Algebraic Topology 2012: Some motivating problems

The following list is a small sampling of questions that will motivate our study this quarter.

(1) **Homeomorphism classification problems:** Find general methods for proving that two given spaces are not homeomorphic. For example:
  (a) How can we prove that the spheres S^n and S^n are homeomorphic if and only if $n = m$?
  (b) Classify all closed (i.e. compact without boundary) n-dimensional manifolds up to homeomorphism. For example, how can we prove that the 2-sphere S^2 is not homeomorphic to the torus? More generally, if S_g denotes the closed, oriented surface of genus g (i.e. the "sphere with g handles"), how to prove that $S_g \approx S_h$ if and only if $g = h$?

(2) **Maps:**
  (a) Given two spaces X, Y, what are all the continuous maps $f : X \to Y$ up to continuous deformation (i.e. homotopy)? $X = Y = S^n$ is already nontrivial. The case $f : S^n \to S^m, m < n$ is open in general.
  (b) Given a space X and a subspace A, does there exist a continuous map $r : X \to A$ so that r fixes A pointwise? Such a map is called a *retraction* of X onto A.
  (c) Borsuk-Ulam Theorem: For any continuous map $f : S^n \to \boldsymbol{R}^n$, there exists $v \in S^n$ so that $f(v) = f(-v)$. So for example, assuming continuity of temperature and barometric pressure on the surface of the Earth, at any fixed time there exists a pair of antipodal points with simultaneously equal temperatures and barometric pressures! Or: take a sphere, crumple it up and throw it on the ground. Then two antipodal points will be be on top of each other.

(3) **Combinatorial applications:**
  (a) Any n finite mass distributions in $\boldsymbol{R}^n$ can be bisected by a single hyperplane.
  (b) The Euler characteristic of any triangulable space X does not depend on the triangulation. The case $X = S^2$ is Euler's famous $V - E + F = 2$ theorem.
  (c) The Euler characteristic of any closed, odd-dimensional manifold equals zero.

(4) **Fixed points and group actions:**
  (a) Let X be a topological space. How to find out when a continuous map $f : X \to X$ has a *fixed point*, that is, a point $x \in X$ such that $f(x) = x$? For which X does any such f have a fixed point? Prove this for $X = D^n$.

   (b) Which finite groups act freely on the n-sphere? Are there any such actions that are not topologically equivalent (i.e. conjugate in Homeo(S^n)) to a linear action (by orthogonal matrices)?

(5) **Knots and links:** How to prove that a knot is actually knotted? that two circles are actually linked?

(6) **Embeddings:** How to prove that neither $\mathbf{RP}^2$ nor the Klein bottle embed in $\boldsymbol{R}^3$? What is the minimal N for which $\mathbf{RP}^n$ embeds in $\boldsymbol{R}^N$?

(7) **Vector fields:** Given a manifold M, e.g. $M = S^n$, does M admit a vector field that is nonzero at each point? If so, what is the maximal number of linearly independent vector fields on M?

(8) **Group theory:** Fix once and for all groups A and B. What are all the groups G such that A is a normal subgroup of G with $G/A = B$. When A and B are finite simple, one can view this as "step 2" of the classification of finite groups.

(9) **Dynamical systems:** Let $f : M \to M$ be a homeomorphism of a manifold M.
   (a) If M has nonzero Euler characteristic then f has a periodic point.
   (b) The number of periodic points of f of period n and index 1 is of the form $\sum_i \alpha_i^n - \sum_j \beta_j^n$ for some fixed (independent of n) algebraic integers α_i, β_j. Note that for typical sets of algebraic integers, such (differences of) sums are rarely integers for all $n > 0$!
   (c) How can one compute a lower bound for the topological entropy of f?

(10) **Arithmetic geometry:** Let X be a finite system of polynomial equations with integer coefficients. How many solutions does the system have mod p^n? Amazingly, the answer is often governed by, and governs, the topology of the space of complex solutions. Also amazingly, the answer is often of the form $\sum_i \alpha_i^n - \sum_j \beta_j^n$ for some fixed (independent of n) algebraic integers α_i, β_j (compare with the dynamical systems example above).

Each of the above questions can be attacked, and in many cases answered, using the machinery of algebraic topology. We will answer some (but not all) of the problems in this course.

There are vastly more applications of this theory, in areas ranging from algebraic geometry to representation theory to differential geometry to combinatorics to number theory to complex analysis to mathematical physics to finite and infinite group theory (not to mention all flavors of topology). Indeed, algebraic topology has become a central and essential tool in each of these areas.