifold of X of codimension $k(k+1)/2$.

Problem 5.9. Assume that $0 \in \mathbb{R}$ is a regular value (see dfn 6.1) of C^∞ function $f : M \to \mathbb{R}$. Define $h(x,t) = f(x) - t^2$ for all $x \in M, t \in \mathbb{R}$. Prove that zero is a regular value of h as well.

remark Let $W = \{x \in M : f(x) \geq 0\}$, let $\partial(W) = \{x \in M : f(x) = 0\}$. *Then W is a manifold with boundary* $\partial(W)$ and $h^{-1}(0)$ is referred to as the *double* of W because it is obtained from $W \times \{\pm 1\}$ by identifying $(x, -1)$ with $(x, 1)$ for all $x \in \partial(W)$.

Problem 5.10.

Show that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ s a C^∞ sub manifold of $\mathbb{R}^2$ off f is itself a C^∞ function.

Problem 5.11. $\|(x_1, ..., x_n)\| = \sqrt{x_1^2 + ... + x_n^2}$. Let $v_1, v_2, ..., v_g$ be distinct points of $\mathbb{R}^n$. Let $r_1, r_2, ..., r_g, R > 0$. Assume

(1) $r_i + r_j < \|v_i - v_j\|$ whenever $1 \leq i < j \leq g$, and
(2) $r_i + \|v_i\| < R$ whenever $1 \leq i \leq g$

Let $f(x) = (R^2 - \|x\|^2)\prod_{i=1}^{i=g}(\|x - v_i\|^2 - r_i^2)$ for all $x \in \mathbb{R}^n$. Let $h(x,t) = f(x) - t^2$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Prove that 0 is a regular value of h.

When $n = 2$, the manifold $h^{-1}(0)$ as an orientable surface of genus g. In general, it is the connected sum of g copies of $S^1 \times S^{n-1}$.

Problem 5.12. With f as in the previous problem, show that

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k : f(x) = \|y\|^2\}$$

is a C^∞ submanifold of $\mathbb{R}^n \times \mathbb{R}^k$ of codimension 1.

Problem 5.13. *map germs* Let M, N be C^∞ manifolds. Let $p \in M$ and $q \in N$. The set of map germs mapgrm$((M,p) \to (N,q))$ is the set of equivalence classes of C^∞ maps $f : (U,p) \to (N,q)$ where U is a nbhd of p in M. The equivalence relation $\sim$ is given as follows: given $f_i : (U_i, p) \to (N,q)$ for $i = 1, 2$,

$f_1 \sim f_2$ if and only if there is a nbhd U of p contained in $U_1 \cap U_2$ such that $f_1(x) = f_2(x)$ for all $x \in U$. The *germ of f* is the equivalence class of f, and will be denoted by $[f]$. The three subsets of mapgrm$((M,p) \to (N,q))$ consisting of germs of f such that $f'(p) : T_pM \to T_qN$ is (i) one-to-one (ii) onto and (iii) an isomorphism are denoted by (i) immgrm$((M,p) \to (N,q))$, (ii) sbmgrm$((M,p) \to (N,q))$ and (iii) diffgrm$((M,p) \to (N,q))$

autgrm(M,p) is defined to be diffgrm$((M,p) \to (M,p))$.

(1) For $[f] \in \text{mapgrm}((M,p) \to (N,q))$ and $g \in \text{mapgrm}((N,q) \to (S,r))$ define $[g] \circ [f] \in \text{mapgrm}((M,p) \to (S,r))$.
(2) Show that the above binary operation $\circ$ turns $\text{autgrm}(M,p)$ into a group.
(3) We have an action of the group $\text{autgrm}(N,q) \times \text{autgrm}(M,p)$ on the set $\text{mapgrm}((M,p) \to (N,q))$ given by

$$([g],[h]).[f] = [g] \circ [f] \circ [h]^{-1}$$

The subsets (i), (ii) and (iii) listed above, of $\text{mapgrm}((M,p) \to (N,q))$, are also taken into themselves by this group action. This group action restricts to an action of the subgroups $\text{autgrm}(N,q)$ and $\text{autgrm}(M,p)$.

There are nine questions:

which of the three groups $\text{autgrm}(N,q), \text{autgrm}(M,p), \text{autgrm}(N,q) \times \text{autgrm}(M,p)$ fails to act transitively on the sets (i) $\text{immgrm}((M,p) \to (N,q))$, (ii) $\text{sbmgrm}((M,p) \to (N,q))$ and (iii) $\text{diffgrm}((M,p) \to (N,q))$?

Problem 5.14. Given (M,p) and (N,q) and a non-negative integer r consider the set of C^∞ maps $f : (U,p) \to (N,q)$ (where U is a nbhd of p in M) such that $\text{rank}(f'(x)) = r$ for all $x \in U$. Denote the set of germs of such maps by $\text{mapgrm}_r((M,p) \to (N,q))$. Which of the three groups $\text{autgrm}(N,q), \text{autgrm}(M,p), \text{autgrm}(N,q) \times \text{autgrm}(M,p)$ fails to act transitively on $\text{mapgrm}_r((M,p) \to (N,q))$? (The rank theorem in [14] is another corollary of the Inverse function thm, and this is useful here.)

Problem 5.15. Let $p : \widetilde{M} \to M$ be the universal covering space of a topological manifold M. Let Γ be the group of covering transformations. Assume that $\widetilde{M}$ has been given a C^∞ structure.

Then M itself acquires the structure of a ringed space: given an open subset U of M, we define $R(U) \subset C^0(U)$ by

$$\forall f \in C^0(U), f \in R(U) \iff p^* f \in C^\infty(p^{-1}(U))$$

Show that the ringed space (M,R) is a C^∞ manifold if and only if every covering transformation is a diffeomorphism of $\widetilde{M}$.

Problem 5.16. Let $p : \mathbb{R}^2 \to M$ be the universal cover of the Mobius band. The covering transformations are $(x,y) \mapsto x+m, (-1)^m y)$ for all $m \in \mathbb{Z}$. The previous problem defines a C^∞ structure on$M$. Let $A = p(\mathbb{R} \times \{0\})$. This A is a subset of M homeomorphic to a circle. Is there a C^∞ function $f : M \to \mathbb{R}$ for which $b \in \mathbb{R}$ is a regular value and $f^{-1}(b) = A$?

6. TRANSVERSALITY

[12, 5, 8] are all excellent references.

Definition 6.1. *$M, N, f : M \to N$ are all C^∞ and A is a closed C^∞ submanifold of N.*

Let $p \in M$. Let $q = f(p)$. f is said to be transverse to A at p if
(a) $q \notin A$ or
(b) $q \in A$ and $T_q A + f'(p) T_p M = T_q N$.

f is transverse to A on a subset $D \subset M$ if f is transverse to A at every point $p \in D$.

If this condition is satisfied for $D = M$, then f *is said to be transverse to* A.

One checks easily that *the set* $B = \{p \in M : f \text{ is not transverse to } A \text{ at } p\}$ *is a closed subset of* $f^{-1}(A)$ *and in particular a closed subset of* M

A point $a \in N$ is *a regular value of* f if $f : M \to N$ is transverse to the submanifold $\{a\}$ of N–in other words, $f'(p)$ is surjective whenever $f(p) = a$.

Proposition 6.2. *If* f *is transverse to* A, *then* $f^{-1}(A)$ *is a* C^∞ *submanifold. Furthermore (with notation as above, for all* $p \in f^{-1}(A), q = f(p) \in A$,

the linear transformation $f'(p) : T_pM \to T_qN$ *takes* $T_pf^{-1}(A)$ *into* T_qA *and induces an isomorphism of quotient-spaces* $\overline{f'(p)} : T_pM/T_pf^{-1}(A) \xrightarrow{\cong} T_qN/T_qA$.

In particular, if $a \in N$ *is a regular value of* f *then* $f^{-1}(a)$ *is a submanifold whose tangent-space at* p *equals the kernel of* $f'(p)$.

7. MISCELLANY

One ought to know the definition of $p : E \to B$ is a *fiber bundle* (see any of [17, 9, 6]).

Straight from the definition, one checks that

(A) if $p_i : E_i \to B$ are fiber bundles for $i = 1, 2$, then $E_1 \times_B E_2 \to B$ is a fiber bundle.

(B) if If $p : E \to B$ is a fiber bundle and if $f : Y \to B$ is continuous, then the the projection $Y \times_B E \to Y$ turns the fiber product into a fiber bundle on Y. This fiber bundle is often denoted f^*E.

(C) It should be easy to guess what the definition of a C^∞ *fiber bundle* is. One ought to check the counterparts of (A) and (B) in this context. One should note that the definition of "$p : E \to B$ is a C^∞ fiber bundle" implies that p is a submersion. This observation shows that the fiber products in (A) and (B) are indeed C^∞ manifolds (see hw 3).

Remark 7.1. A *section of a fiber bundle* $p : E \to B$ is a continuous $s : B \to E$ such that $p(s(x)) = x$ for all $x \in B$. If the fiber bundle and the section s are both C^∞ then the tangent-space $T_{s(x)}E$ is the direct sum of the tangent-spaces of the submanifolds $p^{-1}(x)$ and $s(B)$ at that point $s(x)$.

One should also know what a *vector bundle* is (see [17, 9, 2]) and also the definition of a C^∞ vector bundle of a C^∞ manifold. Facts (A) and (B) above are also valid for vector bundles.

Remark 7.2. Pointwise addition of C^∞ sections of a C^∞ vector bundle $p : V \to M$ turns the set of C^∞ sections, denoted by $\Gamma(M, V)$ henceforth, into a commutative group. Given a C^∞ real-valued function f and $s \in \Gamma(M, V)$ we get another section fs given by $(fs)(x) = f(x)s(x)$. We see that $\Gamma(M, V)$ now inherits the structure of a module over the ring of C^∞ functions on M.

In particular, every vector bundle $p : V \to M$ *has the zero section, which we'll denote by* $0_V : M \to V$. The tangent-space of V at $0_V(x)$ is the direct sum of the tangent-spaces of $p^{-1}(x)$ and $0_V(M)$ at that point, by 7.1.

Two examples have been encountered so far:

Example 7.3. The *tangent bundle* TM of a C^∞ manifold M (see [18, 10, 7, 3]).

Given a C^∞ map, we have $Tf : TM \to TN$ given by $Tf(v) = f'(x)v$ for all $v \in T_xM$.

The complement of the zero section of the tangent bundle TM will be denoted henceforth by $T'M$.

Example 7.4. The normal bundle N of a *locally closed* C^∞ submanifold Y of $\mathbb{R}^m$. It is given by

$$N = \{y, v) \in Y \times \mathbb{R}^m :< w, v >= 0 \forall w \in T_yY\}$$

Now N is a closed C^∞ submanifold of $Y \times \mathbb{R}^m$. Denoting $(y, v) \mapsto y$ by $p : N \to Y$ one checks that $p : N \to Y$ is a C^∞ vector bundle on Y.

For $\epsilon > 0$, let $N_\epsilon = \{(y, v) \in N : \|v\| < \epsilon\}$. This is an open subset of N that contains the zero section $0_N(Y) = \{(y, 0) : y \in Y\}$. Define $A : N \to \mathbb{R}^m$ by $A(y, v) = y + v$ for all $(y, v) \in N$. A simple computation (an application of 7.1 and/or 7.2) shows that $A'(p)$ is an isomorphism for every $p \in 0_V(Y)$. From the inverse function thm and the local compactness of Y one deduces
the tubular nbhd thm there are open subsets $\Omega \subset N$ and $U \subset \mathbb{R}^m$ such that
(a) $0_V(Y) \subset \Omega$ and
(b) $A|_\Omega : \Omega \to U$ is a diffeomorphism.

When Y is *compact* then we may take $\Omega = N_\epsilon$ for some positive ϵ.

Corollary 7.5. *Every C^∞ submanifold Y of $\mathbb{R}^m$ is contained in an open subset $U \subset \mathbb{R}^m$ that is equipped with a C^∞ map $r : U \to Y$ such that $r(y) = y$ for all $y \in Y$.*

In fact, if $B : U \to \Omega$ is the inverse of the diffeomorphism $A|_\Omega$ of the tubular nbhd thm, then $r = p \circ B$ is the desired retraction (with $p : N \to Y$ as in the notn of the tubular nbhd thm).

It is easy (and shown in class) to prove that every compact (Hausdorff) C^∞ manifold can be embedded in $\mathbb{R}^N$ for some N. The same statement is true for arbitrary second countable Hausdorff C^∞ manifolds Y. We will assume this and proceed to rewrite the previous corollary as

Corollary 7.6. *Let Y be a C^∞ second countable Hausdorff manifold. Then there exists an open subset $U \subset \mathbb{R}^m$ and C^∞ mappings $i : Y \to U$ and $r : U \to Y$ such that $r(i(y)) = y$ for all $y \in Y$.*

8. Sard's theorem and applications

The proof of Sard's thm given in class is taken from [12]. Sard's thm (both statement and proof) for C^k maps is due to Whitney– see [13].

Theorem 8.1. ***Sard's theorem*** *If $f : M \to N$ is C^∞ then the complement in N of the set of regular values of f is a set of measure zero.*

Remark 8.2. If $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, then the complement of the set of regular values is a finite set.

If $f \in \mathbb{C}[x_1, ..., x_n]$ then $f : \mathbb{C}^n \to \mathbb{C}$ given by $x \mapsto f(x)$ has its regular values as the complement of a finite subset of $\mathbb{C}$. The proof of such statements require some rudimentary algebraic geometry.

Corollary 8.3. *If M, N and $f : M \to N$ are all C^∞ and if* $\dim(M) < \dim(N)$*, then $f(M)$ has measure zero. In particular, f is not surjective.*

Proof. In fact $a \in N$ is a regular value of f iff $f^{-1}(a)$ is empty.

□

The four applications of Sard's theorem we are concerned with follow a fixed pattern. The simplest of all is the standard and important lemma below. See 7.2 for the definition of a section of a vector bundle.

Lemma 8.4. *Let $p : V \to M$ be a C^∞ bundle. Let $s_1, ...s_r$ be C^∞ sections of V. Assume that for every $x \in M$, the vector space $p^{-1}(x)$ is spanned by $s_1(x), s_2(x), ..., s_r(x)$.*

Then for almost all $(a_1, ..., a_r) \in \mathbb{R}^r$ the section $a_1s_1 + a_2s_2 + ...a_rs_r$ of V meets the zero-section of V transversally. In particular, for such $(a_1, ..., a_r)$ the set

$$Z(a_1, ..., a_r) = \{x \in M : a_1s_1(x) + ...a_rs_r(x) = 0\}$$

is a C^∞ submanifold of M.

Proof. Define $A : M \times \mathbb{R}^r \to V$ by $A(x, a_1, ..., a_r) = a_1s_1(x) + ...a_rs_r(x)$.
Step 1: Show A is a submersion. In particular, A is transverse to the zero section $0_V(M)$ of V. Its inverse image $Z = A^{-1}0_V(M)$ is therefore a C^∞ sbmanifold of V. Let $q = p_2 \circ i$ where $i : Z \hookrightarrow M \times \mathbb{R}^r$ denotes the inclusion and $p_2 : M \times \mathbb{R}^r \to \mathbb{R}^r$ is the projection. Note that $q^{-1}(a_1, ..., a_r) = Z(a_1, ..., a_r)$ for all $(a_1, ..., a_r)$ in $\mathbb{R}^r$.
Step 2: Prove that $(a_1s_1 + ... + a_rs_r) : M \to V$ is transverse to $0_V(M)$ iff $(a_1, ..., a_r)$ is a regular value of q (an application of the frequently employed lemma below). Now appeal to Sard's theorem.

□

Lemma 8.5. *Let $p : P \to S$ be a submersion and let $f : P \to N$ be transverse to a C^∞ submanifold $W \subset N$. Let $B = f^{-1}(W)$. Let $q = p|_B : B \to S$. Now let $s \in S$. Then*

s is a regular value of q iff $f|_{p^{-1}(s)} : p^{-1}(s) \to N$ is transverse to W.

We shall discuss next perturbations/deformations of a given C^∞ map $f_0 : X \to Y$. The definition is given below. The properties (a,b,c) listed in 8.6 below ensure that there exist f_s with desirable properties (see 8.7, 8.9 and 8.10).

Definition 8.6. $X, Y, f_0 : X \to Y$ are all C^∞. A *deformation* of f_0 is a C^∞ map $F : X \times S \to Y$ where S is a nbhd of 0 in $\mathbb{R}^k$ and $F(x, 0) = f_0(x)$ for all $x \in X$. For $s \in S$, the C^∞ map $f_s : X \to Y$ is then given by $f_s(x) = F(x, s)$ for all $x \in X$.

There are three conditions on F that we are concerned with:
(a) the linear transformation $F'(x, 0) : T_{(x,0)}(X \times S) \to T_{f_0(x)}Y$ is onto for all $x \in X$

Define $\widetilde{F} : T'X \times S \to TY$ by $\widetilde{F}(v, s) = T(f_s)v$ for all $v \in T'X, s \in S$ (see 7.3 for the notation $T'X$ and $T(f_s)$.
(b) $\widetilde{F}$ is transverse to the zero-section of TY on the subset $T'X \times \{0\}$. See 6.1 for notation.

Let $(X \times X)'$ be the complement of the diagonal ΔX in $X \times X$.

Define $G : (X \times X)' \times S \to Y \times Y$ by

$$G(x_1, x_2, s) = (f_s(x_1), f_s(x_2))\forall(x_1, x_2) \in (X \times X)', \forall s \in S.$$

(c) G is transverse to ΔY on the subset $(X \times X)' \times \{0\}$.

The proposition below will lead to a proof of Thom's transversality lemma.

Proposition 8.7. *Assume that X is compact and that F satisfies 8.6(a). Let $A \subset Y$ be a closed C^∞ submanifold. Then for almost all $s \in S'$, where S' is a nbhd of zero in S, the map $f_s : X \to Y$ is transverse to A.*

Proof. $X \times \{0\}$ is contained in the open subset $\{(x,s) \in X \times S :$ the linear transformation $F'(x,s)$ is onto $\}$ of $X \times S$. The compactness of X assures us that this open subset contains $X \times S'$ where S' is a nbhd of 0 in S. It will simplify notation to replace the given (S,F) by $(S', F|_{X\times S'})$. We proceed doing so. Thus we now assume that $F'(x,s)$ is surjective for all $(x,s) \in X \times S$. In other words, F is a submersion. It follows that F is transverse to every C^∞ submanifold of Y, in particular to A. Thus $F^{-1}(A)$ is a C^∞ submanifold of $X \times S$. By Sard's theorem, the set of regular value s of the projection $F^{-1}(A) \to S$ is the complement of a set of measure zero. By lemma 8.5 the set of regular values of that projection coincides with set of $s \in S$ for which $f_s : X \to Y$ is transverse to A.

□

Definition 8.8. If V is a vector space over a field F, its projective space $\mathbb{P}(V)$ is the set of one-dimensional linear subspaces of V.

Let $p : V \to Z$ be a C^∞ vector bundle on a C^∞ manifold. Every $z \in Z$ gives rise to a real vector space $p^{-1}(z)$ and therefore to a real projective space $\mathbb{P}(p^{-1}(z))$ as well. The disjoint union of these projective spaces, indexed by $z \in Z$, is the set $\mathbb{P}(V)$. Let V' denote the complement in V of its zero section $0_V(Z)$. There is a surjective map $q : V' \to \mathbb{P}(V)$ given as follows. For $z \in Z$ and $0 \neq v \in p^{-1}(z)$, we define $q(v)$ to be the one-dimensional subspace $\mathbb{R}v$ of $p^{-1}(z)$.

Both the topology and C^∞ structure on $\mathbb{P}(V)$ are dictated by $q : V' \to \mathbb{P}(V)$.

The proposition below will lead to a proof of Whitney's immersion theorem.

Proposition 8.9. *Assume that X is compact and that F satisfies 8.6(b). Then the collection S^0 of $s \in S$ for which $Tf_s|_{T'X} : T'X \to TY$ is transverse to the zero section of TY contains the complement of a set of measure zero in a nbhd of 0 in S.*

For $s \in S^0$, the closed subset $K_s = \{(x,L) : x \in X, L \in \mathbb{P}(T_xX), f'(x)L = 0\}$ of $\mathbb{P}(TX)$ is a C^∞ submanifold whose dimension is $2\dim(X) - \dim(Y) - 1$.

In particular, f_s is an immersion for $s \in S^0$ when $2\dim(X) \leq \dim(Y)$

Proof. $B = \{(v,s) \in T'X \times S : \widetilde{F}$ is not transverse to the zero section of TY at $(v,s)\}$ is a closed subset of $T'X \times S$ (see 6.1). Now $\widetilde{F}(tv,s) = t\widetilde{F}(v,s)$ for all $t \in \mathbb{R}, (v,s) \in T'X \times S$. It follows that there is a closed subset $\overline{B} \subset \mathbb{P}(TX) \times S$ such that $B = \widetilde{q}^{-1}\overline{B}$ where $\widetilde{q} = q \times id_S : T'X \times S \to \mathbb{P}(TX) \times S$ with $q : T'X \to \mathbb{P}(TX)$ as in 8.8.

The assumption 8.6(b) says that $B \cap T'X \times \{0\}$ is empty. It follows that $\overline{B} \cap \mathbb{P}(TX) \times \{0\}$ is empty. Furthermore $\mathbb{P}(TX)$ is compact (because $\mathbb{P}(TX) \to X$ is proper and X is compact). Replacing S by a suitable nbhd of 0 in S we may assume that $\overline{B}$ is empty, in other words that $\widetilde{F}$ is transverse to the zero section $0_{TY}(Y)$ of TY.

We take S^0 to be the set of regular values of C^∞ map $\widetilde{F}^{-1}0_{TY}(Y) \to S$. The first assertion of the proposition now follows from Sard's theorem and 8.5 as before. For $s \in S^0$ we see that $\widetilde{K}_s = f_s^{-1}0_{TY}(Y)$ is a closed C^∞ submanifold of $T'Y$ whose dimension is $\dim T'X - (\dim TY - \dim 0_{TY}Y)$ which is $2\dim(X) - \dim(Y)$. Because $q : T'X \to \mathbb{P}(TX)$ is a submersion with one-dimensional fibers and $\widetilde{K}_s = q^{-1}K_s$

the second assertion of the proposition follows. That assertion also shows that K_s is empty for $s \in S^0$ if "$\dim K_s < 0$" and that is exactly the same as saying that $f_s : X \to Y$ is an immersion. This completes the proof. □

The proposition below will lead to a proof of the Whitney embedding theorem. But if this is the only application we are concerned with, the proof could be much shorter (see 8.14 below).

Proposition 8.10. *Assume that X is compact and that the deformation F of f_0 satisfies 8.6(b) and (c). Let S^{00} be the collection of $s \in S$ that satisfies*

(1) *$f_s|_{T'X} : T'X \to TY$ is transverse to the zero section of TY*
(2) *$g_s : (X \times X)' \to Y \times Y$ given by $g_s(x_1, x_2) = (f_s(x_1), f_s(x_2))$ is transverse to ΔY.*

Then S^{00} contains the complement of a set of measure zero in a nbhd of 0 in S.
If $2\dim(X) < \dim(Y)$ then f_s is an embedding for all $s \in S^{00}$.

Proof. As in the proof of 8.9 we may assume that $\widetilde{F}$ is transverse to the zero section of TY. The set $C = \{(x, x', s) \in (X \times X)' \times S : G$ is not transverse to ΔY at $(x, x', s)\}$ is a closed subset of $(X \times X)' \times S$ (as has been remarked in 6.1).
Claim: *C is a closed subset of $X \times X \times S$.*

We assume the claim and proceed. 8.6(c) says that $C \cap X \times X \times \{0\}$ is empty. By the compactness of $X \times X$ we may shrink S once again to a smaller nbhd of 0 and now assume that C is empty. It follows that $M = G^{-1}\Delta Y$ is a C^∞ submanifold of $(X \times X)' \times S$. We intersect the set of regular values of $\widetilde{F}^{-1}(0_{TY}Y) \to S$ with the set of regular values of $M \to S$ and observe (appealing to 8.5 again) that all s in this intersection are members of S^{00}. This proves the first assertion of the proposition. Now for the second assertion. By 8.9 we know f_s is an immersion for $s \in S^{00}$. That f_s is one-to-one follows from $\dim g_s^{-1}(\Delta(Y)) < 0$.

We now address the claim. The statement is local in nature, and we may therefore assume that X and Y are open subsets of vector spaces V and W respectively. We have natural identifications $TX \cong X \times V$ and $TY \cong Y \times W$ (and we simplify notation by replacing these $\cong$ by $=$). Let $V' = V \setminus \{0\}$.

Define $A : X \times V' \times S \to W$ by $A(x, v, s) = f_s'(x)$. The assumption that $\widetilde{F}$ is transverse to the zero section of TY is clearly equivalent to the assumption that 0 is a regular value of the C^∞ map A.

Now consider the open subset $\Omega = \{(x, v, t) \in X \times V' \times \mathbb{R} : x + tv \in X\}$. The function $(x, v, t, s) \mapsto F(x + tv, s) - F(x, s)$ defined on $\Omega \times S$ vanishes when $t = 0$ and by 1.8 we see that it equals tR where $R : \Omega \times S \to W$ is C^∞. We see that $R|_{t=0} = A$ from the definition of derivative. Thus our assumption is that 0 is a regular value of $R|_{t=0}$. It follows that for compact subsets $K_1 \subset X, K_s \subset V', K_3 \subset S$ there is a nbhd I of 0 in $\mathbb{R}$ such that $K_1 \times K_2 \times I$ is contained in Ω and $R|_{t=h}$ is transverse to $0 \in W$ on the subset $K_1 \times K_2 \times K_3$ for all $h \in I$. In other words,

$$(x, v, s) \mapsto \frac{F(x + hv, s) - F(x, s)}{h}$$

is transverse to $0 \in W$ at all $(x, v, s) \in K_1 \times K_2 \times K_3$ for all $h \in I$.

It follows that $(x, v) \mapsto F(x + v, s) - F(x, s)$ is transverse to $0 \in W$ for all $x \in K_1, v \in \{h \in I : h > 0\}K_2, s \in K_3$. We take K_2 to be a sphere centered at zero in

V. We deduce that $\{h \in I : h > 0\}K_2$ contains a punctured nbhd U^* of zero in V. It follows that if $H(x, x', s) = F(x', s) - F(x, s)$ then H is transverse to $0 \in W$ at all (x, x', s) when $s \in K_3, x \in K_1, x' - x \in U^*$. But C is simply the subset at which H is not transverse to the submanifold $\{0\}$. We have now shown the closure of C does not intersect $\Delta K_1 \times K_3$. It follows that no point of $\Delta X \times S$ is in the closure of C. This proves the claim, and therefore completes the proof of the proposition as well.

□

Problem 8.11. Let X be a locally closed C^∞ submanifold of a finite dimensional vector space V, We have $V' = V \setminus \{0\}$ and $q : V' \to \mathbb{P}(V)$. (a) Prove that the closure of $\{(x, x', q(x' - x)) : x, x' \in X, x \neq x'\}$ in $X \times X \times \mathbb{P}(V)$ is C^∞ submanifold B (referred to as the blow-up of $X \times X$ along the diagonal). (b) Prove that B has an open subset U that is diffeomorphic to $(X \times X)'$. (c) Prove that the complement of U in B is a closed C^∞-submanifold that is diffeomorphic to $\mathbb{P}(TX)$.

This construction may be employed (i) to make the proof of the above proposition more conceptual and (ii) also to show that S^{00} is open.

Problem 8.12. Let $f_0 : S^1 \to \mathbb{R}^2$ be the inclusion. Show that there is a sequence of C^∞ maps $f_n : S^1 \to \mathbb{R}^2$, none of which is an embedding, but which converge uniformly to f_0.

Problem 8.13. Parametrise the circle by $(\cos(\theta), \sin(\theta))$. Let $f_n : S^1 \to \mathbb{R}^2$ be a sequence of maps that converges uniformly to a C^∞ map $f_0 : S^1 \to \mathbb{R}^2$. Assume furthermore that $\frac{\partial}{\partial\theta} f_n$ converges uniformly to $\frac{\partial}{\partial\theta} f_0$. Show that if f_0 is an embedding, then the f_n are embeddings as well for all sufficiently large n.

Problem 8.14. In proposition 8.10 assume that $f_0 : X \to Y$ is an immersion. An earlier hw problem shows that by shrinking S we may assume that f_s is an immersion for all $s \in S$. Now give a self-evident proof of the claim in the proof of 8.10. Next deduce Whitney's embedding theorem as stated in thm. 8.17 from the immersion thm.

Lemma 8.15. *If $X, Y, f_0 : X \to Y$ are all C^∞ there exist deformations F of f_0 that satisfy 8.6 (a), (b) and (c).*

Remark 8.16. The proof given below is for X is compact. We will also rely on 7.6 which in turn depends on embedding Y as a locally closed C^∞ submanifold of Euclidean space. Thus the proof given in these notes (so far) is complete only when (a) Y is compact or Y is open in Euclidean space and (b) X is compact.

Proof. We will assume that (i) X is a compact C^∞ submanifold of a finite dimensional real vector space V and (ii) we have i, r, U as in 7.6 with U an open subset of a finite dimensional vector space W.

Hom(V, W) is the space of linear transformations from V to W. We define

$$D : X \times W \times \mathrm{Hom}(V, W) \to W \text{ by } D(x, w, L) = i(f_0(x)) + w + L(x)$$

forall $x \in X, w \in W, L \in \mathrm{Hom}(V, W)$.

$$D(x, 0, 0) = i(f_0(x)) \in U \text{ for all } x \in X. \quad (3)$$

By the compactness of X we obtain a nbhd S of 0 in $W \times \mathrm{Hom}(V, W)$ such that $D(X \times S)$ is contained in U. We define F to be $r \circ D|_{X \times S}$. We may assume that

$S = S' \times S''$ where S' and S'' are nbhds of 0 in W and $\mathrm{Hom}(V, W)$ respectively. Equation (3) and the identity $r \circ i = id_Y$ show that F is a deformation of f_0.

Fix some $x \in X$. Put $y = f_0(x)$. Then consider the map $S' \to Y$ given by $w \mapsto F(x, w, 0)$. The derivative of this map at $0 \in S'$ is the linear transformation $w \mapsto r'(y)w$ which is a surjection $W \to T_yY$. This proves that F satisfies (a).

We now check that F satisfies 8.6(c). With $x \in X$ and $y = f_0(x)$ as above, consider next the map $S'' \to Y$ given by $L \mapsto F(x, 0, L)$. Its derivative at $0 \in S''$ is the linear transformation $L \mapsto r'(y)L(x)$. Now suppose $x' \in X$ also satisfies $f_0(x') = y$. Assume that $x' \neq x$. It follows that the derivative of $S'' \to Y \times Y$ given by $L \mapsto G(x, x', 0, L)$ at $0 \in S''$ is the linear transformation $\mathrm{Hom}(V, W) \to T_yY \times T_yY$ gievn by the formula $L \mapsto (r'(y)L(x), r'(y)L(x'))$. Because $0 \neq x - x' \in V$ we see that $L \mapsto r'(y)L(x) - r'(y)L(x') = r'(y)L(x - x')$ is surjective. This implies that G is transverse to ΔY at $(x, x', 0, 0)$. This proves that F satisfies (c).

We now come to (b). We have $D : X \times S \to U$ and therefore the map $\widetilde{D} : T'X \times S \to T(U)$ defined in an analogous manner. We fix $x \in X, 0 \neq u \in T_xX$ and consider the map $E : S \to T(U)$ given by $(w, L) \mapsto \widetilde{D}(u, w, L)$. Now the tangent-spaces at all points of U are identified canonically with W, and this gives a natural identification $R : T(U) \to U \times W$. We see that

$$E(w, L) = (if_0(x) + w + L(x), (if_0)'(x)u + L(u))$$

This is a constant plus a linear transformation, so its derivative at any point is the linear transformation

$$(w, L) \mapsto (w + L(x), L(u))$$

which is clearly surjective. We have shown that $\widetilde{D}$ is a submersion.

Now $\widetilde{F} = T(r) \circ \widetilde{D}$ (see 7.3 for the definition of $T(r)$). Now $r \circ i = id_Y$ implies that $T(r) \circ T(i)$ is the identity on TY. It follows that $\widetilde{F}$ induces a surjection on tangent-spaces at $T'X \times \{0\}$ which more than checks the desired transversality condition of (b).

□

The above lemma and the three preceeding propositions now finish the proofs of the following three theorems.

Theorem 8.17. *Let $X, Y, f_0 : X \to Y$ be C^∞. Assume X is compact. Let A be a closed C^∞ submanifold of Y. Let $\epsilon > 0$. Then there exists a C^∞ map $f : X \to Y$ with $d(f_0(x), f(x)) < \epsilon$ for all $x \in X$ with the property*

1. *(the transversality lemma) f is transverse to A*
2. *(Whitney's immersion theorem) f is an immersion if if* $2\dim(X) \leq \dim(Y)$
3. *(Whitney's embedding theorem) f is an embedding if* $2\dim(X) < \dim(Y)$.

Remark 8.18. There are formulations with "approximation" replaced by homotopy. The precise relation between these concepts when the domain is compact is a hw 3 problem.

Problem 8.19. A closed subset A of a C^∞ manifold Y has the property that there is an increasing sequence of closed subsets $\emptyset = A_{-1} \subset A_0 \subset ... \subset A_r = A$ such that for all $0 \leq i \leq r$, $A_i \setminus A_{i-1}$ is C^∞ submanifold of dimension i of $Y \setminus A_{i-1}$. Let X be a compact C^∞ manifold. Assume that $\dim(X) + r < \dim(Y)$. Prove that every map

C^∞ map $f : X \to Y$, where X is a compact C^∞ manifold of is homotopic to a map $g : X \to Y$ such that $g(X) \cap A$ is empty.

Under the weaker assumption $\dim(X) + r \leq \dim(Y)$ prove that there is a g such that $g(X) \cap A$ is a finite set.

9. DEGREE

[12, 8, 5] are excellent references.

A very nice self-contained account of Duality is given in [4].

We will need the following theorem(see[4],[16])

Theorem 9.1. *If X is a compact connected (non-empty) orientable n-manifold, then* $\mathrm{H}_n(X) \cong \mathbb{Z}$. *For every $x \in X$, the natural arrow* $\mathrm{H}_n(X) \to \mathrm{H}_n(X, X \setminus \{x\})$ *is an isomorphism.*

If one works with singular homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients, then the above statements are true even after dropping the hypothesis of orientability.

Definition 9.2. An *orientation* θ_X of a (compact connected orientable nonempty) X is a generator of $H_n(X)$.

Let $x \in X$. The image of θ_X in $\mathrm{H}_n(X, X \setminus \{x\})$ is denoted by $\theta_X(x)$. Now let U be a nbhd of $x \in X$. We obtain a unique $\theta_U(x) \in \mathrm{H}_n(U, U \setminus \{x\})$ that is taken to $\theta_X(x)$ under the arrow $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ (recall that this arrow is an isomorphism by excision).

Definition 9.3. Let $f : X \to Y$ be a continuous map where X, Y are compact connected nonempty oriented n-manifolds. $\deg(f)$ is the unique integer such that $\mathrm{H}_n(f)\theta_X = \deg(f)\theta_Y$.

Let $x \in X$. Assume that x is an isolated point of the closed set $f^{-1}f(x)$. We then obtain nbhds U_x of x in X and U_y of $y = f(x)$ in Y such that $f(U_x) \subset U_y$ and $f(U_x \setminus \{x\}) \subset U_y \setminus \{y\}$. We define $\deg_x f$ to be the unique integer such that $\theta_{U_x} x \mapsto (\deg_x f)\theta_{U_y} y$ under $\mathrm{H}_n(U_x, U_x \setminus \{x\}) \to \mathrm{H}_n(U_y, U_y \setminus \{y\})$.

It is easy to check that $\deg_x f$ is in fact independent of the choice of U_x, U_y.

If the hypothesis of orientability is dropped, then $\deg(f)$ and $\deg_x f$ are still well defined in $\mathbb{Z}/2\mathbb{Z}$.

Theorem 9.4. *Let $f : X \to Y$ be a continous map, where both (X, θ_X) and (Y, θ_Y) are compact connected nonempty oriented manfolds. Assume that $y \in Y$ has the property that $f^{-1}(y)$ is finite. Then*

$$\deg(f) = \Sigma\{\deg_x f : x \in f^{-1}(y)\}$$

Proof. Put $S = f^{-1}(y)$. First show that there is a nbhd U_y of y whose inverse image is the disjoint union pairwise disjoint nbhds U_x of x for all $x \in S$. Let $U = \sqcup_{x \in S} U_x$.

Consider

(1) $X \xrightarrow{f} Y \to (Y, Y \setminus \{y\})$
(2) $X \to (X, X \setminus S) \xrightarrow{f} (Y, Y \setminus \{y\})$
(3) $(U, U \setminus S) \to (X, X \setminus S) \xrightarrow{f} (Y, Y \setminus \{y\})$
(4) $(U, U \setminus S) \xrightarrow{f} (U_y, U_y \setminus \{y\}) \to (Y, Y \setminus \{y\})$
(5) $(U_x, U_x \setminus \{x\}) \xrightarrow{i(x)} (U, U \setminus S) \xrightarrow{f} (U_y, U_y \setminus \{y\})$

We are concerned with the effect of the above arrows on n-th relative homology.

The effect of the arrows in (1) on θ_X is:

$\theta_X \mapsto \deg(f)\theta_Y \mapsto \deg(f)\theta_Y(y)$.

Now the composite of the arrows in (1) and (2) is the same. Define $\theta_X S$ by $\theta_X \mapsto \theta_X S$ under $H_n(X) \to \mathrm{H}_n(X, X\setminus S)$ We see therefore that $f : (X, X\setminus S) \to (Y, Y\setminus\{y\})$ on n-the relative homology takes $\theta_X S$ to $\deg(f)\theta_Y(y)$.

By excision, $(U, U \setminus S) \to (X, X \setminus S)$ induces an isomorphism on all homologies. Thus we get $\theta_U(S) \in \mathrm{H}_n(U, U \setminus S)$ that is taken to $\theta_X(S)$ under $\mathrm{H}_n(U, U \setminus S) \to \mathrm{H}_n(X, X \setminus S)$. We now see that the composite of the arrows in (3) takes $\theta_U(S)$ to $\deg(f)\theta_Y(y)$. The composite of the arrow in (3) is no different from the composite of the arrows in (4). By the definition of $\theta_{U_y}(y)$ we deduce that $\theta_U(S) \mapsto \deg(f)\theta_{U_y}y$ under the map $(U, U \setminus S) \xrightarrow{f} (U_y, U_y \setminus \{y\})$.

By the defintion of $\deg_x f$, the composite of the arrows in (5) takes $\theta_{U_x}x$ to $(\deg_x f)\theta_{U_y}y$. Thus the theorem is a consequence of the claim:

$$\sum_{x\in S} i(x)\theta_{U_x}x = \theta_U(S)$$

with the $i(x)$ as in (5).

We now verify the above claim. Consider the sequence of arrows below.

$$\bigoplus_{x\in S} \mathrm{H}_n(U_x, U_x \setminus \{x\}) \to \mathrm{H}_n(U, U \setminus S) \to \mathrm{H}_n(X, X \setminus S) \to \bigoplus_{x\in S} \mathrm{H}_n(X, X \setminus \{x\})$$

The first is evidently an isomorphism. the second is an isomorphism by excision. The composite is given by a $S \times S$ matrix. The diagonal entries of this matrix are the isomorphisms

$$\mathrm{H}_n(U_x, U_x \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$$

for all $x \in S$. The off-diagonal entries are

$$\mathrm{H}_n(U_x, U_x \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x'\})$$

for $x \neq x'; x, x' \in S$. Because U_x is contained in $X \setminus \{x'\}$ these off-diagonal entries are zero. It follows that the composite of the second and third arrow is an isomorphism, and this composite takes both

$$\sum_{x\in S} i(x)\theta_{U_x}x \text{ and } \theta_U(S)$$

to

$$\oplus_{x\in S}\theta_X(x) \in \oplus_{x\in S}\mathrm{H}_n(X, X \setminus \{x\})$$

This shows the equality of those two elements that crop up in the claim. □

Remark 9.5. Now assume that the X, Y, f in the theorem are all C^∞. Let y be a regular value of f. Then $f^{-1}(y)$ is discrete, being a 0-submanifold, and compact, and is therefore finite. Wth notation as above the $U_x \to U_y$ are diffeomorphsms and therefore the $\deg_x f \in \{\pm 1\}$.

Problem 9.6. Let X and Y be compact oriented surfaces of genus g_1 and g_2 respectively. Suppose $f : X \to Y$ is a map with $\deg(f) \neq 0$. Show that $g_1 \geq g_2$. Hint: Consider the ring homomorphism f^* on cohomology with rational coefficients.

Problem 9.7. Let $P \in \mathbb{C}[X]$. Then $z \mapsto P(z)$ gives a map $\mathbb{C} \to \mathbb{C}$ which extends to a continuous map $P^* : \mathbb{C} \cup \infty \to \mathbb{C} \cup \infty$ of one-point-compactifications. Show that $\deg_a(P^*)$ is the highest power of $(X-a)$ that divides $P(X) - P(a)$ in the ring $\mathbb{C}[X]$ for every $a \in \mathbb{C}$. What is $\deg \infty(P^*)$?

With $P \in \mathbb{R}[X]$ and $P^* : \mathbb{R} \cup \infty \to \mathbb{R} \cup \infty$ figure out a formula for $\deg_a(f)$ for all $a \in \mathbb{R} \cup \infty$.

Problem 9.8. Let $n > 1$. Fix a point $v_0 \in S^{n-1} = \{x \in \mathbb{R}^n :< x, x >= 1\}$. For every $v \in S^{n-1}$ that is orthogonal to v_0, define $g_v : \mathbb{C} \to \mathbb{R}^n$ by $g_v(a+ib) = av_0 + bv$ for all $a, b \in \mathbb{R}$.

Show there is a unique map $f : S^{n-1} \to S^{n-1}$ such that $f(g_v(z)) = g_v(z^2)$ for all $z \in \mathbb{C} : |z| = 1$ and for all $v \in S^{n-1}$" $< v, v_0 >= 0$.

Prove that $\deg(f) = 1 + (-1)^n$.

10. Direct sums, tensor products, exterior products, duals, etc of vector bundles

Any construction of a 'new vector space from some old ones' has an analogue for vector bundles. For example, given vector bundles $p_i : V_1 \to B$ and $p_2 : V_2 \to B$ there is a vector bundle $p : V_1 \otimes V_2$ on B whose fiber $p^{-1}(x)$ over $x \in B$ is canonically identified with $p_1^{-1}(x) \otimes p_2^{-1}(x)$. If the V_i are C^∞ bundles then so s $V_1 \otimes V_2$.

One may construct such things by (a) patching, (b) principal bundles and associated fiber spaces, (c) sheaves. These notes have a brief version of (b). Both (b) and (c) work in great generality (beyond C^∞ situations).

See [17, 9] for the definition below:

Definition 10.1. a *principal G-bundle on a space B* consists of the data

(1) a map $\pi : P \to B$
(2) a right G-action on P such that
    (a) $\pi(xg) = \pi(x)$ for all $x \in P$
    (b) X is covered by its open subsets U for which there is a homeomorphism $f : U \times G \to \pi^{-1}(U)$ such that

$$\pi f(z, g) = z \text{ and } f(z, hg) = f(z, h)g \text{ for all } z \in U; h, g \in G$$

Given an action of G on a space F the *associated fiber space* is the quotient of $P \times F$ by the equivalence relation $(zg, y) \equiv (z, gy)$ for all $z \in P, g \in G, y \in F$. The associated fiber space is denoted $P \times_G F$. The composite $P \times F \xrightarrow{p_1} P \xrightarrow{\pi} B$ factors through $P \times_G F \to B$. It is easily checked that $P \times_G F \to B$ is a fiber bundle with fiber F.

Of special interest are representations $\rho : G \to \mathrm{GL}_m(\mathbb{R})$. Such a ρ gives an action of G on $\mathbb{R}^m$. The associated fiber space $P \times_G \mathbb{R}^m$ is then a vector bundle of rank m on B.

Example 10.2. See [**?**, 10]. A real rank k vector bundle $p : V \to B$ gives rise to its bundle of frames: P is the open subset of the k-fold fiber product

$$V_B^k = V \times_B V \times_B V \times ... \times_B V$$

consisting of those $(v_1, v_2, ..., v_k)$ which form a basis for the vector space $p^{-1}(x)$. We then have $\pi(v_1, ..., v_k) = p(v_1)$ which gives $\pi : P \to B$.

Next notice that for all $x \in B$, every point $(v_1, ..., v_k)$ of $\pi^{-1}(x)$ gives rise to an isomorphism $\phi : \mathbb{R}^k \to p^{-1}(x)$, namely $\phi(a_1, a_2, ..., a_k) = a_1 v_1 + a_2 v_2 + ... + a_k v_k$. This leads to the description of P as $\sqcup_{x \in B} \mathrm{Iso}(\mathbb{R}^k, p^{-1}(x))$. Now given $\phi \in \mathrm{Iso}(\mathbb{R}^k, p^{-1}(x))$ and $g \in \mathrm{GL}_k(\mathbb{R})$ we get $\phi \circ g \in \mathrm{Iso}(\mathbb{R}^k, p^{-1}(x))$. This gives the right action of $\mathrm{GL}_k(\mathbb{R})$ on P.

Remark 10.3. These constructions make it possible to construct new bundles from old ones in the following manner. Given a real rank k vector bundle $p : V \to B$ we consider its bundle of frames $\pi : P \to B$; this is a principal $\mathrm{GL}_k(\mathbb{R})$-bundle. The fiber space associated to a representation $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$ is a real rank m bundle on B. Let us denote this bundle by ${}^{\rho}V$. When $m = k$ and $\rho = id$ then ${}^{\rho}V$ is canonically identified with V.

To obtain the dual bundle V^*, we take $\rho(A) = ({}^t A)^{-1}$, for all $A \in \mathrm{GL}_k(\mathbb{R})$.

To obtain $\Lambda^2(V)$ we consider $\rho : \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}(\Lambda^2(\mathbb{R}^k))$ given by $\rho(A) = A \Lambda A$.

Given vector bundles $p_i : V_i \to B$ for $i = 1, 2$, we may construct $V_1 \otimes V_2$ as follows. We have the frame bundles $\pi : i : P_i \to B$ of the vector bundles $p_i : V_i \to B$. Denoting the fiber prodct $P_1 \times_B P_2$ by P we see that P is a principal G-bundle on B, where $G = \mathrm{GL}_{k_1}(\mathbb{R}) \times \mathrm{GL}_{k_2}(\mathbb{R})$. We consider the representation $\rho : G \to \mathrm{GL}(\mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2})$ given by $(S, T) \mapsto S \otimes T$.

Remark 10.4. Let B be a C^∞ manifold. We have the sheaf of $\mathbb{R}$-algebras C_B^∞ on B. It has already been remarked that $\Gamma(B, V)$ is a module over the ring of C^∞ functions on B. The presheaf defined by $U \mapsto \Gamma(U, V|_U)$ for all open subsets U of B is in reality a locally free sheaf of modules over C_B^∞. Let us denote this sheaf by $C^\infty(V)$.

One shows that $V \mapsto C^\infty(V)$ establishes an equivalence of the category of C^∞ bundles on B and the category of locally free sheaves of C_B^∞-modules.

11. Fundamental groupoid

12. Thom isomorphism

Given a fiber bundle $E \to B$ with fiber F the cohomology of E can be computed from the cohomology of B, that of F, and plenty of other information recorded by the Leray spectral sequence.

The Thom isomorphism is a special case, and works better with sheaf cohomology. It will be stated first in this form. The version **??** for singular cohomology is stated later.

The data is a real rank k bundle $p : V \to B$, as before V' is the complement of the the zero section $0_V(B)$. The Thom isomorphism for sheaf cohomology states

There is a natural isomorphism $\mathrm{H}^i(B, \mathrm{Or}_V) \cong \mathrm{H}^{i+k}_{0_V(B)}(V)$

Identification of the above groups with those arising from singular cohomology requires some hypotheses on B.

* *We assume that the collections of contractible open subsets forms a basis for the topology of B.*

For an open subset U of B, denote the pair $(p^{-1}(U), p^{-1}(U) \cap V')$ by (V_U, V'_U). To each such U, we associate the Abelian group $\mathrm{H}^k(V_U, V'_U)$. Cohomology is a contravariant functor, and so we see that this defines a presheaf on B. Its sheafification is the Orientation sheaf Or_V. The hypothesis * on B ensures that all the stalks of this sheaf are isomorphic to $\mathbb{Z}$ (why?) An *orientation* of V is a global section s of Or_V

such that $s(x)$ is a *generator* the stalk of Or_B at x for all $x \in B$. If an orientation exists, then V is said to be orientable. It is clear that V is orientable if and only if its restriction to each connected component of B is orientable. If B is connected and s is an orientation, then the only other orientation is $(-s)$.

The term $\mathrm{H}^{i+k}_{0_V(B)}(V)$ is simply $\mathrm{H}^{i+k}(V, V')$. In view of the fact that cohomology(either sheaf cohomology or singular cohomology) with values in locally constant sheaves has still be defined, the Thom isomorphism will be stated below in a weaker form which is adequate when V is orientable.

Theorem 12.1. (1) $\mathrm{H}^i(V, V') = 0$ *for all* $i < k$ *(where* k *is the rank of* V*).*

(2) *The natural arrow* $H^k(V, V') \to \Gamma(B, \mathrm{Or}_V)$ *is an isomorphism.*

(3) *Let* $s \in \Gamma(B, \mathrm{Or}_V)$ *be an orientation. By the above statement* s *has a pre-image* θ *in* $\mathrm{H}^k(V, V')$.
$a \in \mathrm{H}^i(B) \mapsto p^*a.\theta \in \mathrm{H}^{i+k}(V, V')$ *yields an isomorphism* $\mathrm{H}^i(B) \to \mathrm{H}^{i+k}(V, V')$.

Some clarification of the notation used in part (3). p^* is a ring homomorphism from the cohomology of B to the cohomology of V. The cohomology of (V, V') is a module over the cohomology of V. A proof of this theorem appears soon after the Leray-Hirsch theorem in ch.5 of [16]. A part of it is sketched below.

Proof. The idea is to prove the statement not just for $p : V \to B$ but also for $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ for all open subsets U of B.

$\mathrm{H}^m(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ is isomorphic to $\mathbb{Z}$ when $m = k$, and is zero for $m \neq k$. The Kunneth formula shows that the theorem is true for the vector bundle $p_1 : U \times \mathbb{R}^k \to U$.

Let $\mathcal{U}$ be the collection of open subsets U of B such that $V|_U$ is isomorphic to the trivial bundle. Thus the theorem is proved for all $U \in \mathcal{U}$. Let $\mathcal{U}_m = \{U_1 \cup ... \cup U_m : U_i \in \mathcal{U} \forall i\}$. Next we will prove the theorem for $U \in \mathcal{U}_m$ by induction on m. For this it will be helpful to recall the form of the Mayer Vietoris sequence below for open sets $J, K \subset B$, with $I = J \cap K$ and $U = J \cup K$

$$\mathrm{H}^{i+k-1}(V_I, V'_I) \to \mathrm{H}^{i+k}(V_U, V'_U) \to \mathrm{H}^{i+k}(V_J, V'_J) \oplus \mathrm{H}^{i+k}(V_K, V'_K) \to \mathrm{H}^{i+k}(V_I, V'_I) \to$$

Assume now that the first two assertions of the theorem have been proved for all $u \in \mathcal{U}_m$ for some $m \geq 1$. Let $U \in \mathcal{U}_{m+1}$. Then $U = J \cup K$ with $J \in \mathcal{U}_m$ and $K \in \mathcal{U}$. Then $I = J \cap K$ is of course in $\mathcal{U}$. Let $q < k$.Then the terms adjacent to $\mathrm{H}^q(V_U, V'_U)$ in the above sequence vanish, and so that proves the first assertion for U. For the second assertion, we note that $i = 0$ in the above sequence gives the left exact seqeunce

$$0 \to \mathrm{H}^k(V_U, V'_U) \to \mathrm{H}^k(V_J, V'_J) \oplus \mathrm{H}^k(V_K, V'_K) \to \mathrm{H}^k(V_I, V'_I)$$

which sits right above the left exact sequence(meaning there is a commutative diagram of left exact sequences with downward vertical arrows)

$$0 \to \Gamma(U, \mathrm{Or}_V) \to \Gamma(J, \mathrm{Or}_V) \oplus \Gamma(K, \mathrm{Or}_V) \to \Gamma(I, \mathrm{Or}_V)$$

The vertical arrows indexed by I,J,K are all isomorphisms, which implies that the vertical arrow for U is also an isomorphism (by the five-lemma).

For the passage to arbitrary open sets, see [16]. But in the situations we are concerned with, the proof is complete. Specifically if B is a manifold of dimension $(m-1)$ then every open subset belongs to $\mathcal{U}_m$ (precise reference to be found).

The assertion of (3) is also proved by induction on m.

□

13. Orientation and the first Stiefel Whitney class

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