Assignment 1: due Friday, October 5.

You may work on each home assignment jointly with another student (but no more than one, please). In that case, please indicate the name of that person at the beginning of your work.

Standard Notation and Definitions:

- $\mathbb{Z}$ denotes the ring of integers.
- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, denote the fields of rational, real, and complex numbers, respectively.
- Given a ring A, we write $\mathrm{M}_n(A)$ for the ring of $n \times n$-matrices with entries in A, resp. $\mathrm{GL}_n(A)$, for the group of invertible elements of the ring $\mathrm{M}_n(A)$.
- $\Bbbk$ always stands for a (nonzero) field. Given $\Bbbk$-vector spaces V, W, let $\mathrm{Hom}_{\Bbbk}(V, W)$ denote the vector space of linear maps $V \to W$.

Throughout the course, all rings are assumed to have a unit. A map $f : A \to B$, between two rings A and B, is called a *ring homomorphism* (or just 'morphism', for short) if $f(1_A) = 1_B$, and one has $f(a_1 + a_2) = f(a_1) + f(a_2)$ and $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$, for any $a_1, a_2 \in A$.

Definition. A vector space A, over a field $\Bbbk$, equipped with a $\Bbbk$-bilinear operation $\cdot : A \times A \to A$ is called a $\Bbbk$-*algebra* provided the operations '+' and '$\cdot$' make A a ring and, in addition, one has $1_{\Bbbk} \cdot a = a$, for any $a \in A$ (it suffices to require $1_{\Bbbk} \cdot 1_A = 1_A$). One defines $\Bbbk$-algebra *morphisms* as $\Bbbk$-*linear* ring morphisms.

A (not necessarily commutative) ring, resp. algebra, A is called a *division ring*, resp. *division algebra*, if any nonzero element of A is invertible.

Let V be an n-dimensional $\Bbbk$-vector space. The vector space $\mathrm{End}_{\Bbbk}V := \mathrm{Hom}_{\Bbbk}(V, V)$ has the natural $\Bbbk$-algebra structure, with multiplication operation given by composition of maps. We write $\mathrm{GL}(V)$ for the group of invertible linear operators and $\mathrm{SL}(V)$ for the subgroup of $\mathrm{GL}(V)$ formed by the operators with determinant 1. We will often identify $\mathrm{End}_{\Bbbk}(\Bbbk^n) \cong \mathrm{M}_n(\Bbbk)$ and $\mathrm{GL}(\Bbbk^n) \cong \mathrm{GL}_n(\Bbbk)$.

Let $(-,-) : V \times V \to \Bbbk$ be a *nondegenerate* symmetric bilinear form. Given a linear operator $a \in \mathrm{End}_{\Bbbk}V$ one defines the adjoint operator $a^* \in \mathrm{End}_{\Bbbk}V$ by the equation

$$(av_1, v_2) = (v_1, a^* v_2) \text{ for all } v_1, v_2 \in V. \tag{$\bigstar$}$$

The assignment $a \mapsto a^*$ gives an *anti-involution* of the ring $A = \mathrm{End}_{\Bbbk}V$, i.e. the following identities hold:

$$(1_A)^* = 1_A, \qquad (a^*)^* = a, \qquad (a \cdot b)^* = b^* \cdot a^*, \ \forall a, b \in A.$$

A linear map $a : V \to V$ is an isometry iff one has $aa^* = a^*a = 1$. Isometries are also called 'orthogonal transformations'. The set of isometries is a subgroup $\mathrm{O}(V) \subset \mathrm{GL}(V)$, called the *orthogonal group*. We will also use the group $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}(V)$. A linear operator $a \in \mathrm{End}_{\Bbbk}V$ is called *symmetric*, resp. *skew-symmetric*, if one has $a^* = a$, resp. $a^* = -a$.

Now, let $\Bbbk = \mathbb{C}$ and V be a $\mathbb{C}$-vector space. A *positive definite hermitian form* (also referred to as a 'hermitian inner product') on V is a map $(-,-) : V \times V \to \mathbb{C}$ which is additive with respect to each of the arguments and such that for any $\lambda, \mu \in \mathbb{C}$ and $v_1, v_2 \in V$ one has

$$(\lambda \cdot v_1,\ \mu \cdot v_2) = \lambda\bar{\mu} \cdot (v_1, v_2), \qquad (v_2, v_1) = \overline{(v_1, v_2)}, \qquad (v, v) \geq 0, \qquad (v, v) = 0 \ \Leftrightarrow \ v = 0.$$

It is known that any n-dimensional vector space V with a positive definite hermitian form admits an *orthonormal basis*, i.e., a basis $v_1, \ldots, v_n$ such that $(v_i, v_j) = \delta_{i,j}$ (=the Kronecker delta).

Given a linear operator $a \in \mathrm{End}_{\mathbb{C}}V$ one defines its (hermitian) adjoint $a^* \in \mathrm{End}_{\Bbbk}V$ by formula ($\bigstar$). In this setting, counterparts of orthogonal, resp. symmetric and skew-symmetric, operators are called *unitary*, resp. *hermitian* and *skew-hermitian*, operators. We write $\mathrm{U}(V)$ for the set of unitary operators and $\mathrm{SU}(V) := \mathrm{U}(V) \cap \mathrm{SL}(V)$ for the set of unitary operators with determinant 1.

Remark. The map $\mathrm{End}_{\mathbb{C}}V \to \mathrm{End}_{\mathbb{C}}V$, $a \mapsto a^*$, is *skew-linear*, i.e. for $\lambda \in \mathbb{C}$ and $a \in \mathrm{End}_{\Bbbk}V$ one has $(\lambda \cdot a)^* = \bar{\lambda} \cdot a^*$. Thus, hermitian, resp. skew-hermitian, operators form a *real* (but not complex) vector subspace of $\mathrm{End}_{\mathbb{C}}V$. An operator $a \in \mathrm{End}_{\mathbb{C}}V$ is hermitian iff $\sqrt{-1} \cdot a$ is skew-hermitian.

Problems:

1. (i) Equip $V = \mathbb{C}^n$ with the standard hermitian inner product $(x, y) = x_1 \cdot \bar{y}_1 + \dots x_n \cdot \bar{y}_n$ for all $x, y \in \mathbb{C}^n$. Show that $\mathrm{U}(\mathbb{C}^n)$ is a subgroup of $\mathrm{GL}(\mathbb{C}^n)$ and that $\mathrm{U}(\mathbb{C}^n)$ is a compact subset of $\mathrm{M}_n(\mathbb{C})$. Furthermore, $\mathrm{SU}(\mathbb{C}^n)$ is a closed normal subgroup of $\mathrm{U}(\mathbb{C}^n)$. Describe the factor group $\mathrm{U}(\mathbb{C}^n)/\mathrm{SU}(\mathbb{C}^n)$. Is $\mathrm{U}(\mathbb{C}^n)$, resp. $\mathrm{SU}(\mathbb{C}^n)$, a connected subset of $\mathrm{M}_n(\mathbb{C})$?
[We view $\mathrm{M}_n(\mathbb{C})$ as a topological space using natural identifications $\mathrm{M}_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$.]
(ii) Equip $\mathbb{R}^n$ with the standard inner product $(x, y) = x_1 \cdot y_1 + \dots x_n \cdot y_n$. Consider similar (to part (i)) questions for the groups $\mathrm{O}(\mathbb{R}^n)$ and $\mathrm{SO}(\mathbb{R}^n)$.
(iii) Find the real dimension of the (real) vector space of skew-hermitian operators on $\mathbb{C}^n$.

In Problems 2-4, we let V be a finite dimensional complex vector space equipped with some hermitian positive definite inner product $(-, -)$. For $v \in V$ put $|v| = \sqrt{(v, v)}$.

2. Define a function $\mathsf{N} : \mathrm{End}_{\mathbb{C}}V \to \mathbb{R}_{\geq 0}$, called *operator norm* on the algebra $\mathrm{End}_{\mathbb{C}}V$, by
$$\mathsf{N}(a) := \max_{\{v \in V \ | \ |v|=1\}} |av|.$$
(i) Check that $\mathsf{N}(a \cdot b) \leq \mathsf{N}(a) \cdot \mathsf{N}(b)$, $\mathsf{N}(a^*) = \mathsf{N}(a)$, and $\mathsf{N}(a \cdot a^*) = (\mathsf{N}(a))^2$ for all $a, b \in \mathrm{End}_{\mathbb{C}}V$.
(ii) Find $\max\limits_{a,b \in \mathrm{U}(V)} \mathsf{N}(a - b)$, the 'diameter' of the group $\mathrm{U}(V)$.

3. We say that $a \in \mathrm{End}_{\mathbb{C}}V$ is *positive* and write $0 \preceq a$ if, for any $v \in V$, one has $(av, v) \geq 0$.
(i) Show that $0 \preceq a$ iff there exists $b \in \mathrm{End}_{\mathbb{C}}V$ such that $a = b^*b$.
(ii) Show that $0 \preceq a$ iff $\mathrm{tr}(ab)$ is a nonnegative real number, for any $0 \preceq b$.
(iii) Show that for any $a \in \mathrm{End}_{\mathbb{C}}V$ there exist a unitary operator $u \in \mathrm{U}(V)$ and a positive operator h such that $a = h \cdot u$. [Hint: generalize the proof in the case $V = \mathbb{C}$.]

4. Check that $X = \{x \in \mathrm{End}_{\mathbb{C}}V \mid 0 \preceq 1 - x^*x\}$ is a convex set, i.e. that $x, y \in X$ implies $\frac{1}{2}(x + y) \in X$. An element $a \in X$ is said to be an *extremal point* of X if the only solution $(x, y) \in X \times X$ of the equation $a = \frac{1}{2}(x + y)$ is $x = y = a$. Find all extremal points of X.

5. The algebra $\mathbb{H}$ of *Quaternions* is defined as a 4-dimensional $\mathbb{R}$-algebra with basis $\{1, i, j, k\}$ and the following multiplication table
$$ij = k = -ji, \ \ ki = j = -ik, \ \ jk = i = -kj, \ \ i^2 = j^2 = k^2 = -1. \qquad (\dagger)$$
We write a quaternion in the form $x = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, \dots, x_3 \in \mathbb{R}$. Put $x^* := x_0 - x_1 i - x_2 j - x_3 k$ and $|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$.
(i) Check that $(x \cdot y)^* = y^* \cdot x^*$, $\forall x, y \in \mathbb{H}$, (i.e. the map $a \mapsto a^*$ is an anti-involution of $\mathbb{H}$).
(ii) Check that $|xy| = |x| \cdot |y|$ and $xx^* = x^*x = |x|^2$, for any $x, y \in \mathbb{H}$.
(iii) Show that $\mathbb{H}$ is a division algebra. Find the center of the ring $\mathbb{H}$.

6. Let $\mathbb{H}_{\mathbb{Z}} := \{x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H} \mid x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$ be the ring of integral quaternions. Find all invertible elements in $\mathbb{H}_{\mathbb{Z}}$.

7. (i) We identify $\mathbb{R}^4$ with $\mathbb{H}$ via $(x_0, x_1, x_2, x_3) \mapsto x = x_0 + x_1 i + x_2 j + x_3 k$. Show that the bilinear form $x, y \mapsto 1/2(x\bar{y} + y\bar{x})$ on $\mathbb{H}$ gets transported, via this identification, to the standard Euclidean inner product $(x, y) = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$ on $\mathbb{R}^4$.
(ii) Let $\mathbb{V} \subset \mathbb{H}$ denote the $\mathbb{R}$-linear span of the elements $\{i, j, k\}$. We identify $\mathbb{V}$ with $\mathbb{R}^3$. Put $[a, b] := ab - ba$. Prove that, for any $x, y \in \mathbb{V}$, the following holds:
   - $(x, y) = -1/2 \cdot (xy + yx)$, and $xy = -(x, y) + 1/2 \cdot [x, y]$
   - $[x, y] \in \mathbb{V}$, moreover, the vector $[x, y]$ is orthogonal to both x and y (with respect to the inner product). Deduce that $1/2 \cdot [x, y]$ goes, via the identification $\mathbb{R}^3 = \mathbb{V}$, to $x \times y \in \mathbb{R}^3$, the vector product in $\mathbb{R}^3$ (pay attention to orientations). Thus, $xy = -(x, y) + x \times y$.

(iii) Fix $u \in \mathbb{V}$ such that $|u| = 1$, and some angles $0 \leq \alpha, \beta \leq \pi$. Let $a := \cos\alpha + \sin\alpha\ u$, resp. $b := \cos\beta + \sin\beta\ u \in \mathbb{H}$. Check that $ab = \cos(\alpha + \beta) + \sin(\alpha + \beta)u$.

8. Let $u, v, w \in \mathbb{R}^3$ be a triple of vectors which form an orthonormal basis in $\mathbb{R}^3$ (with the standard orientation). Identify u, v, w with quaternions in $\mathbb{V}$ and, given an angle $0 \leq \theta \leq \pi$, put $q := \cos\theta + \sin\theta\ u \in \mathbb{H}$. Describe in geometric terms the linear map $\operatorname{Ad} q:\ \mathbb{V} \to \mathbb{V},\ \ x \mapsto qxq^{-1}$.
9. We have an imbedding $\mathbb{C} \hookrightarrow \mathbb{H},\ 1 \mapsto 1,\ i \mapsto i$. Right multiplication by elements of the field $\mathbb{C} \subset \mathbb{H}$ makes $\mathbb{H}$ a 2-dimensional vector space over $\mathbb{C}$ with basis $\{1, j\}$. Therefore, associated with any quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$, there is a $\mathbb{C}$-linear map $\widehat{x}:\ \mathbb{H} \to \mathbb{H},\ y \mapsto xy$, which we view as a map $\mathbb{C}^2 \to \mathbb{C}^2$.
   (i) Find $\operatorname{tr}(\widehat{x})$, $\det(\widehat{x})$, and $\mathsf{N}(\widehat{x})$.
   (ii) Let $\mathbb{H}_{\mathbb{C}}$ be a 4-dimensional $\mathbb{C}$-algebra with $\mathbb{C}$-basis $\{1, i, j, k\}$ and multiplication table (†). Prove an algebra isomorphism $\mathbb{H}_{\mathbb{C}} \cong \mathrm{M}_2(\mathbb{C})$.
10. Let A be a finite dimensional division algebra over $\mathbb{R}$ (thus, A is an associative, not necessarily commutative, $\mathbb{R}$-algebra such that any nonzero element of A is invertible). Prove that A is isomorphic either to $\mathbb{R}$, or to $\mathbb{C}$, or else $A \cong \mathbb{H}$. [Hint: Show that if $\dim_{\mathbb{R}} A > 1$ then A contains a copy of the field $\mathbb{C}$ and then mimic the construction of Problem 9.]
11. Let $\mathbb{U} := \{x \in \mathbb{H} \mid |x| = 1\}$. This is a group under multiplication.
   (i) Show that $\mathbb{U}$ is a compact and connected subset of $\mathbb{H} \cong \mathbb{R}^4$.
   (ii) Show that the assignment $x \mapsto \widehat{x}$ yields a group isomorphism $\rho:\ \mathbb{U} \xrightarrow{\sim} \mathrm{SU}(\mathbb{C}^2)$ and, moreover, the map ρ is a homeomorphism of topological spaces.
12. (i) Prove that any element of the group $\mathrm{SO}(\mathbb{R}^3)$ is a rotation about an axis. Construct a continuous and surjective group homomorphism $\pi:\ \mathrm{SU}(\mathbb{C}^2) \twoheadrightarrow \mathrm{SO}(\mathbb{R}^3)$ with kernel $\{\pm \mathrm{Id}\}$.
   (ii) If you know the notion of fundamental group of a (path-connected) topological space, find the fundamental groups of the spaces $\mathrm{SU}(\mathbb{C}^2)$ and $\mathrm{SO}(\mathbb{R}^3)$.
13. Let $\mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. The *stereographic projection* with center $(0, 0, 1) \in S^2$ is an explicit bijection between the unit sphere and the Riemann sphere provided by the following map:
$$p:\ S^2 \xrightarrow{\sim} \mathbb{C} \cup \{\infty\}, \quad (x, y, z) \mapsto \frac{x + iy}{1 - z}.$$
   For any invertible matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ there is an associated fractional-linear transformation $\phi_g:\ \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, of the Riemann sphere, given by
$$\phi_g(z) := \frac{az + b}{cz + d} \qquad \text{(in particular, we have } \phi_g(\infty) = \frac{a}{c} \text{ and } \phi_g(-\frac{d}{c}) = \infty, \text{ if } c \neq 0).$$
   One may use the bijection p to transport the map ϕ_g to the unit sphere. That is, we consider the following composite map
$$F_g:\ \ S^2 \xrightarrow{p} \mathbb{C} \cup \{\infty\} \xrightarrow{\phi_g} \mathbb{C} \cup \{\infty\} \xrightarrow{p^{-1}} S^2.$$
   (i) Show that if $g \in \mathrm{SU}(\mathbb{C}^2)$ then the map F_g is a rotation of the sphere S^2.
   (ii) Relate the homomorphism $\pi:\ \mathrm{SU}(\mathbb{C}^2) \twoheadrightarrow \mathrm{SO}(\mathbb{R}^3)$ of Problem 12 to the map $g \mapsto F_g$.

Assignment 2: due Friday, October 12

In Problems 1-2, we fix a (possibly infinite) set S, an integer $n \geq 1$, and write $[1,n] := \{1, \ldots, n\}$. Let A be a PID and $\mathcal{A}: S \times [1,n] \to A$, $(s,j) \mapsto a_{s,j}$ a function, thought of as a 'rectangular' matrix $\mathcal{A} = \|a_{s,j}\|_{s \in S, j \in [1,n]}$ with infinitely many rows, in general. We define *elementary transformations* $\mathcal{A} := \|a_{s,j}\| \rightsquigarrow \mathcal{A}' := \|a'_{s,j}\|$ of such matrices of the following 3 types:

- For any choice of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$ and any pair $1 \leq i, j \leq n$, $i \neq j$ put

$$\text{column}'_i := a \cdot \text{column}_i + b \cdot \text{column}_j, \quad \text{column}'_j := c \cdot \text{column}_i + d \cdot \text{column}_j. \tag{ET1}$$

- For any choice of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$ and a pair $s, t \in S$, put

$$\text{row}'_s := a \cdot \text{row}_s + b \cdot \text{row}_t, \quad \text{row}'_t := c \cdot \text{row}_s + d \cdot \text{row}_t. \tag{ET2}$$

- For any choice of an element $t \in S$ and of a map $S \to A$, $s \mapsto \alpha_s$ such that $\alpha_t = 0$, put

$$\text{row}'_s := \text{row}_s + \alpha_s \cdot \text{row}_t, \quad \forall s \in S. \tag{ET3}$$

Each transformation of type (ET1), resp. (ET2), affects only two columns, resp. two rows. On the other hand, a single transformation of type (ET3) may affect infinitely many rows, in general.

1. (i) Show that an elementary transformation of a matrix $\mathcal{A}$ does not change the ideal $I(\mathcal{A}) \subset A$ generated by the entries of $\mathcal{A}$.
(ii) Let $d(\mathcal{A})$ be a generator of the ideal $I(\mathcal{A})$. Show that there is a finite subset $S' \subset S$ such that $d(\mathcal{A})$ belongs to the ideal generated by the entries of the matrix $\|a_{s,j}\|_{s \in S', j \in [1,n]}$, of finite size.
2. (i) Let $u, v \in A \setminus \{0\}$ and put $d = \mathsf{gcd}(u,v)$. Find a matrix $g \in \mathrm{M}_2(A)$ such that one has
$$\begin{pmatrix} d \\ 0 \end{pmatrix} = g \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \det g = 1.$$
(ii) Fix a matrix $\mathcal{A}$, as above, and an element $s \in S$. Show that the matrix $\mathcal{A}$ can be transformed using a finite sequence of elementary transformations of types (ET1)-(ET2), to a matrix $\mathcal{A}' = \|a'_{s,j}\|$ such that $a'_{s,1} = d(\mathcal{A})$ and such that, moreover, one has $d(\mathcal{A}) | a'_{t,j}$, for any $t \in S$ and any $j \in [1,n]$.
(iii) Fix $s_1, s_2, \ldots, s_n \in S$. Show that any matrix $\mathcal{A} = \|a_{s,j}\|$ can be transformed, using a finite sequence of elementary transformations of types (ET1)-(ET3), to a matrix $\mathcal{A}' = \|a'_{s,j}\|$ such that
$$a'_{s,j} = \begin{cases} d_j & \text{if } s = s_j, \ j = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } d_i \in A \text{ satisfy } d_j | d_{j+1} \quad \forall 1 \leq j \leq n-1.$$
(iv) Prove the following result stated in class:
Let M be a rank n free A-module and $L \subset M$ a submodule. Then there exists a basis $m_1, \ldots, m_n$, of M, and elements $d_1, \ldots, d_k \in A$, where $k \leq n$ and $d_1 | d_2 | \ldots d_{k-1} | d_k$, such that the elements $d_1 m_1, \ldots, d_k m_k$, form a basis of L.

In Problems 3-6, we let $\Bbbk$ be a field of characteristic zero with an algebraic closure $\bar{\Bbbk}$, and V a finite dimensional $\Bbbk$-vector space. We fix a linear map $a: V \to V$ and write $A_a \subset \mathrm{End}_{\Bbbk} V$ for the $\Bbbk$-subalgebra generated by a (that is, A_a is the $\Bbbk$-span of $1, a, a^2, \ldots$). The imbedding $A_a \hookrightarrow \mathrm{End}_{\Bbbk} V$ makes V an A_a-module.

3. Show that the following 5 properties of the operator a are equivalent:
   - V is a direct sum of simple A_a-modules;
   - The algebra A_a is isomorphic to a direct sum of fields;
   - One has an isomorphism $A_a \cong \Bbbk[x]/\Bbbk[x]\cdot f$, where $f \in \Bbbk[x]$ is such that $\mathsf{gcd}(f, \frac{df}{dx}) = 1$;
   - The algebra A_a has no nonzero nilpotent elements, i.e., no elements $b \in A_a$ such that $b \neq 0$ and $b^m = 0$ for some $m > 1$.
   - The matrix of a can be diagonalized over $\bar{\Bbbk}$.
4. Show that the following 4 properties of the operator a are equivalent:
   - V is a cyclic A_a-module;
   - We have $\dim_{\Bbbk}(\mathrm{End}_{A_a} V) = \dim_{\Bbbk}(V)$;
   - We have $\mathrm{End}_{A_a} V = A_a$;
   - In Jordan normal form of a over $\bar{\Bbbk}$, any two different Jordan blocks have different diagonal entries.

An element $a \in \mathrm{End}_{\Bbbk} V$ that satisfies the equivalent properties of Problem 3, resp. Problem 4, is called semisimple, resp. regular.

5. Let $s, u \in \mathrm{M}_m(\Bbbk)$ be a pair of *commuting* matrices such that s is a diagonal matrix and u is a strictly upper triangular matrix (with zeros at the diagonal). Put $a = s + u$. Show that there exists a polynomial $f(x) = c_1 \cdot x + \ldots + c_d \cdot x^d \in \Bbbk[x]$, without constant term and such that one has $s = f(a)$ (a matrix equality), where $f(a) := c_1 \cdot a + \ldots + c_d \cdot a^d$.
6. (i) Show that for any $a \in \mathrm{End}_k V$, there exists a semisimple element $s \in A_a$ and a nilpotent element $u \in A_a$ such that $a = s + u$, and we have $su = us$. Hint: assume first that the field $\Bbbk$ is algebraically closed. Then, reduce the general case to the case of an algebraically closed field using one of the following approaches.
   - Approach 1: Use Galois theory.
   - Approach 2: Let $f \in \Bbbk[x]$ be such that $\mathsf{gcd}(f, \frac{df}{dx}) = 1$. Use the Chinese remainder theorem to construct inductively polynomials $g_r \in \Bbbk[x]$, $r = 1, 2, \ldots$, such that, setting
   $$p_r := x + f \cdot g_1 + f^2 \cdot g_2 + \ldots + f^r \cdot g_r \in \Bbbk[x],$$
   we have $f(p_r(x)) \in (f(x))^r \cdot \Bbbk[x]$. Deduce in particular that, for any irreducible polynomial $f \in \Bbbk[x]$ and any $\ell \geq 1$, there is a $\Bbbk$-algebra imbedding $\Bbbk[x]/(f) \hookrightarrow \Bbbk[x]/(f^\ell)$.

   (ii) Let $s + u = s' + u'$, where s, s' are semisimple, resp., u, u' are nilpotent, and such that we have $su = us$ and $s'u' = u's'$. Prove that $s = s'$ and $u = u'$.
7. Give a classification of all conjugacy classes in the following groups: (i) $\mathrm{GL}_2(\mathbb{R})$, (ii) $\mathrm{GL}_2(\mathbb{Q})$.
8. Let $\Bbbk = \mathbb{R}$ and view $\mathrm{M}_m(\mathbb{R})$ as a topological space using the natural identification $\mathrm{M}_m(\mathbb{R}) \cong \mathbb{R}^{m^2}$. Let $\mathbb{O}_a = \{g \cdot a \cdot g^{-1} \mid g \in \mathrm{GL}_n(\mathbb{R})\}$ denote the conjugacy class of an element $a \in \mathrm{M}_m(\mathbb{R})$.
   (i) Show that the set $\mathbb{O}_a$ is closed in $\mathrm{M}_m(\mathbb{R})$ iff the element a is semisimple.
   (ii) Show that for any $a \in \mathrm{M}_m(\mathbb{R})$ there exists a *regular* (in the above sense) element $b \in \mathrm{M}_m(\mathbb{R})$ such that the set $\mathbb{O}_a$ is contained in the closure of the set $\mathbb{O}_b$.

Notation: Let $\Bbbk(t) = \{\frac{f}{g} \mid f, g \in \Bbbk[t],\ g \neq 0\}$ denote the field of rational functions in a variable t.

9. (i) Let R be the $\Bbbk$-subalgebra of $\Bbbk(t)$ generated by the set $\Bbbk[t]$, of all polynoimals, and a pair of rational functions: $\frac{1}{t-1}$ and $\frac{1}{t-2}$. Is the ring R a PID ?
   (ii) Is the ring $\mathbb{Z}[x]$ a PID ? Justify your answers.
10. Let $n > 1$ and $A = \Bbbk[x]$. Let $a_1, \ldots, a_n \in A$ be such that $A \cdot a_1 + \ldots + A \cdot a_n = A$. Prove that there exists an invertible matrix $\|r_{ij}\| \in \mathrm{M}_n(A)$ such that $r_{1,j} = a_j$ for all $j = 1, \ldots, n$.

(A similar result for $A = \Bbbk[x_1, \ldots, x_k]$ where $k > 1$ was a famous conjecture of Serre proved by Suslin and Quillen in 1976).

11. Let V be a finite dimensional vector space over $\Bbbk$ and $A = \mathrm{End}_{\Bbbk} V$.
    (i) Find all two-sided, resp. left, right, ideals in the ring A.
    (ii) Prove that V is a simple A-module.
    (iii) Prove that any A-module is a (possibly infinite) direct sum of copies of V.
    (iv) Let D be a division ring. Describe all two-sided, resp. left, right, ideals in the ring $\mathrm{M}_r(D)$. View $V = D^r$ as an $\mathrm{M}_r(D)$-module of column vectors. Do analogues of statements (ii) and (iii) hold in this setting ?
12. Let $A = \Bbbk(t)\langle \frac{d}{dt} \rangle$ be the $\Bbbk$-algebra of differential operators with coefficients in $\Bbbk(t)$. Thus, an element of A is an expression
$$F = f_n \frac{d^n}{dt^n} + f_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \ldots + f_1 \frac{d}{dt} + f_0, \quad f_i \in \Bbbk(t),\ i = 1, \ldots, n,$$
thought of as an operator $\Bbbk(t) \to \Bbbk(t)$. The sum $F + G$, of differential operators F and G, is defined by taking the sum of the corresponding coefficients f_k and g_k, respectively, in front of $\frac{d^k}{dt^k}$. Multiplication in the ring $\Bbbk(t)\langle \frac{d}{dt} \rangle$ is defined as the *composition* of differential operators (it is easy to see that such a composition is a differential operator again).
One has a ring imbedding $\Bbbk(t) \hookrightarrow A = \Bbbk(t)\langle \frac{d}{dt} \rangle$, where a function $f \in \Bbbk(t)$ is identified with the multiplication operator $m_f :\ g \mapsto f \cdot g$. Note that for $f \in \Bbbk(t)$, we have
$$\frac{d}{dt} \circ m_f - m_f \circ \frac{d}{dt} = m_{f'}, \qquad \text{where } f' := \frac{df}{dt}.$$
So, the elements f and $\frac{d}{dt}$ do *not* commute in A, in general.
    (i) Show that the ring A has no zero divisors.
    (ii) Show that any left ideal of the ring A has the form $A \cdot a$ for some $a \in A$.
    (This fact plays an important role in Drinfeld's notion of '*Shtuka*' used in his proof of the Langlands conjecture for the group GL_2).
    (iii) Show that the rings A and A^{op} are isomorphic.
13. Let $\Bbbk$ be a field of $\mathrm{char}\, \Bbbk \neq 2$. For any $n \geq 1$, define a $\Bbbk$-algebra $A_n(\Bbbk)$, on $2n$ generators, as a quotient of the free algebra $\Bbbk\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ by the two-sided ideal generated by the set
$$\{x_i y_j - y_j x_i - \delta_{ij},\ \ x_i x_j - x_j x_i,\ \ y_i y_j - y_j y_i,\ \ i, j = 1, \ldots, n\}, \qquad \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$
    (i) Find the center and all two-sided ideals of the algebra $A(\Bbbk)$ in the case $\mathrm{char}\, \Bbbk = 0$.
    (ii) Find the center of the algebra $A_n(\Bbbk)$ in the case $\mathrm{char}\, \Bbbk > 0$.
    We define an $A_n(\Bbbk)$-action on the vector space $M := \Bbbk[t_1, \ldots, t_n]$ as follows:
$$x_i \text{ acts as } \frac{\partial}{\partial t_i}, \quad \text{resp. } y_i \text{ acts as multiplication by } t_i, \quad \forall i = 1, \ldots, n.$$
    (iii) Show that this action makes M a cyclic $A_n(\Bbbk)$-module and write this module in the form $M = A_n(\Bbbk)/I$ for an appropriate left ideal $I \subset A_n(\Bbbk)$.
    (iv) Prove that M is a simple $A_n(\Bbbk)$-module if $\mathrm{char}\, \Bbbk = 0$ and that it is not simple if $\mathrm{char}\, \Bbbk > 0$.
    (v) Construct a simple $A_n(\Bbbk)$-module in the case $\mathrm{char}\, \Bbbk > 0$.

Assignment 3: due Friday, October 19

Definition: Let A be a not necessarily commutative ring. An A-module M is said to be of *finite length* n if, for any chain of A-submodules $0 \neq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_m = M$, we have $m \leq n$ and, moreover, there exists a chain as above with $m = n$.

1. Let M be an A-module of finite length and let $u : M \to M$ be an A-module morphism. Put
$$\mathrm{Image}(u^\infty) := \bigcap_{k=1}^{\infty} \mathrm{Image}(u^k), \quad \text{resp.} \quad \mathrm{Ker}(u^\infty) = \bigcup_{k=1}^{\infty} \mathrm{Ker}(u^k).$$
Show that $\mathrm{Image}(u^\infty)$ and $\mathrm{Ker}(u^\infty)$ are A-submodules in M and, moreover, we have
$$M = \mathrm{Image}(u^\infty) \oplus \mathrm{Ker}(u^\infty).$$

2. Fix a ring A and an idempotent $e \in A$. The subset $eAe \subset A$ is a ring with unit e (thus, eAe is *not* a subring of A, according to our definitions). Prove the following:
(i) For any A-module M, there is a natural eAe-module structure on the subgroup $eM \subset M$ (here $eM = \{em \mid m \in M\}$). Furthermore, for any A-modules M and N there is a natural morphism of additive groups $f :\ \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{eAe}(eM, eN)$;
(ii) Multiplication on the right gives an algebra isomorphism $(eAe)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Hom}_A(Ae, Ae)$;
Assume in addition that $AeA = A$, i.e., that any element of A can be written (not necessarily uniquely) in the form $\sum a_i \cdot e \cdot b_i$, for some $a_i, b_i \in A$. Prove that, in such a case, statements (iii)-(iv) below hold:
(iii) There is an algebra isomorphism $A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Hom}_{eAe}(eA,\ eA)$;
(iv) For any A-modules M, N the map f in (i) is a bijection.

3. (i) Find all group homomorphisms $f : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$.
Find all *continuous* group homomorphisms f in the following cases:
(ii) $f :\ (\mathbb{R}, +) \to (\mathbb{R}, +)$; (iii) $f :\ (\mathbb{R}^n, +) \to (\mathbb{R}, +)$;
(iv) $f :\ (\mathbb{R}, +) \to S^1$; (v) $f :\ S^1 \to S^1$.
Here, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, the unit circle viewed as a group with respect to the operation of multiplication of complex numbers.

4. Recall the setting of Problem 2 of Assignment 1. The function N gives a metric on the vector space $A := \mathrm{End}_{\mathbb{C}} V$. For $r > 0$, let $B_r := \{a \in A \mid \mathsf{N}(a) \leq r\}$ be a ball in A of radius r. Show that
(i) The series $e^a := \mathrm{Id} + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \ldots$ converges (absolutely) on B_r to a continuous function $A \to \mathrm{GL}(V),\ a \mapsto e^a$.
(ii) For any fixed $r < 1$, the series $\log(\mathrm{Id} + a) := a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \ldots$ converges absolutely to a continuous function $B_r \to A,\ a \mapsto \log(\mathrm{Id} + a)$.
(iii) One has $\log(e^a) = a$, resp., $e^{\log(\mathrm{Id}+a)} = \mathrm{Id} + a$, for any $a \in \mathrm{M}_n(\mathbb{C})$ such that $\mathsf{N}(a)$ is sufficiently small, say $\mathsf{N}(a) < 1/10$.

5. (i) Show that any matrix $g \in GL_n(\mathbb{C})$ can be written in the form $g = e^a$ for some $a \in \mathrm{M}_n(\mathbb{C})$.
(ii) For fixed g, consider the following system of two equations on the matrix x:
$$\begin{cases} e^x = g \\ \mathrm{tr}\, x = 0. \end{cases}$$
Give an example of a matrix $g \in SL_2(\mathbb{C})$ such that the above system has no solution $x \in \mathrm{M}_2(\mathbb{C})$. Give an example of a matrix $g \in SL_2(\mathbb{R})$ such that the above system has a solution $x \in \mathrm{M}_2(\mathbb{C})$ but has no solution $x \in \mathrm{M}_2(\mathbb{R})$.

6. (i) Prove that, for any $a \in \mathrm{M}_n(\mathbb{R})$, the map $f : (\mathbb{R}, +) \to GL_n(\mathbb{R}),\ t \mapsto e^{t \cdot a}$ is a continuous group morphism.
   (ii) Prove that any *continuous* group morphism $f : (\mathbb{R}, +) \to GL_n(\mathbb{R})$ has the form $f(t) = e^{t \cdot a}$, for some fixed $a \in \mathrm{M}_n(\mathbb{R})$.
7. Fix $n > 1$ and identify $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Let dx be the standard euclidean Lebesgue measure on the vector space $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Find a continuous function $f : GL_n(\mathbb{R}) \to \mathbb{R}_{>0}$ such that the measure $f(x)\,dx$ is a left invariant measure on the group $G = GL_n(\mathbb{R})$.
8. Let $G \to GL(V)$ be a finite dimensional irreducible representation of a finite group G in a complex vector space V. Let $\beta_1, \beta_2 : V \times V \to \mathbb{C}$ be a pair of nonzero hermitian (not necessarily positive definite) G-invariant forms on V. Prove that there exists a nonzero constant $c \in \mathbb{R}$ such that one has $\beta_2(v_1, v_2) = c \cdot \beta_1(v_1, v_2)$, for any $v_1, v_2 \in V$.
9. Let $X \subset \mathbb{R}^n$ be a compact set. Let $\mathbb{C}_{\text{cont}}(X)$ be the algebra of *continuous* functions $f : X \to \mathbb{C}$, with pointwise operations. Define a metric on $\mathbb{C}_{\text{cont}}(X)$ by $\mathit{dist}(f, g) := \max_{x \in X} |f(x) - g(x)|$, for any $f, g \in \mathbb{C}_{\text{cont}}(X)$. An ideal $I \subset \mathbb{C}_{\text{cont}}(X)$ is called a *closed ideal* if I is a closed subset of $\mathbb{C}_{\text{cont}}(X)$ viewed as a metric space.
   (i) Show that the closure $\overline{I}$, of a proper ideal $I \subsetneq \mathbb{C}_{\text{cont}}(X)$, is a proper ideal again (in particular, one has $\overline{I} \neq \mathbb{C}_{\text{cont}}(X)$). Deduce that any maximal ideal is closed.
   (ii) Show that, for any subset $Y \subset X$, the set $I_Y := \{f \in \mathbb{C}_{\text{cont}}(X) \mid f(y) = 0\ \forall y \in Y\}$ is a closed ideal in $\mathbb{C}_{\text{cont}}(X)$. Prove that, for any $x \in X$, the ideal $I_{\{x\}}$ is maximal.

   Let $X := \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$ be a closed segment.

   (iii) Give an example of a nonclosed ideal in $\mathbb{C}_{\text{cont}}(X)$.
   (iv) Prove that the ideal $I_{\{1\}}$ is not generated by any finite collection of elements of $\mathbb{C}_{\text{cont}}(X)$; in particular, it is not a principal ideal.
   (v) Show that any closed ideal I in $\mathbb{C}_{\text{cont}}(X)$ has the form I_Y for some closed subset $Y \subset X$.
10. For each $c \in \mathbb{C}$, analyze the existence of nonzero two-sided ideals $J \subsetneq A_c$ in the $\mathbb{C}$-algebra $A_c := \mathbb{C}\langle x, y\rangle / I_c$ where I_c is a two-sided ideal genrated by the element $xy - yx - c \cdot x - 1$.

Assignment 4: due Friday, October 26

1. Let G be a cyclic group of order 7 written multiplicatively, and let g be a generator of G. Let $A = \mathbb{C}G$.

   (i) Find $\operatorname{spec} a$, where $a = 3g + 16 \in A$.

   (ii) Find the number of solutions $x \in \mathbb{C}G$ of the equation $x^5 = x$. Give explicit formulas for the solutions.

2. Let $G = \mathbb{Z}/36\mathbb{Z}$ be a cyclic group of order 36. Define a function $f \in \mathbb{C}\{G\}$ as follows:

$$f(x) = \begin{cases} e^{\frac{2\pi i n}{6}} & \text{if } x = 6n \bmod 36, \ n = 0, 1, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

   Find the support of the function $\mathcal{F}_G(f) \in \mathbb{C}\{\widehat{G}\}$ (that is, the set of all $\chi \in \widehat{G}$ such that $\mathcal{F}_G(f)(\chi) \neq 0$) where $\mathcal{F}_G$ denotes the normalized Fourier transform.

3. Let G be the group of orientation preserving euclidean motions of $\mathbb{R}^3$ which take a cube $C \subset \mathbb{R}^3$ into itself. Let X be the set of faces of C. The group G acts naturally on X, hence, also on $\mathbb{C}\{X\}$.

   (i) Decompose $\mathbb{C}\{X\}$ into a direct sum of irreducible G-subrepresentations (describe those subrepresentations explicitly).

   (ii) Let $u : \mathbb{C}\{X\} \to \mathbb{C}\{X\}$ be a $\mathbb{C}$-linear map defined by the formula

$$(uf)(x) := \sum_{\{y \in X \ | \ y \ \text{ is adjacent to } \ x\}} f(y)$$

   (summation is taken over the 4 faces which have a common edge with the face x).
   Find the eigenvalues of the map u and describe the corresponding eigenspaces.

4. Let $G \to GL(V)$ be a finite dimensional continuous representation of a compact group G and let $S \subset V$ be a convex G-stable subset. Show that there exists a G-fixed point $s \in S$.

5. Let $\Bbbk$ be a field. For each $a \in \Bbbk^\times$ and each $b \in \Bbbk$, let $g_{a,b} : \ \Bbbk \to \Bbbk$ be an affine-linear map given by $g_{a,b}(x) = a \cdot x + b$. The transformations $\{g_{a,b}, \ a \in \Bbbk^\times, b \in \Bbbk\}$ form a group $G(\Bbbk)$ with respect to the composition operation.

   (i) Give an example of a representation $\rho : \ G(\Bbbk) \to GL(V)$ in a finite dimensional vector space V **over** $\Bbbk$ (not, over $\mathbb{C}$) which is not irreducible but which can not be decomposed as a direct sum of irreducible subrepresentations.

   (ii) Let $\Bbbk = \mathbb{Z}/5\mathbb{Z}$ be the field of residues modulo 5. Classify all irreducible finite dimensional representations $\rho : \ G(\Bbbk) \to GL(V)$ in **complex** vector spaces V, up to isomorphism. Hint: show that the action of the element $g_{1,1}$ in V can be diagonalized and look at the action of other elements of the group $G(\Bbbk)$ in the basis formed by eigenvectors of the element $g_{1,1}$.

   (iii) Let $\Bbbk = \mathbb{R}$. We identify the group $G(\mathbb{R})$ with the set $\{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$, an open subset of $\mathbb{R}^2$, via the map $g_{a,b} \mapsto (a, b)$.
   Find a nonzero continuous function $\phi : \ G(\mathbb{R}) \to \mathbb{R}^\times, \ (a, b) \mapsto \phi(a, b)$, resp. $\psi : \ G(\mathbb{R}) \to \mathbb{R}^\times, \ (a, b) \mapsto \psi(a, b)$, such that $\phi(a, b) \cdot da\, db$ is a left invariant, resp. $\psi(a, b) \cdot da\, db$ is a right invariant, measure on the group $G(\mathbb{R})$. Check that the function ψ is not a constant multiple of the function ϕ.

   (iv) Classify all (complex) finite dimensional continuous irreducible representations of the group $G(\mathbb{R})$ up to isomorphism.

6. Let dx be the standard Lebesgue measure on $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and view $G = \mathrm{GL}_n(\mathbb{R})$ as an open subset of $\mathrm{M}_n(\mathbb{R})$. Find a positive continuous function $\phi : \ G \to \mathbb{R}^{>0}$, such that $\phi(x)dx$ is a left invariant measure on the group G.

7. Let G be the multiplicative group (i.e. the group of invertible elements) of the ring $\mathbb{H}_{\mathbb{Z}} = \{x_0 + x_1 i + x_2 j + x_3 k \,|, x_0, \ldots, x_3 \in \mathbb{Z}\}$, of interger Quaternions. Classify irreducible complex representations of G up to isomorphism.

8. Let G be the following group of upper triangular 3×3-matrices
$$G = \left\{ g_{x,y,z} = \begin{vmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{vmatrix}, \;\; x, y \in \mathbb{R}, \;\; z \in \mathbb{R}/\mathbb{Z} \right\}.$$
Thus, the assignment $g_{x,y,z} \mapsto (x, y, z)$ gives a bijection $G \xrightarrow{\sim} \mathbb{R} \times \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$. We equip G with the natural topology on the set $\mathbb{R} \times \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$.

(i) Check that elements of the form $g_{0,0,z}, \;\; z \in \mathbb{R}/\mathbb{Z}$, form a central subgroup of G which is isomorphic to S^1.

(ii) Show that the group G does not have a faithful continuous finite dimensional complex representation, i.e., any continuous homomorphism $G \to GL(V)$, where V is a finite dimensional complex vector space, is not injective.

9. Let $\mathbb{F}$ be a finite field with q elements, and let $G = SL_2(\mathbb{F})$. The group G acts linearly on the 2-dimensional vector space $\mathbb{F}^2$ and fixes the origin 0. Hence, G acts on the set $X := \mathbb{F}^2 \smallsetminus \{0\}$, the complement of the origin. Below, we are interested in the natural G-representation in the vector space $\mathbb{C}\{X\}$. To this end, for any group homomorphism $\chi : \mathbb{F}^\times \to S^1 \subset \mathbb{C}^\times$, in $\mathbb{C}\{X\}$, we define a subspace
$$\mathbb{C}\{X\}^\chi := \{f \in \mathbb{C}\{X\} \mid f(z \cdot x) = \chi(z) \cdot f(x), \; \forall z \in \mathbb{F}^\times\}.$$

(i) Check that $\mathbb{C}\{X\}^\chi$ is a G-stable subspace of $\mathbb{C}\{X\}$ and there is a vector space direct sum decomposition $\mathbb{C}\{X\} = \bigoplus_{\chi \in \widehat{H}} \; \mathbb{C}\{X\}^\chi$, where we put $H := \mathbb{F}^\times$.

(ii) For which $\chi \in \widehat{H}$ the G-representation $\mathbb{C}\{X\}^\chi$ is irreducible ?
(iii) Give an explicit $\mathbb{C}$-basis of the $\mathbb{C}$-algebra $A := \mathrm{End}_{\mathbb{C}G}\mathbb{C}\{X\}$, of G-intertwiners $\mathbb{C}\{X\} \to \mathbb{C}\{X\}$, and write an explicit multiplication table for the basis elements.
(iv) Decompose $\mathbb{C}\{X\}$ into simple G-representations, and compute dimensions of these simple representations.
(v) Show that there exist simple representations of G which do not occur in the decomposition of $\mathbb{C}\{X\}$.

10. Write $\partial := \frac{d}{dt}$, fix $m, n > 0$, and let
$$F = \partial^n + f_1(t)\partial^{n-1} + \ldots + f_{n-1}\partial + f_0, \quad G := \partial^m + g_1(t)\partial^{m-1} + \ldots + g_{m-1}\partial + g_0$$
be a pair of ordinary differential operators with rational coefficients (i.e. with coefficients in $\mathbb{C}(t)$). Assume that F and G commute (as operators). Then, for any polynomial $p = \sum_{i,j} \; c_{i,j}\, x^i y^j \in \mathbb{C}[x, y]$, there is a well defined differential operator $p(F, G) = \sum_{i,j} \; c_{i,j} F^i \circ G^j$, where $F^i := F \circ \ldots \circ F$ (i times), resp. $G^j := G \circ G \circ \ldots \circ G$ (j times).

(i) Check that the following operators provide an example of a commuting pair:
$$F = \frac{d^2}{dt^2} - 2\frac{1}{t}, \quad G = \frac{d^3}{dt^3} - 3\frac{1}{t^2}\frac{d}{dt} + 3\frac{1}{t^3}.$$
Show that $F^3 = G^2$, i.e. one has $p(F, G) = 0$, where $p = x^3 - y^2 \in \mathbb{C}[x, y]$.

It turns out that one has the following general result: *If F and G commute then there exists a nonzero polynomial $p \in \mathbb{C}[x, y]$ such that one has $p(F, G) = 0$.*

(ii) Prove this result in the special case where $n = 2$. Hint: For each $\lambda \in \mathbb{C}$, let V_λ be the space of solutions $\psi \in C^\infty(a, b)$ of the differential equation $F(\psi) = \lambda \cdot \psi$, on a small enough segment

$[a, b] \subset \mathbb{R}$. The operator G acts on V_λ. Use that there exists a polynomial $p_\lambda \in \mathbb{C}[y]$ such that $p_\lambda(G|_{V_\lambda}) = 0$. Then, analyze the dependence of solutions ψ on the initial condition at the point a to show that one can find p_λ in such a way that the function $(\lambda, \mu) \mapsto p_\lambda(\mu)$ gives a polynomial $p \in \mathbb{C}[x, y]$ such that $p(F, G) = 0$.

Remark. The set of polynomials $p \in \mathbb{C}[x, y]$ such that $p(F, G) = 0$ is an ideal in $\mathbb{C}[x, y]$. Basic algebraic geometry implies that this is a principal ideal generated by some polynomial f. Thus, the algebra $\mathbb{C}[F, G]$, of differential operators generated by F and G, is isomorphic to $\mathbb{C}[x, y]/(f)$.

The complex curve $C = \{(\lambda, \mu) \in \mathbb{C}^2 \mid f(\lambda, \mu) = 0\}$ is called the *spectral curve* for (F, G). This name is motivated by the following result:

(iii) Given a commuting pair F, G, and the polynomial f, as above. Then, for $\lambda, \mu \in \mathbb{C}$, we have

$$f(\lambda, \mu) = 0 \quad \Longleftrightarrow \quad \text{there exists } \psi \in C^\infty(a, b) \text{ such that } F(\psi) = \lambda \cdot \psi \quad \& \quad G(\psi) = \mu \cdot \psi.$$

This is a starting point of the theory of *integrable systems*. The example in (i) is related to the Korteweg-de Vries equation which was, historically, the first example of an integrable system. There are many deep connections between integrable systems, algebraic geometry and representation theory. For example, Drinfeld's notion of '*Shtuka*' was strongly inspired by his earlier works on integrable systems, on the Korteweg-de Vries equation in particular.

Assignment 5: due Monday, November 5

Notation. Throughout, for a tensor product over a field $\Bbbk$ we use unadorned notation $\otimes := \otimes_{\Bbbk}$. Let $Z(A)$ denote the *center* of a ring A.

Definition. A ring A is called *simple* if the only nonzero two-sided ideal in A is A itself.

A ring A is called a *central simple* $\Bbbk$-algebra if A is simple, $\Bbbk := Z(A)$ is a field, and A has finite dimension over $\Bbbk$.

1. Given $m, n \geq 1$ and a pair of rings A, B, which of the ring isomorphisms below hold ?

$$\begin{aligned} &(1) \quad \Bbbk\langle t_1, \ldots, t_n\rangle \otimes \Bbbk\langle s_1, \ldots, s_m\rangle \cong \Bbbk\langle t_1, \ldots, t_n, s_1, \ldots, s_m\rangle \\ &(2) \quad \Bbbk[t_1, \ldots, t_n] \otimes \Bbbk[s_1, \ldots, s_m] \cong \Bbbk[t_1, \ldots, t_n, s_1, \ldots, s_m] \\ &(3) \quad \mathrm{M}_m(A) \otimes_{\mathbb{Z}} \mathrm{M}_m(B) \cong \mathrm{M}_m(A \otimes_{\mathbb{Z}} B). \end{aligned}$$

Justify your answer.

2. (i) Prove that, for any $\Bbbk$-algebras A and B, the natural imbedding $Z(A) \otimes Z(B) \hookrightarrow A \otimes B$ induces an algebra isomorphism $Z(A) \otimes Z(B) \xrightarrow{\sim} Z(A \otimes B)$.
(ii) Give an example where the algebra $A \otimes B$ has a two-sided ideal that does not have the form $I \otimes J$, where I and J are two-sided ideals in A and B, respectively.
(iii) Suppose A is simple and we have $Z(A) = \Bbbk$. Prove that any two-sided ideal in $A \otimes B$ has the form $A \otimes J$, for some two-sided ideal $J \subset B$.
(iv) Prove that if A and B are central simple $\Bbbk$-algebras then so is $A \otimes B$.

3. Let A be a central simple $\Bbbk$-algebra. Prove that
(i) There is an algebra isomorphism $A \otimes_{\Bbbk} A^{op} \cong \mathrm{M}_r(\Bbbk)$, where $r = \dim_{\Bbbk} A$.
(ii) $\mathrm{M}_n(A)$ is a central simple $\Bbbk$-algebra for any $n \geq 1$.

4. Let A be a ring. (i) For any left, resp. right, A-module M give the abelian group $\mathrm{Hom}_A(M, A)$ the structure of a right A-module (to be denoted M^*), resp. left A-module (to be denoted ${}^*M$). Give an example where $M \neq 0$ but $M^* = 0$. Construct canonical morphisms

$$M \to {}^*(M^*), \quad M \to ({}^*M)^*,$$

of left, resp. right, of A-modules.
(ii) Let M be an (A, A)-bimodule. Give the right A-module M^*, resp. left A-module ${}^*M$, an additional structure of left, resp. right, A-module such that the maps defined in (i) become bimodule morphisms.
(iii) Given a pair of left A-modules M, N construct a canonical morphism $M^* \otimes_A N \to \mathrm{Hom}_A(M, N)$, of abelian groups. For M an (A, A)-bimodule, give $\mathrm{Hom}_A(M, N)$ an additional structure of a left A-module such that the canonical morphism that you've constructed becomes a morphism left A-modules, where $M^* \otimes_A N$ is viewed as a left A-module via the left action of A on M^* defined in (ii).
(iv) Let $M_1, \ldots, M_n$ be (A, A)-bimodules. Construct a canonical bimodule morphism

$$M_1^* \otimes_A M_2^* \otimes_A \ldots \otimes_A M_n^* \to (M_1 \otimes_A M_2 \otimes_A \ldots \otimes_A M_n)^*.$$

Warning: make sure that the maps you define are indeed well-defined.

5. Let e be an idempotent in a ring A. Multiplication in A induces a natural morphism $\phi : eA \otimes_A Ae \to eAe$, of eAe-bimodules, and a natural morphism $\psi : Ae \otimes_{eAe} eA \to A$, of A-bimodules.
(i) Prove that ϕ is an isomorphism.
(ii) Prove that if $AeA = A$ then ψ is an isomorphism.
(iii) Prove the following generalization of (ii): if $AeA = A$ then, for any right A-module M and left A-module N, one has a canonical isomorphism $Me \otimes_{eAe} eN \xrightarrow{\sim} M \otimes_A N$.

6. Let A be a semisimple finite dimensional $\Bbbk$-algebra. Prove that each of the morphisms in Problem 4(ii)-(iv) above is, in fact, an isomorphism in the case where all the bimodules involved have finite dimension over $\Bbbk$.
7. Let $\Bbbk$ be a field of characteristic zero and A a $\Bbbk$-algebra. For any $n \geq 1$, the Symmetric group S_n acts naturally on the algebra $A^{\otimes n} := A \otimes A \otimes \ldots \otimes A$ (n factors) by permutation of the tensor factors. This is an action by algebra automorphisms, so $(A^{\otimes n})^{S_n}$, the space of S_n-invariants in $A^{\otimes n}$, is a subalgebra.

   (i) Show that, if A is commutative then the algebra $(A^{\otimes n})^{S_n}$ is generated by elements of the form
$$a \otimes 1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes a \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1 + \ldots 1 \otimes \ldots \otimes 1 \otimes a, \quad a \in A.$$
   (ii) Let S_n act on the algebra $\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by simultaneous permutations of x's and y's, e.g., for any $s \in S_n$, $p, q \geq 0$, we have $s(x_i^p y_j^q) = x_{s(i)}^p y_{s(j)}^q$. Show that the algebra $\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$, of S_n-invariant polynomials, is generated by elements of the form
$$x_1^p y_1^q + \ldots + \ldots + x_n^p y_n^q, \quad p, q \geq 0.$$
8. Let $G \subset \mathrm{GL}(V)$ be a finite subgroup, and Q_G the McKay quiver for G. We have constructed in class an algebra isomorphism $\Phi : \ p(TV\#G)p \xrightarrow{\sim} \mathbb{C}Q_G$. Assume now that $V = \mathbb{C}^2$ and that G is contained in $\mathrm{SL}(V)$ (not only in $\mathrm{GL}(V)$). Let x, y be a $\mathbb{C}$-basis of V. Thus, one has algebra isomorphisms $TV = \mathbb{C}\langle x, y\rangle$ and $\mathrm{Sym}\, V = \mathbb{C}[x, y]$. Let $\omega := (x \otimes y - y \otimes x)\#1$. This is an element of $TV\#G$ which is independent of the choice of basis of V, up to a nonzero constant factor.

   (i) Find (up to a constant factor) the image of the element $p\omega p \in p(TV\#G)p$ under the map Φ.

   (ii) Derive a description of $\mathbb{C}[x, y]^G$, the algebra of G-invariant polynomials, in terms of the McKay quiver Q_G.
9. Let G, a finite group, act on a finite set X. This action induces an action of G on $\mathbb{C}\{X\}$, the algebra of functions on X, by algebra automorphisms. Let $A = \mathbb{C}\{X\}\#G$.

   (i) Take $X = G$, where G acts on X by left translations. Construct an *explicit* isomorphism between the algebra A and the matrix algebra $\mathrm{M}_n(\mathbb{C})$, where $n = \#G$.

   (ii) Take $X = G/H$, where H is a subgroup of G and G acts on G/H by left translation. Let $1_o \in \mathbb{C}\{G/H\}$ be the characteristic function of the coset H/H. Prove an algebra isomorphism $1_o A 1_o \cong \mathbb{C}H$. Deduce a natural correspondence between G-equivariant $\mathbb{C}\{X\}$-modules and representations of the group H. Reformulate this correspondence in the language where $\mathbb{C}\{X\}$-modules are identified with collections $\{V_x, \ x \in X\}$, of vector spaces.

   (iii) Prove that, for any set X as above, the algebra $A = \mathbb{C}\{X\}\#G$ is semisimple. Describe the center of A.
10. Fix $n > 1$. Construct the McKay quiver for the subgroup $G \subset \mathrm{GL}(\mathbb{C}^2)$ generated by the matrices
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \text{ for all } \zeta \in \mathbb{C} \text{ such that } \zeta^{2n} = 1.$$
11. The symmetric group S_4 acts naturally on $\mathbb{R}^4$ by permutation of coordinates. The 3-dimensional subspace $E := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_i x_i = 0\}$ is S_4-stable. The action on E gives group imbeddings $A_4 \hookrightarrow S_4 \hookrightarrow \mathrm{SO}(E) \cong \mathrm{SO}(\mathbb{R}^3)$, where A_4 is the subgroup of even permutations.
    Recall the two-to-one homomorphism $\pi : \ \mathrm{SU}(\mathbb{C}^2) \twoheadrightarrow \mathrm{SO}(\mathbb{R}^3)$, from Problem 12(i) of Assignment 1. Let $G := \pi^{-1}(A_4)$. This is a finite subgroup of $\mathrm{SU}(\mathbb{C}^2)$, in particular, we have $G \subset \mathrm{GL}(\mathbb{C}^2)$. Construct the McKay quiver for G.
    [Remark: This graph is known as an '*extended Dynkin graph of type* $\widetilde{E}_6$'.
    One can replace, in the above construction, the group A_4 by S_4. The resulting McKay quiver is then an extended Dynkin graph of type $\widetilde{E}_7$.]

Assignment 6: due Friday, November 9

Definition. Fix a ring B. Given a subring $A \subset B$, we define $A^! := \{b \in B \mid ab = ba,\ \forall a \in A\}$, the *centralizer of A in B*. This is a subring of A, so one can iterate $A^{!!} := (A^!)^!,\ A^{!!!} := (A^{!!})^!$, etc.

1. (i) Check that one has an inclusion $A \subset (A^!)^!$ and give an example of a pair $A \subset B$ where the inclusion is strict.
(ii) Prove that there is a stabilization of all even, resp. odd, terms of the sequence of iterated centralizers, e.g. we have: $A^{!!!} = A^!$ and $A^{!!!!} = A^{!!}$.

2. Let V be a finite dimensional vector space over $\Bbbk = \bar{\Bbbk}$ and $B := \operatorname{End}_{\Bbbk} V$. Let $A \subset B$ be a semisimple subalgebra. We may view V either as an A-module and as an $A^!$-module. For each simple A-module L, introduce the notation $L^\diamondsuit := \operatorname{Hom}_A(L, V)$.
(i) Show that $A^! \cong \oplus_{L \in S_A} \operatorname{End}_{\Bbbk} L^\diamondsuit$, in particular, the algebra $A^!$ is semisimple and every nonzero $L^\diamondsuit$ is a simple $A^!$-module.
(ii) Show that we have the following equalities $(A^!)^! = A$ and $Z(A) = A \cap A^! = Z(A^!)$.
(iii) Show that $[V : L] \neq 0$ for any $L \in S_A$ and the map $L \mapsto L^\diamondsuit$ provides a bijection $S_A \xrightarrow{\sim} S_{A^!}$.
The actions of A and $A^!$ on V commute, hence they give V the structure of a left $A \otimes A^!$-module.
(iv) Decompose V into a direct sum of simple $A \otimes A^!$-modules.

3. Let $\operatorname{char} \Bbbk = 0$ and $n \geq 2$. Show that $\Bbbk[x_1, \ldots, x_n]$ is a free $\Bbbk[x_1, \ldots, x_n]^{S_n}$-module with basis
$$x_2^{m_2} x_3^{m_3} \ldots x_{n-1}^{m_{n-1}} x_n^{m_n}, \quad m_2 \in [0,1],\ m_3 \in [0,2], \ldots, m_n \in [0, n-1].$$

4. Let $n > 1$ and $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Let $G \in \mathrm{SL}_2(\mathbb{C})$ be a cyclic subgroup of order n generated by the diagonal matrix $g = \operatorname{diag}(\zeta, \zeta^{-1})$. The group G acts on $\mathbb{C}[x, y]$.
(i) Describe the algebra $\mathbb{C}[x, y]^G$ by generators and relations.
(ii) Let $\mathbb{C}[x, y]^\zeta := \{f \in \mathbb{C}[x, y] \mid g(f) = \zeta \cdot f\}$. This space is stable under the $\mathbb{C}[x, y]^G$-action by multiplication. Check whether or not $\mathbb{C}[x, y]^\zeta$ is a free $\mathbb{C}[x, y]^G$-module.

5. Let H be a subgroup of a finite group G and let $\rho : H \to GL(V)$ be a representation of H. Construct *explicit* isomorphisms between each pair of the following three G-representations:
$$M_1 = \operatorname{Ind}_H^G V = \{f : G \to V \mid f(gh^{-1}) = \rho(h)(f(g)),\ \forall g \in G, h \in H\},$$
$M_2 = \operatorname{Hom}_H(\Bbbk\{G\}, V)$, and $M_3 = \Bbbk G \otimes_{\Bbbk H} V$.
[Remark: *The correct equation for M_1 reads $f(gh^{-1}) = \rho(h)(f(g))$ rather than $f(gh) = \rho(h)(f(g))$.*]

6. Let $\mathcal{U}$ be an associative $\mathbb{C}$-algebra with three generators E, H, F, and three defining relations
$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H. \qquad (\heartsuit)$$
(i) Show that the formulas
$$E(f) := x\tfrac{\partial f}{\partial y}, \quad H(f) := x\tfrac{\partial f}{\partial x} - y\tfrac{\partial f}{\partial y}, \quad F(f) := y\tfrac{\partial f}{\partial x} \qquad (\dagger)$$
give the vector space $\mathbb{C}[x, y]$ the structure of an $\mathcal{U}$-module.
(ii) Prove that, for each $m \geq 0$, the space $\mathbb{C}^m[x, y]$, of homogeneous polynomials of degree m, is a simple $\mathcal{U}$-submodule of $\mathbb{C}[x, y]$.
For $m = 1$, compute the 2×2-matrices of the action of the elements E, H, F in the basis x, y.
(iii) Let $S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C})$, a matrix with zero trace. Associated with S, we have a group homomorphism $\mathbb{C} \to \mathrm{SL}_2(\mathbb{C}),\ t \mapsto \exp(t \cdot S) = e^{t \cdot S}$. For any $f \in \mathbb{C}[x, y]$, verify the formula
$$\left[\tfrac{d}{dt}\left(\exp(t \cdot S)(f)\right)\right]_{t=0} = (b \cdot E + a \cdot H + c \cdot F)(f).$$
This formula means that the natural action of the group $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$ induces, infinitesimally, the action of the algebra $\mathcal{U}$ given by formulas $(\dagger)$.

7. Let $\mathcal{U}_+$ be an associative $\mathbb{C}$-algebra with two generators E, H, and one defining relation $HE - EH = 2E$. Let M be an $\mathcal{U}_+$-module.
   (i) Show that if $v \in M$ is a nonzero eigenvector of the operator $H : M \to M$, then $E(v)$ is either zero or is an eigenvector of H again.
   (ii) Show that if M is a finite dimensional $\mathcal{U}_+$-module then there exists a nonzero eigenvector $v \in M$, of H, such that $E(v) = 0$.
8. Let V be a finite dimensional $\mathcal{U}$-module. Let $\lambda \in \mathbb{C}$ and $v \in V$ a nonzero element such that $E(v) = 0$ and $H(v) = \lambda \cdot v$. We put $v_i := \frac{1}{i!}F^i(v),\ i \geq 1$.
   (i) For each $i = 1, 2, \ldots$, find an explicit formula for the element $E(v_i)$ as a function of λ. Deduce that if the $\mathcal{U}$-module V is simple then the complex number λ must be a nonnegative integer.
   (ii) Prove that any simple $\mathcal{U}$-module of dimension $m+1$ is isomorphic to $\mathbb{C}^m[x, y]$.
9. Fix $n \geq 3$. We define a triple of $\mathbb{C}$-linear operators $\Delta, \mathsf{eu}, \mathbf{r^2} : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$. The operator $\mathbf{r^2}$ is given by $f \mapsto r^2 \cdot f$, where $r^2(x) := x_1^2 + \ldots + x_n^2$ is a quadratic polynomial (here, '$\mathbf{r^2}$' and 'r^2' are symbols which do not mean a square of something). The other two operators are the Laplace operator and the Euler operator operator defined as follows:
$$\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}, \qquad \mathsf{eu} := x_1 \frac{\partial}{\partial x_1} + \ldots + x_n \frac{\partial}{\partial x_n}.$$
   (i) Verify that the triple $(\frac{1}{2}\Delta, \mathsf{eu} + \frac{n}{2}, \frac{1}{2}\mathbf{r^2})$ satisfies the relations $(\heartsuit)$, i.e. the assignment $E \mapsto \frac{1}{2}\Delta,\ H \mapsto \mathsf{eu} + \frac{n}{2},\ F \mapsto \frac{1}{2}\mathbf{r^2}$, extends to a homomorphism from the algebra $\mathcal{U}$ to the algebra of differential operators on $\mathbb{R}^n$ with coefficients in $\mathbb{C}[x_1, \ldots, x_n]$.

   For $k \geq 0$, let $\mathbb{C}^k[x_1, \ldots, x_n]$ be the space of homogeneous polynomials of degree k and $H^k[x_1, \ldots, x_n] \subset \mathbb{C}^k[x_1, \ldots, x_n]$ be the subspace of polynomials f such that $\Delta(f) = 0$.
   (ii) Prove that $H^k[x_1, \ldots, x_n] \cap (r^2 \cdot \mathbb{C}^{k-2}[x_1, \ldots, x_n]) = 0$, for any $k \geq 2$.
   [Hint: Use (i) and an adaptation of arguments from Problems 7 and 8.]

Assignment 7: due Friday, November 16

1. Fix $n \geq 1$ and $k \in [1, n]$. For each $r = 1, 2, \ldots, n$, let $e_{r,k} := e_{r-1}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$, an elementary symmetric function in the variables $\{x_1, \ldots, x_n\} \smallsetminus \{x_k\}$.
   (i) Prove the identity:
$$H_n(t) \cdot E_k(-t) = \frac{1}{1 - x_k t}, \quad \text{where} \quad H_n(t) := \prod_{1 \leq i \leq n} \frac{1}{1 - x_i t} \text{ and } E_k(t) := 1 + \sum_{1 \leq r \leq n} e_{r,k} t^r.$$
   Fix $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and consider the following $n \times n$-matrices with polynomial entries:
$$A_\alpha = |a_{i,j}|,\ a_{ij} := (x_j)^{\alpha_i}, \quad B_\alpha = |b_{i,j}|,\ b_{ij} := h_{\alpha_i - n + j}, \quad C = |c_{i,j}|,\ c_{ij} := (-1)^i \cdot e_{n-i,j}.$$
   (ii) Prove the matrix identity: $A_\alpha = B_\alpha \cdot C$. Deduce that $\det A_\alpha = \det B_\alpha \cdot \det C$.
   (iii) Check that, for $\rho = (n-1, n-2, \ldots, 1)$, we have $\det B_\rho = 1$.
   (iv) Prove the determinantal identity: $a(x^\alpha) = a(x^\rho) \cdot \det B_\alpha$.
2. Fix $n \geq 1$ and let $D_n(x)$ and $D_n(y)$ denote the Vandermonde determinants in the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, respectively. Let $D(x, y)$ be an $n \times n$-matrix with entries $d_{ij} := (x_i - y_j)^{-1}$. (i) Check that the rational function $f(x, y)$ below is actually a polynomial:
$$f(x, y) := \det D(x, y) \cdot \prod_{1 \leq i,j \leq n} (x_i - y_j).$$
   Show that the polynomial $D_n(x) \cdot D_n(y)$ divides $f(x, y)$ in $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.
   (ii) Show that $f(x, y) = c \cdot D_n(x) \cdot D_n(y)$ for some nonzero constant $c \in \mathbb{Q}$.
   (iii) Deduce the Cauchy identity: $\det D(x, y) = D_n(x) \cdot D_n(y) \cdot \prod_{i,j} (x_i - y_j)^{-1}$.
3. Fix $n \geq k > 0$. Let $s_\lambda \in R_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ be the Schur polynomial associated with a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$. Show that for any nonzero $q \in \mathbb{C}$, such that q is not a root of unity, one has
$$s_\lambda(1, q, q^2, \ldots, q^{n-1}) = \prod_{1 \leq i < j \leq n} \frac{q^{\lambda_i - i} - q^{\lambda_j - j}}{q^{-i} - q^{-j}}.$$
4. Fix $n, r \geq 0$, and put $M = \mathbb{C}^n$. The group $S_r \times \mathrm{GL}(M)$ acts naturally on $M^{\otimes r} = M \otimes M \ldots \otimes M$, where S_r acts by permutation of the r tensor factors, and $g \in \mathrm{GL}(M)$ acts by $g : m_1 \otimes \ldots \otimes m_r \mapsto g(m_1) \otimes \ldots \otimes g(m_r)$. The subspace $\mathrm{Sym}^r M = (M^{\otimes r})^{S_r}$, resp. $\wedge^r M = (M^{\otimes r})^{\mathrm{sign}}$, of symmetric, resp. skew-symmetric, tensors is $\mathrm{GL}(M)$-stable.
   (i) Show that, for $g = \mathrm{diag}(z_1, \ldots, z_n)$, the diagonal matrix with eigenvalues $z_1, \ldots, z_n \in \mathbb{C}^\times$, one has $\mathrm{tr}_{\,\mathrm{Sym}^r M}(g) = h_r(z_1, \ldots, z_n)$, resp. $\mathrm{tr}_{\,\wedge^r M}(g) = e_r(z_1, \ldots, z_n)$.
   Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$, be a partition of r and for each $\ell > 0$ put $d_\ell(\lambda) := \#\{i \mid \lambda_i = \ell\}$. Let $s \in S_r$ be an element in the conjugacy class with cycle type λ and let $g = \mathrm{diag}(z_1, \ldots, z_n) \in \mathrm{GL}(M)$, as in (i). The elements s and g give a pair of commuting linear operators on $M^{\otimes r}$.
   (ii) Prove that the trace of their composition is given by the formula
$$\mathrm{tr}_{\,M^{\otimes r}}(g \circ s) = \prod\nolimits_\ell \ p_\ell(z_1, \ldots, z_n)^{d_\ell(\lambda)}, \qquad \text{where } \ p_\ell(x) := x_1^\ell + \ldots + x_n^\ell.$$
5. Let the group S_n be imbedded into $\mathrm{GL}(\mathbb{C}^n)$ via permutation matrices and $E := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1 + \ldots + z_n = 0\}$. Thus, E is an S_n-stable subspace of $\mathbb{C}^n$ of dimension $n - 1$ and, for each $r = 1, 2, \ldots, n$, one has a natural S_n-action on $\wedge^r E$.
   We claim that there is an isomorphism between the S_n-representation $\wedge^r E$ and the Specht module V_λ for an appropriate partition $\lambda = \lambda(r)$, of n. (In particular, $\wedge^r E$ is an irreducible representation).
   (i) Prove the claim and find $\lambda(r)$ in the special cases $r = 1, 2$. (ii) Prove the general case.

6. Let $a, b \in \mathrm{M}_n(\mathbb{R})$. Show that the function $f : \mathbb{R}^2 \to \mathrm{M}_n(\mathbb{R})$, $(t, s) \mapsto e^{t \cdot a} e^{s \cdot b} e^{-t \cdot a} e^{-s \cdot b}$ is differentiable any number of times and one has

$$\left(\frac{\partial^2 f}{\partial t\, \partial s}\right)\Big|_{t=s=0} = ab - ba.$$

Below, V is a finite dimensional $\mathbb{C}$-vector space. Write $\mathcal{D}(V)$ for the algebra of differential operators on V with constant coefficients. Given a subgroup $G \subset \mathrm{GL}(V)$, let $\mathbb{C}^i[V]^G$, resp. $\mathcal{D}^i(V)^G \cong (\mathrm{Sym}^i\, V)^G$, be the vector space of G-invariant homogeneous polynomials, resp. G-invariant homogeneous differential operators with constant coefficients, of degree i. One has direct sum decompositions:

$$\mathbb{C}[V]^G = \oplus_{i \geq 0}\ \mathbb{C}^i[V]^G \quad \text{and} \quad \mathcal{D}(V)^G = \oplus_{i \geq 0}\ \mathcal{D}^i(V)^G.$$

Definition. A polynomial $f \in \mathbb{C}[V]$ is called *G-harmonic* if, for all $i > 0$ and $u \in \mathcal{D}^i(V)^G$, we have $u(f) = 0$. Let $\mathit{Harm}(G, V) \subset \mathbb{C}[V]$ be the vector space of G-harmonic polynomials on V and $\mathit{Harm}^k(G, V) \subset \mathit{Harm}(G, V)$ the subspace of homogeneous G-harmonic polynomials of degree k.

7. The special orthogonal group $G = \mathrm{SO}(\mathbb{C}^n)$ acts naturally on $V = \mathbb{C}^n$. We identify $\mathbb{C}[V]$ with, $\mathbb{C}[x_1, \ldots, x_n]$ and let $\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ be the Laplace operator.
   (i) Show that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is G-harmonic iff one has $\Delta(f) = 0$.
   (ii) Prove that, for any $k \geq 2$, the Laplace operator gives a surjective linear map $\Delta : \mathbb{C}^k[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}^{k-2}[x_1, \ldots, x_n]$. Find $\dim \mathit{Harm}^k(V, G)$.
   (iii) In the setting of Problem 9 of Assignment 6, prove that one has a direct sum decomposition
   $$\mathbb{C}^k[x_1, \ldots, x_n] = \mathit{Harm}^k(V, G)\ \oplus\ r^2 \cdot \mathbb{C}^{k-2}[x_1, \ldots, x_n], \qquad k \geq 2.$$
   (iv) Prove that multiplication in the algebra $\mathbb{C}[x_1, \ldots, x_n]$ induces a vector space isomorphism
   $$\mathbb{C}[r^2] \otimes_{\mathbb{C}} \mathit{Harm}(V, G)\ \xrightarrow{\sim}\ \mathbb{C}[V].$$

8. Let the group $G \subset \mathrm{GL}(V)$ be compact and let $\int_G$ be an invariant integral on G. Show that a polynomial $f \in \mathbb{C}[V]$ is G-harmonic iff the following 'mean-value property' holds
$$f(v_0) = \frac{1}{\mathsf{vol}(G)} \int_G f(v_0 + g(v)), \qquad \forall v_0, v \in V.$$

9. Let $S_n \hookrightarrow \mathrm{GL}(\mathbb{C}^n)$ be the embedding via permutation matrices. Let $V_\lambda \subset \mathbb{C}[x_1, \ldots, x_n]$ be the Specht module associated with a partition λ, of n.
   Prove that any element $f \in V_\lambda$ is an S_n-harmonic polynomial on $\mathbb{C}^n$.

Notation: Let $I(V, G) \subset \mathbb{C}[V]$, resp. $I^\vee(V, G) \subset \mathcal{D}(V)$, be an ideal generated by the set of all homogeneous G-invariant elements of the algebra $\mathbb{C}[V]$, resp. $\mathcal{D}(V)$, of (strictly) positive degree.

10. (i) Show that a polynomial $f \in \mathbb{C}[V]$ is harmonic iff one has $u(f)(0) = 0$ for all $u \in I^\vee(V, G)$.
   (ii) Let the group G be compact. Construct a G-invariant positive definite hermitian inner product $(-,-)_{\mathbb{C}[V]}$, on $\mathbb{C}[V]$, such that $(\mathbb{C}^i[V], \mathbb{C}^j[V])_{\mathbb{C}[V]} = 0$ for all $i \neq j$ and, in addition, one has
   $$\mathit{Harm}(V, G) = I(V, G)^\perp, \quad \text{where} \quad I(V, G)^\perp := \{f \in \mathbb{C}[V] \mid (f,\, I(V, G))_{\mathbb{C}[V]} = 0\}.$$

Assignment 8: due **Wednesday**, November 28

Notation: Given a $\Bbbk$-algebra A and an A-module M, let $\mathrm{act}_M : A \to \mathrm{End}_{\Bbbk} M$ denote the action map. If M has finite dimension over $\Bbbk$ then, for $a \in A$, we let $\mathrm{Tr}_M(a)$ be the trace of the $\Bbbk$-linear operator $\mathrm{act}_M(a) : M \to M$.

1. Let A be a $\Bbbk$-algebra and $V_1, \ldots, V_n$ a finite collection of pairwise nonisomorphic finite dimensional simple A-modules. Show that
   (i) There exists an element $a \in A$ such that $\mathrm{act}_{V_1}(a) = \mathrm{Id}_{V_1}$ and $\mathrm{act}_{V_i}(a) = 0$, for any $i \neq 1$.
   (ii) If, in addition, $\Bbbk = \bar{\Bbbk}$ then the following algebra homomorphism is surjective
$$A \to \mathrm{End}_{\Bbbk}(V_1) \oplus \ldots \oplus \mathrm{End}_{\Bbbk}(V_n), \quad a \mapsto \mathrm{act}_{V_1}(a) \oplus \ldots \oplus \mathrm{act}_{V_n}(a).$$

2. Prove that, for a finite dimensional $\Bbbk$-algebra A, the following three properties are equivalent:
   (1) A is simple;
   (2) There is an integer $r \geq 1$ and an isomorphism $A \cong \mathrm{M}_r(D)$, where D is a finite dimensional division algebra over $\Bbbk$. [Hint to (1) $\Rightarrow$ (2): consider the action of A in a simple A-module.]
   (3) The center of A is a field K, which is a finite extension of $\Bbbk$ (so $K \subset \bar{\Bbbk}$), and there is a $\bar{\Bbbk}$-algebra isomorphism $A \otimes_K \bar{\Bbbk} \ \cong\ \mathrm{M}_n(\bar{\Bbbk})$, for some integer $n \geq 1$.

In Problems 3-6, $\Bbbk$ is a field of characteristic zero.

3. Let A be a $\Bbbk$-algebra. Let M and N be a pair of finite dimensional A-modules such that, for all $a \in A$, one has $\mathrm{Tr}_M(a) = \mathrm{Tr}_N(a)$. Show that if both M and N are semisimple modules then $M \cong N$. Give an example of not necessarily semisimple modules such that the statement fails.

4. For any $n \geq 1$, define the *Clifford algebra* $C_{2n}(\Bbbk)$, on $2n$ generators, as a quotient of the free associative algebra $\Bbbk\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ by a two-sided ideal generated by the set
$$\{x_i y_j + y_j x_i - \delta_{ij},\ \ x_i x_j + x_j x_i,\ \ y_i y_j + y_j y_i,\ \ i, j = 1, \ldots, n\}, \qquad \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$
   (i) Prove that $C_{2n}(\Bbbk)$ is a central simple algebra.
   (ii) Construct an explicit example of a simple $C_{2n}(\Bbbk)$-module L such that $\dim_{\Bbbk} L = 2^n$.
   (iii) Prove an algebra isomorphism $C_{2n}(\Bbbk) \cong \mathrm{M}_{2^n}(\Bbbk)$.

5. Let A be a $\Bbbk$-algebra. Associated with any finite dimensional A-module M, there is a symmetric $\Bbbk$-bilinear form $\beta_M :\ A \times A \to \Bbbk,\ a, b \mapsto \mathrm{Tr}_M(a \cdot b)$. Given a $\Bbbk$-vector subspace $V \subset A$, let $V^\perp := \{a \in A \mid \mathrm{Tr}_M(a \cdot v) = 0,\ \forall v \in V\}$ be the annihilator of V with respect to β_M.
   (i) Show that for any two-sided ideal $I \subset A$, the set $I^\perp$ is also a two-sided ideal;
   (ii) Prove that for any element a in $A^\perp$, the radical of the bilinear form β_M, the operator $\mathrm{act}_M(a) : M \to M$ is nilpotent.

6. Prove that the following properties of a finite dimensional algebra A are equivalent:
   (1) A is a semisimple algebra;
   (2) A is a direct sum of simple $\Bbbk$-algebras;
   (3) There exists a finite dimensional A-module M such that the bilinear form β_M is nondegenerate;
   (4) The bilinear form β_A is nondegenerate;
   (5) The algebra A has no nonzero two-sided ideals consisting of nilpotent elements;
   (6) $J(A) = 0$.

7. Let $\mathcal{P}_r$ be the set of partitions of $r \geq 1$, and $\lambda, \nu \in \mathcal{P}_r$. Let p_ν, resp. s_ν, denote the corresponding power sum, resp. Schur, polynomial, and $C_\nu \subset S_r$ the conjugacy class with cycle type ν. Let $\chi_{V^\lambda} \in \mathbb{C}\{S_r\}^{S_r}$ be the image of $V^\lambda \in K_{\mathbb{C}}(S_r)$ under the character map $\chi : K_{\mathbb{C}}(S_r) \to \mathbb{C}\{S_r\}^{S_r}$.
   (i) Prove that one has an equality of symmetric functions:
$$p_\nu = \sum\nolimits_{\lambda \in \mathcal{P}_r} \chi_{V^\lambda}(C_\nu) \cdot s_\lambda.$$
   (ii) Fix an integer $n \geq r$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n) \in \mathcal{P}_r$ and $\mu := \lambda + \rho$, where $\rho = (n-1 \geq n-2 \geq \ldots \geq 1 \geq 0)$. Put $\mu! := \mu_1! \cdot \ldots \cdot \mu_n!$ and $x^\mu = x_1^{\mu_1} \ldots x_n^{\mu_n}$. Deduce that the integer $\chi_{V^\lambda}(1)$ equals the coefficient in front of the monomial x^μ in the polynomial $(x_1 + \ldots + x_n)^r \cdot a(x^\rho)$.
   (iii) For any $i, j \in [1, n]$, let $a_{i,j} := \mu_i(\mu_{i-1} - 1) \cdot \ldots \cdot (\mu_i - n + j + 1)$. Prove the formulas
$$\chi_{V^\lambda}(1) = \sum_{\{s \in S_n \mid \mu_i \geq n - s(i)\}} \operatorname{sign}(s) \cdot \frac{r!}{\prod_{1 \leq i \leq n} (\mu_i - n + s(i))!}$$
$$= \frac{r!}{\mu!} \cdot \det \|a_{i,j}\| = \frac{r!}{\mu!} \cdot \prod_{1 \leq i < j \leq n} (\mu_i - \mu_j).$$
   From the inequality $\chi_{V^\lambda}(1) > 0$ conclude that $V^\lambda = [V_\lambda]$ and the dimension of the Specht module V_λ is given by the above formula for $\chi_{V^\lambda}(1)$.

8. Let M be a vector space of dimension $n \geq r$. The groups S_r and $\mathrm{GL}(M)$ act naturally on $M^{\otimes r}$.
   (i) Show that for any $s \in S_r$ and $g \in \mathrm{GL}(M)$, we have
$$\mathrm{Tr}_{M^{\otimes r}}(s \circ g) = \sum\nolimits_{\lambda \in \mathcal{P}_r} \mathrm{Tr}_{V_\lambda}(s) \cdot \mathrm{Tr}_{L_\lambda}(g).$$
   (ii) Use Problem 7(i) above and Problem 4 of Assignment 7 to prove *Weyl's character formula*:
$$\mathrm{Tr}_{L_\lambda}(g) = s_\lambda(z_1, \ldots, z_n), \qquad \forall\, \lambda \in \mathcal{P}_r,\ g = \mathrm{diag}(z_1, \ldots, z_n) \in \mathrm{GL}(M).$$
   (iii) Let $\mu := \lambda + \rho$, as in Problem 7. Deduce the dimension formula:
$$\dim L_\lambda = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \quad \left(= \prod_{1 \leq i < j \leq n} \frac{\mu_i - \mu_j}{\rho_i - \rho_j} \right)$$

9. Let $n > 1$ and let $\Bbbk$ be a field that contains a primitive n-th root of unity $\zeta \in \Bbbk$. Given $a, b \in \Bbbk \smallsetminus \{0\}$, let $\mathbb{H}_{a,b}$ be an associative $\Bbbk$-algebra with two generators x, y, and the following defining relations:
$$x^n = a, \quad y^n = b, \quad yx = \zeta \cdot xy.$$
   Show that $\mathbb{H}_{a,b}$ is a central simple $\Bbbk$-algebra (sometimes called '*generalized Quaterion algebra*').

10. (Optional problem) Given central simple $\Bbbk$-algebras A and B we write $A \sim B$ if there exist $m, n \geq 1$ such that $\mathrm{M}_m(A) \cong \mathrm{M}_n(B)$.
   (i) Show that '$\sim$' is an equivalence relation.
   Let $Br(\Bbbk)$ be the set of equivalence classes of central simple $\Bbbk$-algebras. Write $[A] \in Br(\Bbbk)$ for the class of a central simple algebra A.
   (ii) Show that the formula $[A] \cdot [B] := [A \otimes B]$ gives a well defined operation on $Br(\Bbbk)$. Furthermore, this operation is associative and commutative, and the class $[\Bbbk]$ is a neutral element.
   (iii) Show that in $Br(\Bbbk)$, we have $[A] \cdot [A^{op}] = [\Bbbk]$.
   Thus, the operation on $Br(\Bbbk)$ makes it an abelian group, called the *Brower group* of $\Bbbk$.
   (iv) Show that $Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.
   Remark: One of central results of Class Field Theory says that $Br(\mathbb{Q}_p)$, the Brower group of the field of p-adic numbers, is isomorphic to the group $(\mathbb{Q}/\mathbb{Z}, +)$.

1. The Jacobson Radical

Let A be an arbitrary ring. Given a left A-module M, and $m \in M$, define $\mathtt{Ann}_M(m) = \{a \in A \mid a \cdot m = 0\}$. This is a left ideal in A. The map $a \mapsto a \cdot m$ gives a morphism $A \to M$, of left A-modules. Hence, we obtain an A-module embedding $A/\mathtt{Ann}_M(m) \to M$.

We put $\mathtt{Ann}_M = \bigcap_{m \in M} \mathtt{Ann}_M(m)$. Equivalently, $\mathtt{Ann}_M$ is the kernel of the ring homomorphism $A \to \mathrm{End}_{\mathbb{Z}} M,\ a \mapsto \mathrm{act}_M(a)$. Thus, $\mathtt{Ann}_M$ is a two-sided ideal in A.

Let S_A be the set of isomorphism classes of simple A-modules. The following result gives various equivalent definitions of the Jacobson Radical:

Theorem 1. *For any ring A, the following 7 sets are equal:*

- *(1):* $\{a \in A \mid \forall x, y \in A,\ 1 + xay$ *is invertible*$\}$
- *(2L)* $\{a \in A \mid \forall x \in A,\ 1 + xa$ *has a left inverse*$\}$
- *(2R)* $\{a \in A \mid \forall x \in A,\ 1 + ax$ *has a right inverse*$\}$
- *(3L) The intersection of all maximal left ideals in A.*
- *(3R) The intersection of all maximal right ideals in A.*
- *(4L)* $\bigcap_{L \in S_A} \mathtt{Ann}_L$
- *(4R) The intersection of the annihilators of all simple right A-modules.*

The set $J(A)$ defined by these conditions is a two-sided *ideal in A, thanks to (4L)-(4R), called the* Jacobson Radical *of A.*

Proof. We will prove $4L \iff 3L \iff 2L \iff 1$. The corresponding claims for 'R' will follow.

($4L \iff 3L$): Let $\mathcal{M} = \{(M, m) \mid M \in S_A,\ m \in M \smallsetminus \{0\}\}$. Define a map from $\mathcal{M}$ to the set of maximal left ideals by $(M, m) \mapsto \mathtt{Ann}_M(m)$. This map is surjective since simple A-modules are precisely the A-modules of the form A/I for a maximal left ideal I of A. Hence, the intersection of all maximal left ideals of A is equal to

$$\bigcap_{(M,m)\in\mathcal{M}} \mathtt{Ann}_M(m) = \bigcap_{M \in S_A} \bigcap_{m \in M \smallsetminus \{0\}} \mathtt{Ann}_M(m) = \bigcap_{M \in S_A} \mathtt{Ann}_M.$$

We note that the rightmost intersection above is clearly a two-sided ideal.

($3L \iff 2L$): $a \in A$ has no left inverse iff it is contained in a proper left ideal iff it is contained in a maximal left ideal.

Let $\mathcal{L}$ be the set of elements $a \in A$ such that $1 + xa$ has a left inverse for all $x \in A$. Let $a \in \mathcal{L}$ and assume there is a maximal left ideal I not containing a. Then $Aa + I = A \Rightarrow 1 = xa + y$ (for some $x \in A, y \in I$) $\Rightarrow y = 1 + xa$. Since $a \in \mathcal{L}$, then y has a left inverse, which is impossible since it is in I. Therefore, $\mathcal{L} \subseteq I$ for all maximal left ideals I.

Conversely, assume a is in all maximal left ideals, but there is x such that $1 + xa$ has no left inverse. Then $1 + xa$ is in some maximal left ideal I. But xa is also in this ideal, so 1 is in this ideal, a contradiction.

From above, we deduce that the set $\mathcal{L}$ is a two-sided ideal. We will use this in the next step:

($2L \iff 1$): Let $a' \in \mathcal{L}$ and for arbitrary $x, y \in A$ set $a = xa'y$. Then $a \in \mathcal{L}$ since $\mathcal{L}$ is a two-sided ideal. Hence, $1 + a$ has a left inverse b, so $b(1 + a) = 1 \Rightarrow b = 1 - ba$. Since $a \in \mathcal{L}$, then b has a left inverse c, so that $cb = 1$. But $c = c \cdot 1 = c(b(1+a)) = cb(1+a) = 1 + a$. Thus $(1+a)b = 1$, so $1 + a$ has a two-sided inverse, whence $(2L) \Rightarrow (1)$. The reverse inclusion is clear. $\square$

Lemma 2 (Nakayama lemma)**.** *Let M be a finitely-generated A-module such that $J(A) \cdot M = M$. Then $M = 0$.*

Proof. Find a minimal generating set $m_1, \ldots, m_n$. Then we have $M = J(A) \cdot M = \sum J(A)m_i$. Then we may write $m_n = j_1 m_1 + \ldots + j_{n-1} m_{n-1} + j_n m_n$ for $j_i \in J(A)$. But then we have $(1 - j_n) m_n = j_1 m_1 + \ldots + j_{n-1} m_{n-1}$. Since $j_n \in J(A)$, then $1 - j_n$ is invertible, so that m_n is actually in the span of $\{m_i \mid i < n\}$, so we have a smaller generating set, a contradiction. $\square$

Assignment 9: due Friday, December 7

1. Let A be a finitely generated $\mathbb{C}$-algebra and let $a \in A$ be a *non*algebraic element. Show that there are uncountably many $\lambda \in \mathbb{C}$ such that the element $a - \lambda$ is not a zero divisor but, at the same time, it is not invertible. (Thus, as opposed to the case of finite dimensional algebras, 'most' of noninvertible elements of A are *not* zero-divisors!)

2. Let A be a central simple $\Bbbk$-algebra. Prove that:
(i) Any two A-modules of the same dimension over $\Bbbk$ are isomorphic (as A-modules).
(ii) Any $\Bbbk$-linear algebra automorphism $\phi : A \to A$ is an *inner* automorphism, i.e., there exists an invertible element $u \in A$ such that $\phi(a) = u \cdot a \cdot u^{-1}$ for all $a \in A$.
[Hint for (ii): Given ϕ, define an A-action on the vector space A by the formula $a(a') := \phi(a) \cdot a'$, $\forall a, a' \in A$. This action gives A the structure of a left A-module. Deduce statement (ii) by observing that the constructed A-module must be a rank 1 free A-module, by (i).]

3. Let A and B be finitely generated $\mathbb{C}$-algebras. Show that for any simple modules M and N, over A and B respectively, the $A \otimes_{\mathbb{C}} B$-module $M \otimes_{\mathbb{C}} N$ is simple. (Recall that the action of $A \otimes_{\mathbb{C}} B$ on $M \otimes_{\mathbb{C}} N$ is defined by the formula $(a \otimes b)(m \otimes n) := am \otimes bn$, for all $a \in A, b \in B, m \in M, n \in N$.)

4. Let $A \subset \mathbb{Q}$ be a subring formed by all rational numbers of the form p/q, where p, q are integers such that $\mathsf{gcd}(p,q) = 1$ and q is *odd*. Let B be the ring of 2×2-matrices of the form $\begin{pmatrix} a & u \\ 0 & v \end{pmatrix}$, $a \in A$, $u, v \in \mathbb{Q}$. Find $J(A)$ and prove that $\cap_{n\geq 1} J(B)^n \neq 0$ using the inclusion:
$$J(B) \supseteq \begin{pmatrix} J(A) & \mathbb{Q} \\ 0 & 0 \end{pmatrix}.$$

5. Associated with an arbitrary direct sum $E = \oplus_{i\geq 0}\, E_i$, of finite dimensional vector spaces E_i, there is a formal power series P_E, with nonnegative integer coefficients, called *Poincaré series*:
$$P_E(t) = \sum_{i=0}^{\infty} \dim E_i \cdot t^i.$$
Given a vector subspace $E \subset \Bbbk[x_1, \ldots, x_n]$, we put $E_i := E \cap \Bbbk^i[x_1, \ldots, x_n]$, $i = 0, 1, \ldots$. We say that E is a *graded* subspace of $\Bbbk[x_1, \ldots, x_n]$ if the natural inclusion $\sum_{i\geq 0}\, E_i \subseteq E$ is an equality. In that case, we have a direct sum decomposition $E = \oplus_{i\geq 0}\, E_i$.
Let $A \subset \Bbbk[x_1, \ldots, x_n]$ be a subalgebra, which is also a graded subspace $A = \oplus_{i\geq 0}\, A_i$. One can write $A = A_0 \oplus A_{>0}$, where we have $A_0 = \Bbbk^0[x_1, \ldots, x_n] = \Bbbk$ and $A_{>0} := \oplus_{i\geq 1}\, A_i$ is a graded ideal of A, called *augmentation ideal.*
(i) Show that the ideal $A_{>0}$ is finitely generated (as an ideal) iff A is finitely generated as a $\Bbbk$-algebra.
(ii) Let $I := \Bbbk[x_1, \ldots, x_n] \cdot A_{>0}$ be an ideal of the algebra $\Bbbk[x_1, \ldots, x_n]$ generated by the set $A_{>0}$. It is clear that I is a graded subspace of $\Bbbk[x_1, \ldots, x_n]$ and one has a vector space decomposition $\Bbbk[x_1, \ldots, x_n]/I = \oplus_{i\geq 0}\, (\Bbbk^i[x_1, \ldots, x_n]/I_i)$. Let $H \subset \Bbbk[x_1, \ldots, x_n]$ be a graded vector subspace such that $\Bbbk[x_1, \ldots, x_n] = I \oplus H$. Show that the following three properties are equivalent:
(1) $\Bbbk[x_1, \ldots, x_n]$ is free as an A-module;
(2) The map $A \otimes_{\Bbbk} H \to \Bbbk[x_1, \ldots, x_n]$ induced by multiplication in the algebra $\Bbbk[x_1, \ldots, x_n]$ is a vector space isomorphism;
(3) One has an equality: $P_{\Bbbk[x_1,\ldots,x_n]}(t) = P_A(t) \cdot P_{\Bbbk[x_1,\ldots,x_n]/I}(t)$, of formal power series.

6. (i) Find closed formulas for Poincaré series $P_{\mathbb{C}[x_1,\ldots,x_n]}$ and $P_{\mathbb{C}[x_1,\ldots,x_n]^{S_n}}$.

Let $Harm(\mathbb{C}^n, S_n) \subset \mathbb{C}[x_1,\ldots,x_n]$ be the space of S_n-harmonic polynomials on $\mathbb{C}^n$ (with respect to the natural representation $S_n \to \mathrm{GL}(\mathbb{C}^n)$ via permutation matrices).

(ii) Use Problem 3 of Assignment 6 and Problem 10 of Assignment 7 to show that the map

$$\mathbb{C}[x_1,\ldots,x_n]^{S_n} \otimes Harm(\mathbb{C}^n, S_n) \to \mathbb{C}[x_1,\ldots,x_n],$$

induced by multiplication, is a vector space isomorphism.

(iii) Find the Poincaré series $P_{Harm(\mathbb{C}^n,S_n)}$. Deduce that $Harm^i(\mathbb{C}^n, S_n) = 0$ for all $i > \frac{n(n-1)}{2}$ and that the Vandermonde polynomial D_n is, up to a constant factor, the only homogeneous harmonic polynomial of degree $\frac{n(n-1)}{2}$.

(iv) Show that $\dim Harm(\mathbb{C}^n, S_n) = n!$.

7. Let A be a Banach $\mathbb{C}$-algebra with norm $\mathsf{N}(-)$ and let $a \in A$.

(i) Show that the set $\operatorname{spec} a$ is a closed subset of the disc $\{z \in \mathbb{C} \mid |z| \leq \mathsf{N}(a)\}$.

(ii) Prove a more precise formula:

$$\max_{z \,\in\, \operatorname{spec} a} |z| \;=\; \limsup_{n\to\infty} \mathsf{N}(a^n)^{\frac{1}{n}}.$$

[Hint: find the radius of convergence of the Taylor expansion of the function $z \mapsto (1 - z \cdot a)^{-1}$.]

8. Let $K_{\mathbb{C}}(G)$ be the Grothendieck group (over $\mathbb{C}$) of finite dimensional representations of a finite group G. Associated with any such representation V, there is a linear operator $t_V : K_{\mathbb{C}}(G) \to K_{\mathbb{C}}(G)$ given by the assignment $[M] \mapsto t_V([M]) := [V \otimes M]$. Show that the element $[\mathbb{C}G] \in K_{\mathbb{C}}(G)$, the class of the regular representation of G, is an eigen-vector of the operator t_V, for any G-representation V.

9. (optional) Let Q be a quiver with finitely many edges and such that the underlying graph is connected. Let $I = \{1,\ldots,n\}$ be the vertex set of Q, so we have $\mathbb{R}\{I\} \cong \mathbb{R}^n$.

For each pair $i, j \in I$, let $a_{i,j}$ be the number of edges $j \to i$, with head i and tail j. The $n \times n$-matrix $A_Q = \|a_{i,j}\|_{i,j\in I}$ is called the *adjacency matrix* of Q. Abusing the notation, we will also write A_Q for a linear map $\mathbb{R}^n \to \mathbb{R}^n$ given by the matrix $\|a_{i,j}\|_{i,j\in I}$ (in the standard basis i.e., the basis formed by the characteristic functions $1_i \in \mathbb{R}\{I\} = \mathbb{R}^n,\ i \in I$). We put

$$\alpha_Q := \max_{z \,\in\, \operatorname{spec} A_Q} |z|, \qquad C_Q(x) := \alpha_Q \cdot \sum\nolimits_{k\in I}\ x_k^2 \;-\; \sum\nolimits_{i,j\in I}\ a_{i,j} x_i x_j,$$

a quadratic form on $\mathbb{R}^n$ associated with the matrix $C_Q := \alpha_Q \cdot \mathrm{Id} - A_Q$.

(i) Prove that there exists a vector $\delta \in \mathbb{R}^n$, with positive coordinates $\delta_i > 0,\ \forall i = 1,\ldots,n$, such that $A_Q(\delta) = c \cdot \delta$, for some real constant $c > 0$. [Hint: Use Brower's fixed point theorem.]

(ii) Show that $c = \alpha_Q$.

(iii) Show that if A_Q, the adjacency matrix, is symmetric then one has an identity

$$C_Q(x) = \frac{1}{2} \sum_{i \neq j}\ a_{i,j}\, \delta_i\, \delta_j \left(\frac{x_i}{\delta_i} - \frac{x_j}{\delta_j}\right)^2, \qquad \forall\, x = (x_1,\ldots,x_n) \in \mathbb{R}^n.$$

Thus, $C_Q(x) \geq 0$ and $C_Q(x) = 0$ holds iff $x \in \mathbb{R} \cdot \delta$, so the vector δ is determined by the quiver.

Let $G \subset \mathrm{GL}(V)$ be a finite subgroup and $Q = Q_G$ the corresponding McKay quiver, so $I = \widehat{G}$. Let $\delta \in \mathbb{R}^n$ be a vector with coordinates $\delta_i := \dim L_i$ ($i \in I$), the dimension of the i-th irreducible G-representation.

(iv) Prove that $A_Q(\delta) = \dim V \cdot \delta$; hence, for McKay quivers, one has $\alpha_Q = \dim V$.

(v) Prove that, for $G \subset \mathrm{SL}(\mathbb{C}^2)$, the adjacency matrix of Q_G is symmetric, furthermore, $a_{i,i} = 0$ for all $i \in I$. Thus, in this case, we get that $C_Q(x) = 2\sum_{k\in I} x_k^2 - \sum_{i\neq j} a_{i,j} x_i x_j$ is a positive semi-definite form.