Course notes in algebraic topology

Benson Farb

CLASS NOTES, Fall 2012
Comments/corrections/suggestions welcome
November 11, 2012

Contents

Chapter 1

Homology

Algebraic topology began in ernest with Poincaré's famous paper *Analysis Situs*. To Poincaré, topology was what we today call "differential topology" . One of the major directions in topology after Poincaré was the development of combinatorial methods, most notably the theory of simplicial complexes, simplicial homology, etc., whereby continuous problems about spaces are converted into purely combinatorial problems that, in principle, can be solved by a computer. The interest in this combinatorialization for its own sake, sometimes called "PL topology", peaked in the 1970's, but it has since become absorbed into many areas of mathematics, from geometric group theory to algebraic combinatorics to number theory.

Because of its wide applicability, and because I believe simplicial homology is the best first example of a homology theory to present, this is how we will begin.

1.1 Construction of simplicial homology

1.1.1 Simplices and Δ-complexes

Our first goal is to define a collection of spaces whose topology can be described in a purely combinatorial way. Classically this was done with simplicial complexes but, as developed in Hatcher's book, there is an even simpler combinatorial structure called a

Δ-complex. While such spaces may seem special, most examples studied in topology admit such structures; examples include smooth manifolds, Eilenberg-MacLane spaces, and algebraic varieties.

Definition 1.1.1 (Affine independence). A set $\{v_0, \ldots, v_n\}$ of vectors in $\mathbb{R}^{n+1}$ is *affinely independent* if one of the following equivalent definitions is satisfied:

1. The set of vectors $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent.
2. If there exist $a_i \in \mathbb{R}$ so that $\sum_{i=0}^{n} a_i v_i = 0$ and $\sum_{i=0}^{n} a_i = 0$ then $a_j = 0$ for each $0 \leq j \leq n$.

Definition 1.1.2 (Simplex). Let $(v_0, \ldots v_n), v_i \in \mathbb{R}^n$ be an ordered $(n+1)$-tuple of affinely independent vectors in $\mathbb{R}^{n+1}$. The *n-dimensional simplex (or simply n-simplex) spanned by* $\{v_0, \ldots v_n\}$, denoted $[v_0 \cdots v_n]$, is defined to be the topological space

$$[v_0 \cdots v_n] := \{\sum_{i=0}^{n} a_i v_i : \text{each } a_i \geq 0 \text{ and } \sum_{i=0}^{n} a_i = 1\}.$$

Fixing the ordered $(n+1)$-tuple v_i, each point $x = \sum_{i=0}^{n} a_i v_i \in [v_0 \cdots v_n]$ is uniquely determined by the a_i; these numbers are called *barycentric coordinates* on $[v_0 \cdots v_n]$.

Thus a 0-simplex $[v_0]$ is a point, a 1-simplex $[v_0 v_1]$ is an edge, a 2-simplex $[v_0 v_1 v_2]$ is a triangle, and a 3-simplex $[v_0 v_1 v_2 v_3]$ is a tetrahedron. Note that $[v_0 \cdots v_n]$ is precisely the convex hull of $\{v_0, \ldots, v_n\}$ in $\mathbb{R}^{n+1}$; that is, $[v_0 \cdots v_n]$ is the smallest convex subset of $\mathbb{R}^{n+1}$ containing $\{v_0, \ldots, v_n\}$.

The ordering of $\{v_i\}$ is part of the data attached to the simplex $\sigma := [v_0 \cdots v_n]$. When we want to emphasize this we will call σ an *ordered simplex.* Note that any reordering of the v_i gives a simplex with span equal to that of σ. The ordering on a simplex σ is useful in a number of respects. For one, for any two n-simplices $\sigma := [v_0 \cdots v_n]$ and $\tau := [w_0 \cdots w_n]$ there is a unique linear homeomorphism $f : \sigma \to \tau$ given by

$$f(\sum a_i v_i) := \sum a_i w_i.$$

Secondly, any subset $\{v_{i_1}, \ldots, v_{i_k}\}$ of $\{v_i\}$, with ordering given by the restriction of the ordering $(v_0, \ldots, v_n)$, gives a $(k-1)$-dimensional simplex $[v_{i_1} \cdots v_{i_k}]$, called a *subsimplex* of $[v_0 \cdots v_n]$. As a special case we have the $(n+1)$ different *faces* of σ, with a different face σ_j of σ for each $0 \leq j \leq n$, given by deleting the j^{th} vertex v_j:

$$\sigma_j := [v_0 \cdots \widehat{v_j} \cdots v_n] \tag{1.1}$$

We would like to build spaces X by taking a finite number of simplices and gluing them together along various subsimplices. Thus such spaces X would have a topology given by the quotient topology. How do we record which simplices glue to which others? A convenient way to do this is to think of each n-simplex in X not as a subset of X, but as the image of a continuous map $f : \Delta^n \to X$ of the standard n-simplex Δ^n into X.

Definition 1.1.3 (**Δ-complex).** Let X be a topological space. A (finite) Δ-*complex structure* on X consists of the following pieces of data:

Decomposition into simplices: A finite collection $\mathcal{S} = \{\Delta_i\}$ of simplices (perhaps of different dimensions), with continuous maps $\sigma_i : \sigma_i \to X$, injective on the interior of σ_i, so that :

1. $\bigcup_i \sigma_i(\Delta_i) = X$.
2. Each $x \in X$ lies in the image of the interior of precisely one simplex.

Closure under taking faces: If $\sigma : \Delta \to X$ is an element of $\mathcal{S}$, then so is the restriction of σ to any subsimplex τ of Δ.

We will also call a Δ-complex structure on X a *triangulation* of X.[1] Sometimes we will simply call X itself a Δ complex. Each pair (σ_i, Δ_i) will be called a *simplex of* X.

One can check that X is homeomorphic to the quotient space given by the disjoint union of the Δ_i with identifications given by $\Delta_i \sim \Delta_j$ precisely when $\sigma_i(\Delta_i) = \sigma_j(\Delta_j)$ as subsets of X.

Notation: We will often denote a simplex $\sigma : \Delta_i \to X$ simply by σ. It is often conceptually simpler to think of σ as its image $\sigma(\Delta)$ in X, in which case one needs to

[1] This differs from the standard usage of the term "triangulation", which is usually reserved for the special case when the Δ complex is actually a simplicial complex.

remember the ordering of the vertices of Δ, which one also thinks of as an ordering of the σ-images of the vertices.

As we will see, Δ-complexes give a huge collection of topological spaces. Later we will allow the collection $\{\sigma_i\}$ to infinite, but this requires us to be more careful about specifying the topology on X.

Actually recording all of the data that determines a Δ-complex gets cumbersome quite quickly. Thus we will use the following shortcut: we simply give a diagram of glued simplices specifying a topological space X, with each simplex endowed with an orientation given by an arrow, with any two identified simplices Δ_i, Δ_j labelled with the same label.

Example 1.1.4. The torus. The Klein bottle. The genus $g \geq 2$ surface. S^n.

Example 1.1.5. Nerve of a cover gives a Δ-complex; indeed a simplicial complex. Same with nerve of a category. To get an ordering on the vertices, first order the elements of the cover, and take the induced ordering on vertices.

1.1.2 Simplicial homology: construction

Our first goal is to attach to any Δ-complex X and any integer $i \geq 0$ a finitely generated abelian group $H_i(X)$, called the i^{th} homology group of X. We want to do this so that at least each of the following simple properties hold:

1. **Computability:** Each $H_i(X)$ is algorithmically computable via basic linear algebra.
2. **Topological invariance:** If X and Y are Δ-complexes, and if X is homeomorphic to Y, then $H_i(X) \approx H_i(Y)$ for each $i \geq 0$.
3. **Nontriviality:** For any n-sphere $S^n, n > 0$, the group $H_n(S^n) \neq 0$.

Of course there are other versions of nontriviality one could ask for. We chose this one because it is simple to state but it already implies the Brouwer Fixed Point Theorem in all dimensions.

Thus the homology groups will give a computable topological invariants for a large collection of topological spaces. They will satisfy many other important properties. The construction will consist of two major steps: one topological, the other algebraic.

Step 1 (topological step): The chain complex of X

Definition 1.1.6 (Simplicial n-chains). Fix a Δ-complex X. For each $n \geq 0$, the *group of (simplicial) n-chains of X*, denoted $C_n(X)$, is the free abelian group on the set of n-simplices of X.

Since X has finitely many simplices, each $C_n(X)$ is finitely generated. Since we are assuming that the set of simplices in the Δ-complex structure of X is closed under taking subsimplices, $C_n(X)$ is nonzero precisely for $n \leq \max \dim(\sigma)$, where the max is taken over all simplices of X.

We write the group operation in $C_n(X)$ as addition. Thus if $\{\sigma_i\}$ denotes the set of all n-simplices of X, then any element $c \in C_n(X)$ can be written uniquely as a formal finite sum

$$c = \sum_i a_i \sigma_i \quad \text{with } a_i \in \mathbb{Z}$$

To be pedantic, if we choose an isomorphism $C_n(X) \to \mathbb{Z}^r$ taking the basis element σ_i to the element $(0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^r$ with a "1" in the i^{th} position, then c corresponds to $(a_1, \ldots, a_r)$ under the isomorphism. As we will see, the notation $c = \sum_{i=1}^r a_i \sigma_i$ will be easier to work with. Yet another equivalent way to view $C_n(X)$ is as the space of integer-valued functions on $\{n - \text{simplices of } X\}$, which is an abelian group under addition of functions.

For each $n \geq 1$, we define the **boundary homomorphism**

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

on any n-simplex $\sigma : [v_0 \cdots v_n] \to X$ via

$$\partial_n \sigma := \sum_{i=0}^n (-1)^i \sigma|_{[v_0 \cdots \widehat{v_i} \cdots v_n]} \tag{1.2}$$

We emphasize that the right hand side of Equation (1.2) is given in the notation for an element of the abelian group $C_{n-1}(X)$. By the universal property of free abelian groups, the map $\partial_n : C_n(X) \to C_{n-1}(X)$ has a unique $\mathbb{Z}$-linear extension of the map defined in Equation (1.2):

$$\partial_n(\sum_i a_i\sigma_i) = \sum_i a_i\partial_n\sigma_i$$

It will also be convenient to let $\partial_0 : C_0(X) \to 0$.

Recall from the theory of finitely generated abelian groups that, once generating sets $\{u_i\}$ for $\mathbb{Z}^n$ and $\{v_j\}$ for $\mathbb{Z}^m$ are chosen and fixed, there is a bijection

$$\{\text{homomorphisms } \phi : \mathbb{Z}^n \to \mathbb{Z}^m\} \longleftrightarrow \{m \times n \text{ matrices with integer entries}\}$$

where the matrix A corresponding to a homomorphism ϕ has entries a_{ij} determined by $\phi(u_i) = \sum_j a_{ij}v_j$. Thus the homomorphism ∂_n can be encoded by an $\dim(C_{n-1}(X)) \times \dim(C_n(X))$ matrix with integer entries.

The idea of making the definition given in Equation 1.2 is that "the boundary of a simplex $[v_0 \cdots v_n]$ is the formal sum of all of its faces, oriented properly." The "oriented properly" part is the reason for the sign in Equation 1.2; we will discuss this below when explaining orientations.

Poincaré already made the key observation that "a boundary has no boundary". The example to think of here is a simplex σ: points in the interior of σ look topologically different from points on $\partial\sigma$, while all points of $\partial\sigma$ look alike (i.e. any two have homeomorphic neighborhoods in $\partial\sigma$. Algebraically, this is reflected in the following proposition, which is simple but crucial for homology theory.

Proposition 1.1.7. *Let X be any Δ complex. For any $n \geq 1$ the homomorphism $\partial_n \circ \partial_{n+1} : C_{n+1}(X) \to C_{n-1}(X)$ is the zero homomorphism.*

Proof. Give proof. Trick: break sum into $i < j$ and $i > j$ parts; they cancel. □

Example 1.1.8. The 3-simplex. The 2-sphere.

Example 1.1.9 (The torus). Let $T^2 = S^1 \times S^1$ be the 2-dimensional torus. We view T^2 as a square with opposite sides identified. GIVE Δ-complex structure. Write out each $C_n(X)$ and the boundary maps.

At this point we have the following: given any finite Δ-complex X, we have a sequence of finitely generated abelian groups and homomorphisms:

$$C_n(X) \stackrel{\partial_n}{\to} C_{n-1}(X) \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_2}{\to} C_1(X) \stackrel{\partial_1}{\to} C_0(X) \stackrel{\partial_0}{\to} 0$$

where $n = \max\dim(\sigma)$ for σ a simplex in X, and where (by Proposition 1.1.7) the composition of any two adjacent homomorphisms is zero. This setup is the basic setup to which we can now apply homological algebra to produce algebraic invariants of X.

Step 2 (algebraic step): Chain complexes

With the previous example in mind, we begin with the general homological theory of chain complexes. This machinery is widely applicable not only within topology, but it is a fundamental tool in number theory, algebraic geometry, and group theory.

Definition 1.1.10 (Chain complex). A *chain complex (of abelian groups)* $\mathcal{C} := \{C_n, \partial_n\}$ is a collection of abelian groups C_n and homomorphisms $\partial_n : C_n \to C_{n-1}$ for each n satisfying $\partial_n \circ \partial_{n-1} = 0$.

Remark 1.1.11. We explicitly allow the C_n to be infinitely generated; indeed we will later see quite important examples where C_n is the free abelian group on an uncountable set. We also explicitly allow n to be an arbitrary integer, for convenience.

Example 1.1.12 (Complex of simplicial chains). Let X be a Δ-complex, and for $n \geq 0$ let $C_n(X)$ denote the group of n-chains on X. The *complex of simplicial chains* of X is the chain complex $\mathcal{C}(X) := \{C_n(X), \partial_n\}$, where $n \geq 0$, and where we set $C_{-1}(X) = 0$ by definition.

Since a chain complex $\mathcal{C} := \{C_n, \partial_n\}$ consists of abelian groups and homomorphisms between them, it is natural to consider kernels and images. We define the **group Z_n of n-cycles of $\mathcal{C}$** to be the kernel

$$Z_n(\mathcal{C}) := \ker(\partial_n : C_n \to C_{n-1})$$

The letter "Z" is used because Vietoris, who first defined Z_n in this way, called Z_n the *n-te Zusammenhangsgruppethe.* [2] We define the **group B_n of n-boundaries of $\mathcal{C}$** to

[2]In his book [Mu], Munkres attributes the "Z" to the fact that the German word for "cycle" is "Zyklus".

be the image

$$B_n(\mathcal{C}) := \operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n)$$

Since $\partial_n \circ \partial_{n-1} = 0$, it follows that $B_n \subseteq Z_n$. The homology group $H_n(\mathcal{C})$ gives a measure of how many i-cycles are not cycles for the trivial reason that they are boundaries.

Definition 1.1.13 (Homology of a chain complex). Let $\mathcal{C} := \{C_n, \partial_n\}$ be a chain complex. For each $i \geq 0$ we define the i^{th} *homology group* of the chain complex, denoted $H_i(\mathcal{C})$ by

$$H_i(\mathcal{C}) := Z_n(\mathcal{C})/B_n(\mathcal{C})$$

If two i-cycles $c, c' \in Z_i(\mathcal{C})$ satisfy $c = c' + \partial d$ for some $(i+1)$-chain d, we say that c is **homologous** to c'. Note that in this case $[c] = [c'] \in H_i(\mathcal{C})$.

Since the quotient of any free abelian group is abelian, each $H_i(\mathcal{C})$ is abelian; it is finitely generated if C_n is finitely generated. By the classification of finitely generated abelian groups, in this case $H_i(\mathcal{C}) = \mathbb{Z}^{b_i} \times T_i$ where $b_i \geq 0$ and T_i is a finite abelian group. The number b_i is called the i^{th} **Betti number of** X.

Classically, the Betti numbers of a space were studied before homology theory was discovered. Forgetting the information that T_i might give, why is the group $H_i(X)/T_i \approx \mathbb{Z}^{b_i}$ better than just the integer b_i? As an invariant they contain the same information. However, we will see that the extra group-theoretic structure of $H_i(X)/T_i$ adds a great deal of extra structure and power to the story. Sometimes the process of realizing a numerical invariant as the dimension of a vector space, or rank of a group, etc., is called *categorification.*

We define the **(simplicial) homology** $H_i(X)$ of a Δ-complex X to be the homology of the complex of simplicial chains of X:

$$H_i(X) := H_i(\mathcal{C}(X))$$

We thus have a procedure:

$$\boxed{\text{Space } X} \rightsquigarrow \boxed{\text{Triangulation of } X} \rightsquigarrow \boxed{\text{Chain complex } \{C_n(X), \partial_n\}} \rightsquigarrow \boxed{H_i(X)}$$

Here the first squiggly arrow uses topology; the second is purely combinatorial; the third is basic linear algebra (in the guise of homological algebra). Sometimes it is useful to think of the homology groups $H_i(X)$ as "linearizations" of the space X.

Later in this class we will give several other ways that take as input a topological space, and as output gives a chain complex. The homology of this chain complex will then give us another kind of "homology theory" for spaces.

Chain maps

As is common with algebraic objects, the maps between objects are just as important as the objects themselves. What are the natural morphisms between chain complexes?

Definition 1.1.14 (Chain map)**.** Let $\mathcal{C} = \{C_n, \partial_n\}$ and $\mathcal{C}' = \{C'_n, \partial'_n\}$ be chain complexes. A **chain map** $\phi : \mathcal{C} \to \mathcal{C}'$ is a collection of homomorphisms $\phi_n : C_n \to C'_n$ that commute with the boundary maps ∂_n, ∂'_n:

$$\phi_{n-1} \circ \partial_n = \partial_n \circ \phi_n \quad \text{for all } n \geq 0$$

One of the fundamental properties of homology groups is the following.

Lemma 1.1.15 (Chain maps induce homology maps). *Let $\phi : \mathcal{C} \to \mathcal{C}'$ be a chain map of chain complexes. Then $\phi_n(B_n) = B'_n$ and $\phi(Z_n) = Z'_n$ for each $n \geq 0$. Thus the map*

$$\begin{aligned} \phi_* : \quad & H_n(\mathcal{C}) \to H_n(\mathcal{C}') \\ & [z] \mapsto [\phi_n(z)] \end{aligned}$$

is a homomorphism. The association $\phi \mapsto \phi_*$ is functorial: $(\phi \circ \psi)_* = \phi_* \circ \psi_*$

Proof. This is easy. □

1.1.3 First computations

The general idea of simplicial homology is that $H_i(X)$ measures the number of nontrivial, "i-dimensional holes in X", where by "hole" we mean "cycle" (i.e. a hole is supposed to be the same as a chain with vanishing boundary, which is a really clever formalization of the notion of hole), and by "nontrivial" we mean "which is not a hole for the stupid reason that it bounds something".

Example 1.1.16. Compute the homology of S^1. Now add the 2-disk, and see how this "filling the hole of S^1" is detected in homology. Compute the homology of the 2-sphere and the torus.

Remark 1.1.17. Fix $n > 2$. The n-sphere can easily be given a Δ-complex structure by taking two n-simplices and $\sigma := [v_0 \cdots v_n]$ and $\tau := [u_0 \cdots u_n]$ and identifying them along their boundaries: for each $0 \leq i \leq n$ identify $[v_0 \cdots \widehat{v}_i \cdots v_n] \sim [w_0 \cdots \widehat{w}_i \cdots w_n]$. However, a direct computation of the simplicial homology groups of this Δ-complex becomes a complicated affair for large n, since in order to compute we need an actual Δ-complex structure, and for this we need to throw in all faces of σ and τ, and the faces of these faces, etc. It is possible to do this but it is also cumbersome. We will soon have other methods, for example cellular homology, whereby the problem of computing the homology groups $H_*(S^n)$ is easy.

The answer of this computation answers a fundamental problem of topology that was open for a long time: the invariance of dimension under homeomorphism; namely $\mathbb{R}^m \approx \mathbb{R}^n$ if and only if $m = n$. This is deduced from the corresponding fact for $S^n \approx S^m$, which in turn follows from the computation that $H_i(S^n) = 0$ except for $i = 0, n$, in which case $H_i(S^n) = \mathbb{Z}$.

In order to do explicit computations on more complicated spaces, we will have to develop more machinery. In terms of general results, we first note that since $C_n(X) = 0$ for each $n > \max\{\dim(\sigma) : \sigma \subset X\}$, it follows trivially that $H_n(X) = 0$ for such n. At the other extreme, $H_0(X)$ detects the number of path components of X.

Proposition 1.1.18. *Let X be a Δ-complex with d path components $X_1, \ldots, X_d$. Then $H_0(X) \approx \mathbb{Z}^d$. In fact, for any $n \geq 0$ we have*

$$H_n(X) \approx \oplus_{r=1}^d H_n(X_r)$$

Proof. We first prove that when X is connected then $H_0(X) \approx \mathbb{Z}$. Let v_0 and v_2 be any two 0-simplices of X. Since X is connected, there is a path in X from v_0 to v_1. Since any simplex is connected, we can clearly find such a path consisting of a union of 1-simplices

$[v_0w_0] \cup [w_0w_1] \cup \cdots \cup [w_rv_1]$. We thus have a one-chain $[v_0w_0] + [w_0w_1] + \cdots + [w_rv_1]$ with

$$\begin{aligned}\partial([v_0w_0] + [w_0w_1] + \cdots + [w_rv_1]) &= [w_0] - [v_0] + [w_1] - [w_0] + \cdots [v_1] - [w_n] \\ &= [v_1] - [v_0]\end{aligned}$$

so that $[v_0] = [v_1] \in H_0(X)$. It follows that for any 0-cycle $\tau := \sum_i a_iv_i$, we have

$$[\tau] = \sum_i a_i[v_i] = \sum_i a_i[v_0] = (\sum_i a_i)[v_0]$$

so that $[v_0]$ generates $H_0(X)$. Thus we need only prove that $n[v_0] \neq 0$ for any $n \neq 0$. Suppose $nv_0 = \partial\sigma$ for some $\sigma = \sum_i a_i\sigma_i \in C_1(X)$. Define a homomorphism $\psi : C_0(X) \to \mathbb{Z}$ to be the unique linear extension of the map with $\psi(v) = 1$ for all 0-simplices v. Then for any 1-simplex $[uv]$ we have

$$\psi(\partial[uv]) = \psi(u - v) = \psi(u) - \psi(v) = 1 - 1 = 0.$$

Linearity implies $\psi(\partial\sigma) = 0$ for any $\sigma \in C_1(X)$. But then

$$n = n\psi(v_0) = \psi(nv_0) = \psi(\partial\sigma) = 0$$

so that $n = 0$. This proves that $[v_0] \in H_0(X)$ generates an infinite cyclic subgroup. We have thus proved that $H_0(X) \approx \mathbb{Z}$ when X is connected. This shows that the second claim in the proposition follows from the first claim, which we now prove.

Since each X_i is connected, any simplex $\sigma \in C_n(X)$ is also a simplex of some (in fact unique) X_i. Thus a triangulation of X is the same thing as a triangulation of each X_i, so for each $n \geq 0$ we have an isomorphism of abelian groups

$$\phi : \bigoplus_{i=1}^{d} C_n(X_i) \to C_n(X)$$

which is just the sum of inclusion maps. The map that associates σ to this simplex in $C_n(X_i)$ is an inverse of ϕ. Thus ϕ is an isomorphism. Since a simplex lies in X_i if and only if any of its subsimplices does, it follows that each boundary operator ∂_n commutes with ϕ, that is $\partial_n \circ \phi = \phi \circ \partial_n$; that is, both ϕ and ϕ^{-1} are chain maps. It follows from Lemma 1.1.15 that ϕ induces a homomorphism

$$\phi_* : \bigoplus_{i=1}^{d} H_n(X_i) \to H_n(X).$$

Similarly for ϕ^{-1}. Thus ϕ_* is an isomorphism. This proves the second claim of the proposition. □

Proposition 1.1.18 thus reduces the computation of homology groups of Δ-complexes to the case of connected Δ-complexes.

1.1.4 Topological invariance of $H_i(X)$

One of our goals in this section is to prove that if X and Y are Δ-complexes and if X is homeomorphic to Y, then $H_i(X) \approx H_i(Y)$ for every $i \geq 0$. This is called the "topological invariance of homology". This problem (or, to be more precise, the corresponding problem for simplicial complexes) was an important open problem in the first few decades of the century.

Let X and Y be simplicial complexes. Let f be any map from the set of 0-simplices of X to the set of 0-simplices of Y with the property that f takes any collection of vertices that span a simplex to a collection of vertices that span a simplex. Then f has a unique linear extension $f : X \to Y$ given by extending using barycentric coordinates. Such a map f is called a **simplicial map**.

As we've seen in an exercise, if there is a homeomorphism $f : X \to Y$ that is simplicial, that is then$f$ induces an isomorphism $f_* : H_i(X) \to H_i(Y)$. In general, however, an arbitrary homeomorphism f will not "see" the Δ-complex structures on X and Y at all. We will describe two strategies to remedy this. But before doing so, it will be convenient to consider a more flexible kind of equivalence than homeomorphism. Indeed, the original attempts at proving topological invariance of homology inspired the basic definitions of the theory of homotopies.

Homotopies.

Define homotopy, homotopy equivalence and homotopy type, contractibility (homotopy type of a point. Of course homotopy equivalence doesn't preserve dimension. Define the category of topological spaces and continuous maps. Want to prove that homology gives a functor from this category. So one first needs to prove the following.

Let X and Y be Δ-complexes. Then:

1. Any continuous map induces for each $i \geq 0$ a homomorphism $f_* : H_i(X) \to H_i(Y)$.
2. If $f, g : X \to Y$ and f is homotopic to g, then $f_* = g_*$.
3. Functoriality: $(f \circ g)_* = f_* \circ g_*$ and $\mathrm{Id}_{\mathrm{X}*} = \mathrm{Id}_{H_i(X)}$.

Even (1) is nontrivial, as continuous maps do not necessarily take simplices to simplices. (2) can be viewed as the "maps version" of invariance of H_i under homotopy equivalence.

Proposition 1.1.19. *(1)-(3) above imply that H_i is a topological invariant; indeed if (1)-(3) hold, then any homotopy equivalence $f : X \to Y$ induces an isomorphism $f : H_i(X) \to H_i(Y)$ for each $i \geq 0$.*

Proof. Give proof. □

Our goal in the rest of this section is to prove (1)-(3).

Strategy 1: Simplicial approximation.

In this approach one tries to find a simplicial map that is "close" to the given continuous map $f : X \to Y$. In order to do this one has to replace X with some iterated barycentric subdivision X' of X, from which one get a simplicial map $f' : X' \to Y$ homotopic to f. The map f' then induces a map on simplicial chains, and therefore induces homomorphisms $f'_* : H_i(X') \to H_i(Y)$. We then need to know two things:

1. $H_i(X') \approx H_i(X)$ for any iterated barycentric subdivision X' of X.
2. The the induced map f'_* doesn't depend on the choice of f': if $f'' : X'' \to Y$ is any other simplicial map then $f''_* = f'_*$.

Given all this, we can then define $f_* : H_i(X) \to H_i(Y)$ to be the composition of the isomorphism $H_i(X') \approx H_i(X)$ with the homomorphism f'_*. One can check functoriality.

Strategy 2: Singular homology.

This strategy has three big steps:

1. (Construction) Construct a new functor from the category of topological spaces and continuous maps to the category of groups and homomorphisms, called **singular homology**, denoted for now by $H_i^s(X)$.

2. (Homotopy functor) Prove that if $f, g : X \to Y$ are homotopic then $f_* = g_* : H_i^s(X) \to H_i^s(Y)$ for each $i \geq 0$.

3. (Simplicial equals singular) Prove for any Δ-complex X that $H_i^s(X) \approx H_i(X)$, so that the singular and the simplicial theories agree.

These three steps immediately imply that simplicial homology groups are invariant under any homotopy equivalence.

1.2 Singular homology

1.2.1 Construction and functoriality

Define singular chain complex, homology. Continuous map induces a chain map, hence a map on singular homology. Check functoriality.

Remark 1.2.1. It is easy to see that H_i^s is a topological invariant, that is, invariant under *homeomorphism*.

1.2.2 Homotopy functor

The key result is the following.

Proposition 1.2.2. *Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous maps. If f is homotopic to g then $f_* = g_* : H_i^s(X) \to H_i^s(Y)$ for each $i \geq 0$.*

By functoriality, we have the following.

Corollary 1.2.3 (Homotopy invariance of singular homology). *If $f : X \to Y$ is a homotopy equivalence of spaces then $f_* : H_i^s(X) \to H_i^s(Y)$ for each $i \geq 0$. In particular H_i^s is a topological invariant of spaces.*

Our goal now is to prove Proposition 1.2.2. The idea is to continue our "dictionary" between topological concepts and homological concepts by giving an algebraic version of homotopies of maps.

Topological object	Homological algebra object
singular n-chains $C_n(X)$	n-chains C_n
boundary map ∂_n	boundary homomorphism ∂_n
continuous map $f : X \to Y$	chain map $f : \oplus C_n \to \oplus C'_n$
homotopy of continuous maps	chain homotopy of chain maps

Of course we also could have presented the left hand column in the simplicial category.

Proof of Proposition 1.2.2. We are given a homotopy $F : X \times [0,1] \to Y$ between $f = F_0$ and $g = F_1$. For each $n \geq 0$ we define a homomorphism, called the **prism homomorphism**, or **prism operator**:

$$P_n : C_n(X) \to C_{n+1}(Y)$$

which "morally" is the unique linear extension of the map that sends the n-chain $\sigma : \Delta^n \to X$ to the $(n+1)$-chain $G : \Delta^n \times [0,1] \to Y$ which is the restriction of F to $\Delta^n \times [0,1]$. As one can see geometrically, this satisfies the key equation

$$\partial P = g_\# - f_\# - P\partial \tag{1.3}$$

which we will use in a moment. One problem here is that $P_n(\sigma)$ is not actually a chain in Y, as $\Delta^n \times [0,1]$ is not an $(n+1)$-simplex. To remedy this we need to chop up $\Delta^n \times [0,1]$ into a signed sum of $(n+1)$-simplices, in a systematic way so that (1.3) holds. One way of doing this is as follows: denote $\Delta^n \times \{0\}$ by $[v_0 \cdots v_n]$ and label the vertices of $\Delta^n \times \{1\}$ by $\{w_i\}$ so that the for each i the "vertical 1-simplex" $v_i \times [0,1]$ has endpoiints v_i and w_i. Then (the unique linear extension of) the formula

$$P_n(\sigma) := \sum_{i=1}^{n} F \circ (\sigma \times \mathrm{Id}) \restriction [v_0 \cdots v_i w_i \cdots w_n]$$

is an $(n+1)$-chain in Y, and a short but somewhat messy computation gives that P_n satisfies (1.3), as desired.

So what does (1.3) do for us? It is the homological algebra version of homotopy.

Definition 1.2.4 (Chain homotopy). Let $\mathcal{C} = \{C_n, \partial_n\}$ and $\mathcal{C}' = \{C'_n, \partial'_n\}$ be chain complexes. Let $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ be chain maps. A **chain homotopy** between ϕ and ψ is a collection of homomorphisms $P_n : C_n \to C'_{n+1}$ satisfying $\psi - \phi = \partial P + P\partial$, or with notation to keep track of dimensions:

$$\psi_n - \phi_n = P_{n-1} \circ \partial_n + \partial'_{n+1} \circ P_n$$

Thus (1.3) gives a chain homotopy between the singular chain maps $g_\#$ and $f_\#$. The proposition then follows from the following lemma, which is a statement in homological algebra.

Lemma 1.2.5 (Fundamental lemma of chain homotopies). *If $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ are chain homotopic chain maps then*

$$\phi_* = \psi_* : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$$

for each $n \geq 0$.

Proof. Give. □

This completes the proof of the proposition.

□

We thus have that singular homology is a homotopy functor on the category of topological spaces and continuous maps; in particular it is invariant under homotopy equivalence, hence under homeomorphism. In order to prove that $H_i^s(X) \approx H_i(X)$ we will need to further develop methods for computing singular homology.

1.2.3 Relative homology

It is often useful to break up computations into smaller ones. In homology theory, one way to do this is to use relative homology.

In this section we develop the relative version of singular homology. However, the theory works basically verbatim for simplicial homology. We will denote singular homology by H_i.

The setup is this: Let X be any topological space, and let $A \subseteq X$ be any (possibly empty) subspace. Denote the natural inclusion map by $i : A \to C$. The map i induces a homomorphism of singular chain groups

$$i_\# : C_n(A) \to C_n(X)$$

The homomorphism $i_\#$ is injective on generators, and since our groups are free abelian it follows that $i_\#$ is injective on all of $C_n(A)$. Sometimes we identify $C_n(A)$ with its image in $C_n(X)$ under $i_\#$.

Clearly $Z_n(A)$ is a subgroup of $Z_n(X)$ and $B_n(A)$ is a subgroup of $B_n(X)$. It follows that $i_\#$ induces a homomorphism $i_* : H_n(A) \to H_n(X)$. Note that i_* is often not injective. For example, let $X = D^2$ and let $A = \partial D^2 = S^1$. We then have $i_* : H_1(S^1) \to H_1(D^2)$, with domain $\mathbb{Z}$ and range 0.

Definition 1.2.6 (**Relative chain group).** Let X be any topological space, and let $A \subseteq X$ be any (possibly empty) subspace. The **relative chain group** of X relative to A is defined to be

$$C_n(X, A) := C_n(X)/C_n(A)$$

The usual boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ clearly preserves $C_n(A)$, and so induces a boundary homomorphism $\partial'_n : C_n(X, A) \to C_{n-1}(X, A)$. The fact that $\partial^2 = 0$ implies that $(\partial')^2 = 0$. Thus $\{C_n(X, A), \partial'_n\}$ is a chain complex.

Definition 1.2.7 (**Relative homology).** Let A be a subspace of a space X. The **relative homology groups** $H_n(X, A), n \geq 0$ are defined to be the homology of the relative chain complex $\{C_n(X, A), \partial'_n\}$.

We have the obvious notions of **relative chain**, **relative cycle**, etc. So, for example, a relative cycle $c \in Z_n(X, A)$ is the same thing as a chain $c \in C_n(X)$ such that $\partial_n(c) \in A$. As we are considering singular homology, and so c is really a continuous map $c : \Delta^n \to X$, what we mean by the statement " $\partial_n(c) \in A$ " is really that $\sigma(\partial\Delta^n) \subset A$. We also have the corresponding theory for maps, where a map of pairs $f : (X, A) \to (Y, B)$ is simply a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. We also have homotopies of pairs, which are continuous families of maps of pairs $F_t : (X, A) \to (Y, B)$.

It is easy to check that if $x \in X$ is any point, then

$$H_i(X, x) \approx \tilde{H}_i(X) \text{ for all } i \geq 0$$

Homology of quotients. One of the most useful aspects of relative homology is its relation to the homology of quotient spaces X/A. However one needs a weak hypothesis on A for this to work. First recall that a space X **deformation retracts** onto a subspace $A \subseteq X$ if there is a homotopy $F : X \times [0, 1] \to A$ so that $F_0 = \mathrm{Id}_X$ and $F_t(a) = a$ for all $a \in A, t \in [0, 1]$. We will say that a nonempty subspace $A \subseteq X$ is **reasonable** if there is a neighborhood $N(A)$ (i.e. an open set containing A) such that $N(A)$ deformation retracts to A.

Examples of reasonable subspaces include any subcomplex of a Δ-complex or a simplicial complex, and any submanifold of a manifold. On the other hand, the topologist's sine curve is not a reasonable subspace of $\mathbb{R}^2$, nor is $\mathbb{Q} \subset \mathbb{R}$.

Let A be a subspace of X. The quotient map $p : X \to X/A$ can also be thought of as a map of pairs $p : (X, A) \to (X/A, A/A)$. The following theorem is gives us access to computations of the singular homology of several nontrivial examples.

Theorem 1.2.8 (Homology of quotients). *Let A be a reasonable subspace of a topological space X. Then the quotient map $p : (X, A) \to (X/A, A/A)$ induces for each $i \geq 0$ an isomorphism*

$$p_* : H_i(X, A) \to H_i(X/A, A/A) \approx \tilde{H}_i(X/A)$$

We will prove Theorem 1.2.8 a little later. The power of Theorem 1.2.8 is in the computability of $H_i(X, A)$, which we continue to develop.

1.2.4 Fundamental theorem of homological algebra

In order to compute $H_i(X, A)$, and hence $H_i(X/A)$, we need some homological algebra. Much of homological algebra is phrased in the language of exact sequences. Let $\{A_n\}$

be a sequence of abelian groups, or of chain complexes, or of modules over a fixed ring. For each n let $\phi_n : A_n \to A_{n-1}$ be a homomorphism in the appropriate category. In terms of diagrams we write

$$\dots \stackrel{\phi_{n+2}}{\to} A_{n+1} \stackrel{\phi_{n+1}}{\to} A_n \stackrel{\phi_n}{\to} A_{n-1} \stackrel{\phi_{n-1}}{\to} \dots$$

We say that this sequence is **exact at A_n** if kernel$(\phi_n) =$ image(ϕ_{n+1}). So for example, in the case that $\mathcal{A} := \{A_n, \phi_n\}$ is a chain complex, we can view $H_n(\mathcal{A})$ as a measure of the non-exactness of $\mathcal{A}$ at A_n. We will soon see that the notion of exactness will also be important in comparing different chain complexes.

Let A, B, C be, as above, abelian groups, chain complexes, or modules over a fixed ring. Let $\phi : A \to B$ and $\psi : B \to C$ be morphisms in the appropriate category, and let 0 denote the trivial object in that category (e.g. trivial group, the zero chain complex, or the trivial module). A sequence

$$0 \to A \stackrel{\phi}{\to} B \stackrel{\psi}{\to} C \to 0$$

is a **short exact sequence** if it is exact at each of A, B and C. Note that exactness at A is equivalent to injectivity of ϕ, while exactness at C is equivalent to surjectivity of ψ. Note too that in the case of modules, the sequence is exact precisely when $C \approx B/A$.

Such exact sequences are ubiquitous in topology. For example, as we showed above, when A is a subspace of a topological space X, we have an exact sequence of chain groups

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

But more than this, denoting we have the full singular chain complexes $\mathcal{C}(A) := \{C_n(A), \partial_n\}$ and $\mathcal{C}(X)$, the relative chain complex $\mathcal{C}(X, A)$, and chain maps $i_\# : \mathcal{C}(A) \to \mathcal{C}(X)$ and $j_\# : \mathcal{C}(X) \to \mathcal{C}(X, A)$. . In other words, we have a short exact sequence of chain complexes give vert-horiz diagrm

$$0 \to \mathcal{C}_n(A) \to \mathcal{C}_n(X) \to \mathcal{C}_n(X, A) \to 0 \tag{1.4}$$

This is one of the easiest examples of three chain complexes with natural chain maps between them. The importance of short exact sequence of chain complexes comes from the following theorem, which allows us to relate the homologies of the complexes in a short exact sequence.

Theorem 1.2.9 (Fundamental Theorem of Homological Algebra). *Let*

$$0 \to \mathcal{A} \overset{\phi}{\to} \mathcal{B} \overset{\psi}{\to} \mathcal{C} \to 0$$

be a short exact sequence of chain complexes. Then for each $n \geq 0$ there is a "connecting homomorphism" ∂ and a "long exact sequence"

$$\cdots \overset{(\phi_{n+1})_*}{\longrightarrow} H_{n+1}(\mathcal{C}) \overset{\partial}{\longrightarrow} H_n(\mathcal{A}) \overset{(\psi_n)_*}{\longrightarrow} H_n(\mathcal{B}) \overset{(\phi_n)_*}{\longrightarrow} H_n(\mathcal{C}) \overset{\partial}{\longrightarrow} \cdots$$

Proof. This proof is the "mother of all diagram chases", and so I will do it out in detail. This will probably be the last diagram chase I work out for you.

Our first goal is to construct the connecting homomorphism. FINISH PROOF. □

Applying Theorem 1.2.9 to the short exact sequence (1.4) gives the following.

Theorem 1.2.10 (Long exact sequence of a pair). *Let X be a topological space and let $A \subseteq X$ be a nonempty subspace. Then there is a long exact sequence*

$$\cdots \longrightarrow H_{n+1}(X,A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow \cdots \longrightarrow H_0(X,A) \longrightarrow 0$$

One can check (homework) that the "connecting homomorphism" $\partial : H_n(X,A) \to H_n(A)$, defined purely via homological algebra, has a geometric meaning here: if $c \in Z_n(X,A)$ is a cycle representing an element of $H_n(X,A)$, then the element $\partial[c] \in H_{n-1}(A)$ is represented by the cycle $\partial c \in Z_{n-1}(A)$.

Finally, when using augmented chain complexes, while we still have the exact sequences

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

for each $n \geq 0$, we have in the augmented case that $C_{-1}(A) = C_{-1}(X) = \mathbb{Z}$ and so $C_{-1}(X,A) = C_{-1}(X)/C_{-1}(A) = 0$. This gives that

$$\tilde{H}_i(X,A) \approx H_i(X,A) \quad \text{for all } i \geq 0 \tag{1.5}$$

Application: Homology of S^n. As a first application of the long exact sequence of a pair, we use it to compute the (reduced) homology of all spheres S^n, as follows. Let

$n \geq 1$. Applying Theorem 1.2.10 to the pair $(X, A) = (D^n, \partial D^n) = (D^n, S^{n-1})$ gives the long exact sequence

$$\cdots \longrightarrow H_{i+1}(D^n, S^{n-1}) \longrightarrow H_i(S^{n-1}) \longrightarrow H_i(D^n) \longrightarrow H_i(D^n, S^{n-1}) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_1(D^n) \longrightarrow H_0(S^{n-1}) \longrightarrow H_0(D^n) \longrightarrow H_0(D^n, S^{n-1}) \longrightarrow 0$$

Now here is where we want to actually use *reduced* homology, replacing each of the homology groups above by their reduced versions. The first thing to note is that $\tilde{H}_i(D^n) = 0$ for $i \geq 0$ (recall $n > 0$). Thus we have for $i \geq 0$ that the pieces of this long exact sequence that are :

$$0 \longrightarrow \tilde{H}_{i+1}(D^n, S^{n-1}) \longrightarrow \tilde{H}_i(S^{n-1}) \longrightarrow 0$$

which, by exactness, proves that

$$\tilde{H}_{i+1}(D^n, S^{n-1}) \approx \tilde{H}_i(S^{n-1}) \quad \text{when } i > 0 \tag{1.6}$$

Now, the pair $(D^n, S^{n-1}), n > 0$ is always a reasonable pair. Theorem 1.2.8 thus implies

$$H_i(D^n, S^{n-1}) = H_i(D^n/S^{n-1}) \quad \text{for each } i \geq 0 \tag{1.7}$$

Of course D^n/S^{n-1} is homeomorphic to S^n. Since singular homology is a topological invariant, combining (1.6) and (1.7) gives

$$\tilde{H}_{i+1}(S^n) \approx \tilde{H}_i(S^{n-1}) \quad \text{for all } n \geq 1, i \geq 0$$

We can now proceed by induction on n, using the fact that in singular homology we easily computed directly that $\tilde{H}_i(S^0) =_0$ for $i > 0$ and $\tilde{H}_0(S^0) = \mathbb{Z}$. It follows from induction on n that for any $n \geq 0$:

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

1.2.5 Isomorphism of the simplicial and singular theories

We now know enough about singular homology to prove that it is isomorphic with singular homology.

THIS STUFF IS EXPLAINED WELL IN HATCHER, p.128-130. PLEASE SEE THAT EXPLANATION.

1.2.6 The Excision Theorem

The only thing I still owe you is a proof of Theorem 1.2.8, namely: For reasonable pairs (X, A) of spaces, the quotient map $p : (X, A) \to (X/A, A/A)$ induces for each $i \geq 0$ an isomorphism

$$p_* : H_i(X, A) \to H_i(X/A, A/A) \approx \tilde{H}_i(X/A)$$

where H_i denotes singular homology. This will follow from a somewhat technical but quite useful theorem, namely the excision theorem in singular homology.

Theorem 1.2.11 (The Excision Theorem). *Let X be a topological space with subspaces A, U with $U \subseteq A \subseteq X$. Assume that the closure of U is contained in the interior of A. Then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism*

$$i_* : H_n(X - U, A - U) \to H_n(X, A) \quad \textit{for all } n \geq 0$$

Theorem 1.2.11 is called the "excision theorem" because it basically says that "excising away U from both X and A doesn't change the relative homology.

Proof of special case. The usual proof of this theorem (see e.g. Hatcher, §2.1) is quite complicated. However, the key idea can be gleaned in the following special case.

Assumption: We assume that X is a Δ-complex and that each of the subspaces $A, X - U$ and $A - U$ of X is actually a subcomplex of X.

The subtlety in the general case: In general, even when X is although (X, A) and $(X - U, A - U)$ may each individually have the structure of a Δ-complex, the Δ-complex structures may have nothing to do with each other. This must be dealt with.

With the assumptions above in hand, consider the chain map ϕ which is the composition of inclusion followed by projection:

$$C_n(X - U) \to C_n(X) \to C_n(X, A)$$

First note that ϕ is surjective. This is because $C_n(X, A) = C_n(X)/C_n(A)$ has as basis the collection of simplices of X not contained in the subcomplex A. Of course any such simplex lies in $X-U$. It is also clear that the kernel of ϕ is $C_n(A-U)$. Thus ϕ induces an isomorphism

$$C_n(X-U, A-U) = C_n(X-U)/C_n(A-U) \approx C_n(X, A) \text{ for each } n \geq 0$$

Since ϕ commutes with boundary operators, the theorem follows.

□

First applications of excision. The excision theorem has many applications. We can finally prove Theorem 1.2.8, which we restate for the reader's convenience.

Theorem 1.2.8 (Homology of quotients). *Let A be a reasonable subspace of a topological space X. Then the quotient map $p : (X, A) \to (X/A, A/A)$ induces for each $i \geq 0$ an isomorphism*

$$p_* : H_i(X, A) \to H_i(X/A, A/A) \approx \tilde{H}_i(X/A)$$

Proof of slightly special case. We prove the theorem under the assumption that A itself is open, so that we can take $V = A$. See Hatcher, Proposition 2.22 for the general case, which is only slightly harder. [It does introduce one new tool, the exact sequence of a triple]

With this assumption in hand, we have a commutative diagram

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\approx} & H_n(X-A, A-A) = H_n(X-A) \\ p_* \downarrow & & \downarrow (p|_{X-A})_* \\ H_n(X/A, A/A) & \xrightarrow{\approx} & H_n(X/A - A/A, A/A) \end{array}$$

Here the horizontal arrows are the excision isomorphisms. Now the right-hand vertical arrow is the map induced by the restriction of p to $X-A$, which is a homeomorphism, so that this right-hand vertical map is an isomorphism. Since the diagram commutes, p_* is itself an isomorphism. □

Theorem 1.2.12 (Invariance of dimension). *Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be open sets. If $V \approx W$ then $n = m$.*

Proof. The theorem is easy if $n = 1$ or $m = 1$, so for convenience we assume $n, m > 1$. Pick any $x \in V$. Applying the excision theorem (Theorem 1.2.11) with $X = \mathbb{R}^n$, $A = \mathbb{R}^n - x$ and $U = \mathbb{R}^n - V$) gives that

$$H_i(V, V - x) \approx H_i(\mathbb{R}^n, \mathbb{R}^n - x)$$

Now $\mathbb{R}^n - x$ deformation retracts to S^{n-1}. The long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n - x)$ thus gives

$$\cdots \to H_i(\mathbb{R}^n) \to H_i(\mathbb{R}^n, \mathbb{R}^n - x) \to H_{i-1}(S^{n-1}) \to H_{i-1}(R^n) \to \cdots$$

So for each $i > 1$ we have

$$H_i(\mathbb{R}^n, \mathbb{R}^n - x) \approx H_{i-1}(S^{n-1})$$

which (since $i, n > 1$) vanishes except when $i = n$. But any homeomorphsim $h : V \to W$ induces a homeomorphism of pairs $h : (V, x) \to (W, h(x))$, which thus induces an isomorphism

$$H_i(\mathbb{R}^n, \mathbb{R}^n - x) \approx H_i(\mathbb{R}^m, \mathbb{R}^m - h(x))$$

and so $n = m$.

□

1.3 Applications and more computational tools

In this section we begin to give some applications of the theory we've built up so far. We also introduce more tools so that we can compute $H_i(X)$ for many more spaces X.

1.3.1 Euler characteristic

We begin with a striking application of the topological invariance of simplicial homology.

Definition 1.3.1 (Euler characteristic). Let X be a finite Δ-complex, and for each $n \geq 0$ let $c_n(X)$ denote the number of n-simplices in X. The **Euler characteristic** of X, denoted by $\chi(X)$, is defined to be the finite alternating sum:

$$\chi(X) := \sum_{n \geq 0} (-1)^n c_n(X)$$

The sum is finite since X is a finite Δ complex. Even for $X = D^2$, there are Δ-complex structures on X with wildly varying terms c_0, c_1, c_2. The following remarkable theorem is a cornerstone of algebraic and combinatorial topology.

Theorem 1.3.2 (Toplogical invariance of $\chi(X)$). *Let X be any Δ-complex. Let $b_n(X)$ denote the n^{th}* **Betti number** *of X; that is, $b_n(X)$ is the rank of the free abelian part of $H_n(X)$. Then*

$$\chi(X) = \sum_{n \geq 0} (-1)^n b_n(X)$$

In particular, since each $b_n(X)$ depends only on the homotopy type (in particular homeomorphism type) of X, the same is true for $\chi(X)$.

Proof. The number $c_n(X)$ is the rank of the free abelian group $C_n = C_n(X)$ of simplicial n-chains. Let Z_n and B_n denote the subgroups of n-cycles and n-boundaries. Since $H_n(X) = Z_n/B_n$ we have that

$$b_n := \text{rank}(H_n(X)) = \text{rank}(Z_n/B_n) = \text{rank}(Z_n) - \text{rank}(B_n) \tag{1.8}$$

The Rank-Nullity Theorem in linear algebra gives

$$\text{rank}(C_n) = \text{rank}(Z_n) + \text{rank}(B_{n-1}) \quad \text{for each } n \geq 1 \tag{1.9}$$

while $\text{rank}(C_0) = \text{rank}(Z_0)$ since $C_0 = Z_0$. Thus

$$\begin{aligned}
\chi(X) &:= \textstyle\sum_{n\geq 0}(-1)^n \,\text{rank}(C_n) & \\
&= \text{rank}(Z_0) + \textstyle\sum_{n \geq 1}(-1)^n[\text{rank}(Z_n) + \text{rank}(B_{n-1})] & \text{by (1.9)} \\
&= \text{rank}(Z_0) - \text{rank}(Z_1) - \text{rank}(B_0) + \text{rank}(Z_2) + \text{rank}(B_1) \cdots & \\
&= [\text{rank}(Z_0) - \text{rank}(B_0)] + [\text{rank}(B_1) - \text{rank}(Z_1)] - \cdots & \text{by regrouping terms} \\
&= \textstyle\sum_{n \geq 0}(-1)^n \,\text{rank}(H_n(X)) & \text{by (1.8)}
\end{aligned}$$

□

1.3.2 The Lefschetz Fixed Point Theorem

The Lefschetz Fixed Point Theorem is probably the most important fixed point theorem in mathematics. It was discovered and proven by Solomon Lefschetz around 1926. In order to explain the theorem and its proof, we need a little bit of algebraic setup.

Let $\mathcal{C} = \{C_n, \partial_n\}$ be any finite chain complex of finitely generated free abelian groups. Any chain map $f : \mathcal{C} \to \mathcal{C}$ induces a homomorphism $(f_n)_* : H_n(\mathcal{C}) \to H_n(\mathcal{C})$ for each $n \geq 0$. Let $T_n(\mathcal{C})$ denote the torsion subgroup of $H_n(\mathcal{C})$. Since any homomorphism of an abelian group preserves its torsion subgroup, f clearly induces for each $n \geq 0$ a homomorphism of free abelian groups

$$(f_n)_* : H_n(\mathcal{C})/T_n(\mathcal{C}) \longrightarrow H_n(\mathcal{C})/T_n(\mathcal{C})$$

Let d denote the rank of the free abelian group $H_n(\mathcal{C})/T_n(\mathcal{C})$. If $d = 0$ then $(f_n)_*$ is the zero map. If $d > 0$, we can choose a basis for $H_n(\mathcal{C})/T_n(\mathcal{C})$ so that $(f_n)_*$ acts as a $d \times d$ integer matrix on $H_n(\mathcal{C})/T_n(\mathcal{C})$. Thus $(f_n)_*$ has a well-defined **trace**, denoted by $\mathrm{Tr}((f_n)_*)$, defined as 0 if $d = 0$ or the sum of the diagonal entries of this matrix if $d > 0$. Since similar matrices have the same trace, $\mathrm{Tr}((f_n)_*)$ does not depend on the choice of basis.

Not that a chain map $f : \mathcal{C} \to \mathcal{C}$ gives by definition homomorphims $f_n : C_n \to C_n$ for all $n \geq 0$. Since each C_n is free abelian, f_n itself has a well-defined trace $\mathrm{Tr}(f_n)$. The Hopf trace formula is a purely homological-algebraic statement that relates the traces of the two different, and *a priori* unrelated, homomorphisms associated to a chain map.

Theorem 1.3.3 (Hopf trace formula). *Let $\mathcal{C} = \{C_n, \partial_n\}$ be any finite chain complex. Let $f : \mathcal{C} \to \mathcal{C}$ be any chain map. Then*

$$\sum_{n\geq 0} (-1)^n \,\mathrm{Tr}(f_n) = \sum_{n\geq 0} (-1)^n \,\mathrm{Tr}((f_n)_*) \tag{1.10}$$

Note that the Hopf Trace Formula is a vast generalization of the topological invariance of Euler characteristic (Theorem 1.3.2): just set $f = \mathrm{Id}$ in Hopf's formula, and note that the trace of the identity matrix equals the rank of the vector space (or the free abelian group).

Proof. We have for each n that $B_n \subseteq Z_n \subseteq C_n$. Pick a basis $\partial_{n+1}(\sigma_1), \ldots, \partial_{n+1}(\sigma_d)$ for B_n, extend it to a basis for Z_n by adding elements $z_1, \ldots, z_r$, then extend that to a basis for C_n. To compute $\mathrm{Tr}(f_n)$, we have to take any basis vector v and determine the coefficient $\lambda(v)$ of v in $f_n(v)$. The key thing to observe is that $\lambda(\partial_{n+1}\sigma_j) = \lambda(\sigma_j)$ since f is a chain map. Then

$$\sum_{n\geq 0}(-1)^n \,\mathrm{Tr}(f_n) = \sum_{n\geq 0}(-1)^n \sum_{k=0}^{r} \lambda(z_k)$$

since all other terms in the sum cancel in pairs. Now since $H_n(\mathcal{C})$ has basis $\{z_1, \ldots, z_r\}$, we have that

$$\mathrm{Tr}((f_n)_*) = \sum_{k=1}^{r} \lambda(z_k)$$

and so we are done. □

The Hopf Trace Formula gives us a useful way to attach a number to any continuous self-map of a Δ-complex.

Definition 1.3.4 (**Lefschetz number).** Now let X be any finite Δ-complex and let $f : X \to X$ be any continuous map. For each $n \geq 0$ let $(f_n)_* : H_n(X)/T_n(X) \to H_n(X)/T_n(X)$ be the induced homomorphism on the free abelian part of the n^{th} homology group of X. The **Lefschetz number** of f, denoted by $\Lambda(f)$, is defined to be the integer

$$\Lambda(f) := \sum_{n\geq 0}(-1)^n \,\mathrm{Tr}((f_n)*)$$

The remarkable Lefschetz Fixed Point Theorem reduces the existence of a fixed point for a self-map to the nontriviality of a single integer attached to that map.

Theorem 1.3.5 (**Lefschetz Fixed Point Theorem).** *Let X be any finite Δ-complex and let $f : X \to X$ be any continuous map. If $\Lambda(f) \neq 0$ then f has a fixed point.*

Proof. Assume that f has no fixed point. We must prove $\Lambda(f) = 0$. Put any metric d on X. Since X is compact and f is continuous, and since $f(x) \neq x$ for any $x \in X$, there exists $\delta > 0$ so that $d(f(x), x) > \delta$ for all $x \in X$. By repeatedly subdividing if necessary, we can make it so the Δ-complex structure on X has the property that each simplex has diameter at most $\delta/100$.

Now homotope f to a simplicial map $h : X \to X$ so that $d(f(x), h(x)) < \delta/2$; this can be done by the Simplicial Approximation Theorem. So $d(h(x), x) \geq \delta/2$ for all $x \in X$. In particular, for every simplex σ we have $h(\sigma) \cap \sigma = \emptyset$. It follows that $\mathrm{Tr}(h_n) = 0$ for each $n \geq 0$. By the Hopf Trace Formula we have $\Lambda(h) = 0$. But since f is homotopic to h we have $f_* = h_* : H_n(X) \to H_n(X)$, so that $\Lambda(f) = \Lambda(h)$, and we are done. □

The Lefschetz Fixed Point gives as a trivial corollary a new proof of the Brouwer Fixed Point Theorem, and indeed a vast generalization of it.

Corollary 1.3.6. *Let X be a finite, connected Δ complex with $H_i(X) = 0$ for each $i > 0$. Then any continuous map $f : X \to X$ has a fixed point.*

Of course one immediately obtains Brouwer by setting $X = D^n, n > 0$.

Proof. Since X is connected, $(f_0)_* : H_0(X) \to H_0(X)$ is the 1×1 identity matrix. Since $H_i(X) = 0$ for $i > 0$, we have that $\mathrm{Tr}((f_i)_*) = 0$ for $i > 0$. Thus $\Lambda(f) \neq 0$. Now apply Lefschetz. □

Since we mention Brouwer, me give an application of it to pure linear algebra.

Corollary 1.3.7. *Let A be an $n \times n$ matrix of real numbers with each entry positive. Then A has some positive eigenvalue λ whose eigenvalue is a vector with all coordinates positive.*

Proof. Give. □

1.3.3 Maps of spheres, with applications

The theory we have developed so far can be used to give us a pretty complete understanding of all homotopy classes of maps $f : S^n \to S^n$ between spheres of the same dimension. The key invariant here is degree.

Definition 1.3.8 (Degree). Recall that $\tilde{H}_n(S^n) \approx \mathbb{Z}$ for each $n \geq 0$. Thus any continuous map $f : S^n \to S^n$ induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. For any homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ there exists a unique $r \in \mathbb{Z}$ so that $\phi(z) = rz$ for all $z \in \mathbb{Z}$. We define the **degree** of f, denoted by $\deg f$, to be the unique integer so that

$$f_*(z) = (\deg f)z \quad \text{for all } z \in H_n(S^n)$$

Since homology is functorial it follows that $\deg(\mathrm{Id}_{S^n}) = 1$, and for each $f, g : S^n \to S^n$ we have for each $z \in H_n(S^n)$:

$$\deg(f \circ g)z = (f \circ g)_*(z) = f_* \circ g_*(z) = f_*((\deg g)(z)) = (\deg f)(\deg g)(z)$$

so that

$$\deg(f \circ g) = (\deg f)(\deg g)$$

Since f homotopic to g implies $f_* = g_*$, degree is an invariant of homotopy classes of maps $S^n \to S^n$. What is remarkable is that this is a complete invariant.

Theorem 1.3.9 (Hopf's classification of maps $S^n \to S^n$). *Two continuous maps $f, g : S^n \to S^n$ are homotopic if and only if $\deg f = \deg g$.*

We will soon see that for $n \geq 1$, there exist maps of any degree $d \in \mathbb{Z}$. Thus the set of homotopy classes of maps $f : S^n \to S^n$ is in bijective correspondence with $\mathbb{Z}$.

We will need some more machinery before proving Theorem 1.3.9. Meanwhile we can see that degree is computable in many examples. Let $f : S^n \to S^n$ be any continuous map.

1. If f is not surjective then $\deg(f) = 0$. This follows since the non-surjectivity of f implies that $f_* : H_n(S^n) \to H_n(S^n)$ factors through $H_n(S^n - x) = 0$ for some $x \in S^n$.

2. Let $\Sigma f : \Sigma S^n \to \Sigma S^n$ denote the suspension map of f. Since the inclusion $h : S^n \to \Sigma S^n \approx S^{n+1}$ induces an isomorphism

$$h_* : \tilde{H}_n(S^n) \to \tilde{H}_{n+1}(S^{n+1})$$

so that $h_*(1) = 1$, it follows that

$$\deg(\Sigma f) = \deg f$$

3. Fix $d \geq 1$. Think of S^1 as the unit circle in the complex plane, and let $\psi_d : S^1 \to S^1$ be the map $\psi_d(z) := z^d$. Let X be the Δ-complex structure on S^1 given by the union of d edges $\sigma_1, \ldots, \sigma_d$, each of length $2\pi/d$, and let Y be the Δ-complex structure on S^1 given by a single edge τ with endpoints identified. Then $\psi_d : X \to Y$ takes each σ_i onto τ as a homeomorphism on the interior, and mapping each endpoint of σ_i onto the common endpoint of τ.

   Now $H_1(X)$ is generated by the homology class of the 1-cycle $\sum_{i=1}^d \sigma_i$, while $H_1(Y)$ is generated by the homology class of the 1-cycle τ. Since $\psi_d(\sigma_i) = \tau$ for each i, we have

   $$\psi_d(\sum_{i=1}^d \sigma_i) = \sum_{i=1}^d \psi_d(\sigma_i) = \sum_{i=1}^d \tau = d\tau$$

   it follows that $(\psi_d)_* : H_1(X) \to H_1(Y)$ is given by multiplication by d, so that $\deg(\psi_d) = d$. We can similarly build a map of any degree $d \leq 1$. Since the constant map has degree 0, we have just proven that there exist maps $f : S^1 \to S^1$ of any degree $d \in \mathbb{Z}$.

   It follows from 2 that for any $n \geq 1$ there exist maps $f : S^n \to S^n$ of any degree $d \in \mathbb{Z}$. For example, the suspension $\Sigma\psi_d : S^2 \to S^2$ has degree d.

   Remark: Implicit in the above discussion is the use of the *naturality* of the isomorphism between the simplicial homology $H_i(X)$ and the singular homology $H_i^s(X) \approx H_i^s(S^1)$, which gives the commutativity of the following diagram:

   $$\begin{array}{ccc} H_i(X) & \overset{(\psi_d)_*}{\longrightarrow} & H_i(Y) \\ \approx\downarrow & & \downarrow\approx \\ H_i^s(S^1) & \overset{(\psi_d)_*}{\longrightarrow} & H_i^s(S^1) \end{array}$$

4. Let $S^n = \{(x_1, \ldots, x_{n+1} : \sum x_i^2 = 1\} \subset \mathbb{R}^{n+1}$, and let $r_n : S^n \to S^n$ be the reflection

   $$r_n(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, x_{n+1})$$

   We claim that

   $$\deg(r_n) = -1 \tag{1.11}$$

   Proof 1. Induct on n. One can check for $n = 1$ this by using the method of 1 above directly. Let $\Sigma : S^n \to S^{n+1}$ be the suspension map. Then the following commutative diagram implies $\deg(r) = -1$ by induction:

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\Sigma_*} & H_{n+1}(S^{n+1}) \\ (r_n)_* \big\downarrow & & \big\downarrow (\Sigma r_n)_* = (r_{n+1})* \\ H_n(S^n) & \xrightarrow{\Sigma_*} & H_{n+1}(S^{n+1}) \end{array}$$

Proof 2. Give S^n the structure of a Δ-complex by gluing two n-simplices Δ_1, Δ^2 along their boundary; we think of Δ_1 as the northern hemisphere and Δ_2 as the southern. Thus $C_n(S^n) = \mathbb{Z}^2$, generated by $\{\Delta_1, \Delta_2\}$. It is easy to check that $Z_n(S^n) = \mathbb{Z}$, generated by $\Delta_1 - \Delta_2$. On the other hand $B_n(S^n) = 0$ since $C_{n+1}(S^n) = 0$. Thus $H_n(S^n) = Z_n(S^n) \approx \mathbb{Z}$ is generated by the homology class $[\Delta_1 - \Delta_2]$.

Now the chain map $r_\#$ induced by r is given by $r_\#(\Delta_1) = \Delta_2$ and $r_\#(\Delta_2) = \Delta_1$. Thus we have

$$r_*([\Delta_1 - \Delta_2]) = [r_\#(\Delta_1 - \Delta_2)] = [\Delta_2 - \Delta_1] = -[\Delta_1 - \Delta_2]$$

Thus $r_*(z) = -z$ for each $z \in H_n(S^n)$, giving $\deg(r) = -1$.

The argument just given of course works almost verbatim for any reflection $r(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$, giving that $\deg(r) = -1$.

5. Now let $A : S^n \to S^n$ be the **antipodal map** $A(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$. Thus A is the composition of $n+1$ reflections. Since $\deg(f \circ g) = (\deg f)(\deg g)$ it follows that

$$\deg(A) = (-1)^{n+1}$$

Note that as an immediate corollary one obtains that when n is even, A is not homotopic to the identity. On the other hand, when n is odd it is not hard to construct a homotopy $A \sim \mathrm{Id}_{S^n}$.

Which self-maps $f : S^n \to S^n$ have fixed points? By the Lefschetz fixed-point theorem we know that

$$\Lambda(f) = 1 + (-1)^n(\deg f)$$

which is nonzero unless $(\deg f) = (-1)^{n+1}$. So if f has no fixed points, it must have the same degree as the antipodal map $A : S^n \to S^n$, which of course has no fixed points.

By Hopf's classification of maps $f : S^n \to S^n$, we know that any such f is actually homotopic to A. Meanwhile we can already prove the following, which shows that up to homotopy, the antipodal map is the only self-map of S^n without a fixed point.

Proposition 1.3.10. *Any continuous map $f : S^n \to S^n$ without fixed points must be homotopic to the antipodal map.*

Proof. We think of S^n as the set of unit vectors v in $\mathbb{R}^{n+1}$. For $t \in [0, 1]$ let

$$f_t(v) := \frac{(1-t)f(v) - tv}{||(1-t)f(v) - tv||}$$

This is well-defined since by assumption $f(v) \neq v$, so the line from v to $f(v)$ does not pass through 0. The maps f_t give the desired homotopy.

□

Application: The Borsuk-Ulam Theorem. See Homework 4.

Application to vector fields on spheres. It is a fundamental question to understand vector fields on manifolds. Perhaps the simplest question along these lines is to ask which manifolds admit a (continuous) nonvanishing vector fields. Recall that a **vector field** on S^n is defined to be a continuous function $V : S^n \to \mathbb{R}^{n+1}$ so that $V(z)$ is perpendicular to z for each $z \in S^n$.

The following theorem generalizes to all even dimensions the "Hairy Ball Theorem", which states that S^2 doesn't admit any nonvanishing vector field.

Theorem 1.3.11 (Nonvanishing vector fields on spheres). *Let $n \geq 1$. Then S^n admits a nonvanishing vector field if and only if n is odd.*

Proof. If n is odd just set

$$f(x_1, \ldots, x_{n+1})) := (-x_2, x_1, -x_3, x_4, \ldots, -x_{n+1}, x_n)$$

which of course works since $n + 1$ is even, and so the terms pair up.

If V is a nonvanishing vector field, by replacing V with $z \mapsto V(Z)/||V(z)||$ we get a continuous map $V : S^n \to S^n$. Assuming this, for $t \in [0, 1]$ let

$$V_t(z) := (\cos \pi t)z + (\sin \pi t)V(z)$$

V_t is a homotopy from the identity to the antipodal map A, so that $1 = \deg(\mathrm{Id}) = \deg(A) = (-1)^n + 1$, which implies that n must be odd. Note that the nonvanishing vector field V was used to give us, at each $z \in S^n$, a direction in which to rotate in order to move z to $-z$.

□

One can rephrase nonvanishing of V as saying that "V is linearly independent". Vector fields $\{V_i\}$ on S^n are **linearly independent** if for each $z \in S^n$ the vectors $\{V_i(z)\}$ are linearly independent in the tangent space TS^n_z. A much deeper problem is to determine, for each $n \geq 1$, the maximal number of linearly independent vector fields on S^n. This famous problem was solved by Frank Adams (in "Vector Fields on Spheres", *Annals of Math.*, 1962). The fact that he got the exact answer is truly remarkable, considering that the exact answer is given by the following.

Theorem 1.3.12 (Adams, 1962). *Let $n \geq 1$. Write*

$$n + 1 = 2^{4a+b}(2k + 1)$$

for a, b, k integers with $0 \leq b \leq 3$. Then the maximal number of linearly independent vector fields admitted by S^n is precisely $2^b + 8a - 1$.

1.4 CW complexes and cellular homology

With a basic understanding of maps $f : S^n \to S^n$ under our belts, we are now able to develop a homology theory that is most computable in practice - CW homology. This will agree with simplicial and singular homology when they are all defined. The CW homology of CW complexes is quite similar to the simplicial homology of Δ-complexes, but many fewer cells are typically needed, and so computations are much easier.

CW complexes. The class of spaces we consider is the class of CW-complexes. Consider the following inductive procedure:

Base step: Let X^0 be a discrete set, that is, a set of points given the discrete topology. We call X^0 the 0**-skeleton** of X.

Inductive step: Suppose for some $n \geq 1$, the $(n-1)$-skeleton X^{n-1} is already defined. Suppose we have a collection of n-disks $\{D^n_\alpha : \alpha \in I\}$, where I is some index set, and for each α we are given a continuous map $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$. The n**-skeleton** of X, denoted X^n, is defined to be the quotient space

$$X^n := [X^{n-1} \coprod_{\alpha \in I} D^n_\alpha]/\sim$$

where $\sim$ is defined by

$$x \sim \phi_\alpha(x) \;\; \forall x \in \partial D^n_\alpha, \;\; \forall \alpha \in I$$

Now set $X = \cup_{n \geq 0} X^n$. The space X is called a **CW complex**. The images $\phi(D^n_\alpha)$ are called the n**-cells** of X, and we say that X^n is obtained from X^{n-1} by "adding n-cells". The maps ϕ_α are called the **attaching maps**. If N is such that $X^n = X^N$ for all $n \geq N$, we say that X is n**-dimensional**. Note that each X^n has a well-defined topology, namely the quotient topology, so when X is finite-dimensional $X = X^N$ has this topology. W

When X is infinite-dimensional we need to specify a topology on X. We do this by declaring a subset $A \subset X$ to be open precisely when $A \cap X^n$ is open for each $n \geq 0$. This is called the **weak topology** on X. Of course one can replace the word "open" by "closed" in this definition.

When a topological space Y is homeomorphic to a specific CW complex X, we refer to X as "a CW structure on Y". We say that a closed subset $Z \subseteq X$ is a **subcomplex** of X if it is a union of cells. Clearly in this case Z is a CW complex in its own right.

The class of CW-complexes includes all Δ-complexes. But CW structures are combinatorially much simpler than Δ-complex structures.

Example 1.4.1 (**First examples of CW complexes).**

1. A 1-dimensional CW complex is called a **graph**.

2. $S^n, n > 0$, has a CW-complex structure with one 0-cell v and one n-cell D^n, with attaching map $\phi : D^n \to \{v\}$ the constant map.

3. The genus $g \geq 1$ surface Σ_g can be given the structure of a CW complex with one vertex, $2g$-edges $a_1, b_1, \ldots, a_g, b_g$ with the obvious attaching maps, and with one 2-cell D^2 with attaching map $\phi : \partial D^2 \to (\cup a_i \cup b_i)$ given by dividing the circle ∂D^2 into $4g$ segments, and mapping the segments in order onto the edges $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

4. The n-dimensional **real projective space**, denoted $\mathbb{RP}^n$, is defined to be the space of lines through the origin in $\mathbb{R}^{n+1}$. Since the set of such lines is in bijective correspondence with the set of (unordered) pairs of antipodal points in S^n, we can topologize $\mathbb{RP}^n$ as a quotient space

$$\mathbb{RP}^n = S^n/(v \sim -v \quad \forall v \in S^n) \tag{1.12}$$

where S^n is here thought of as the set of unit vectors in $\mathbb{R}^{n+1}$. From (1.12) it follows that $\mathbb{RP}^n$ is the quotient of the upper hemisphere D^n of S^n by the equivalence relation $v \sim -v$ for all $v \in \partial D^n$ (the equator). Let $p : D^n \to \mathbb{RP}^n$ be the quotient map. By induction on n it follows that $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by adding the single n-cell D^n, with attaching map the restriction of p to ∂D^n. Thus $\mathbb{RP}^n$ is a CW complex with one cell in each dimension $0 \leq i \leq n$. The space $\mathbb{RP}^\infty = \cup_{n \geq 1} \mathbb{RP}^n$ is a CW complex with one cell in each dimension $i \geq 0$.

5. Let relatively prime numbers $q \geq 1, p \geq 2$ be given. Think of the standard $S^n \subset \mathbb{R}^{n+1}$. Let Rot_p be the rotation of S^n by angle $2\pi q/p$ around the vertical line passing through the north and south poles (i.e. the x_{n+1}-axis). Let $r : S^n \to S^n$ denote the reflection $r(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$. The **$n$-dimensional lens space** $L(p, q)$ is defined to be the quotient of D^n by the equivalence relation

$$v \sim \mathrm{Rot}_p \circ r(v) \;\text{ for all } v \in \partial D^n$$

Note that $L(2, 1) \approx \mathbb{RP}^n$. We claim that $L(p, q)$ has a CW complex structure with one cell in each dimension $0 \leq i \leq n$.

6. Let X and Y be CW complexes. The **product CW structure** on $X \times Y$ has one n-cell $D^p \times D^q$ for every pair consisting of a p-cell D^p of X and a q-cell D^q of Y with $n = p + q$. discuss attaching maps

Cellular homology. The construction of CW homology rests on two facts: the fact that we understand and can compute the degree of maps $f : S^n \to S^n$, and the the fact that CW complexes are built inductively by gluing in cells along their boundary spheres. The latter is summarized by the following, following Lemma 2.34 of [Ha].

Proposition 1.4.2 (Relative homology of CW complexes). *Let X be a CW complex. Then for each $n \geq 0$:*

1.
$$H_i(X^{(n)}, X^{(n-1)}) = \begin{cases} 0 & i \neq n \\ \mathbb{Z}^d & i = n \end{cases}$$
where d is the number of n-cells in X.

2. $H_i(X^{(n)}) = 0$ *for* $i > n$.

3. *The inclusion* $i : X^{(n)} \to X$ *induces an isomorphism* $i_* : H_k(X^{(n)}) \to H_k(X)$ *when* $k < n$.

Proof. Look at the long exact sequence of the pair $(X^{(n)}, X^{(n-1)})$. Since the pair is reasonable,
$$H_i(X^{(n)}, X^{(n-1)}) \approx H_i(X^{(n)}/X^{(n-1)}).$$
But $X^{(n)}/X^{(n-1)}$ is a wedge of n-spheres, one for each n-cell of X. The proposition follows. □

Note that the proof of Proposition 1.4.2 gives that the basis for $H_n(X^{(n)}, X^{(n-1)})$ is given by the set of n-cells $\{\sigma_r\}$ of X, where the basis element corresponding to $\sigma_r : D^n \to X^{(n)}$ is given by the composition
$$D^n \xrightarrow{r} X^{(n)} \xrightarrow{p} X^{(n)}/X^{(n-1)}$$
where p is the natural projection.

Proposition 1.4.2 gives us a hint of how to construct a new kind of homology theory for CW complexes. Let X be a CW complex. We define the group of **cellular n-chains** $C_n^{\mathrm{CW}}(X)$ by setting
$$C_n^{\mathrm{CW}}(X) := H_n(X^{(n)}, X^{(n-1)}) \approx \mathbb{Z}^{\#\{\sigma_\alpha\}}$$

where H_n denotes singular homology and where $\{\sigma_\alpha\}$ is the set of n-cells of X_n. Note that in most of the examples we've seen, the rank of $C_n^{\mathrm{CW}}(X)$ is much smaller than the rank of the corresponding simplicial chain groups. This will make calculations much easier (see below).

In order to get a true chain complex we will need to find homomorphisms $d_n : C_n^{\mathrm{CW}}(X) \to C_{n-1}^{\mathrm{CW}}(X)$ such that $d_{n-1} \circ d_n = 0$. Well, the long exact sequence of the pair $(X^{(n)}, X^{(n-1)})$ gives a boundary homomorphism

$$\partial_n : H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}).$$

The long exact sequence of the pair$(X^{(n-1)}, X^{(n-2)})$ also gives a homomorphism

$$i_{n-1} : H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

We now let d_n be the composition $d_n = i_{n-1} \circ \partial_n$, that is

$$d_n : H_n((X^{(n)}, X^{(n-1)}) \xrightarrow{\partial_n} H_{n-1}(X^{(n-1)}) \xrightarrow{i_{n-1}} H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

It is straightforward to check (see [Ha], page 139) that $d_{n-1} \circ d_n = 0$. Thus $\mathcal{C}^{\mathrm{CW}}(X) := \{C_n^{\mathrm{CW}}(X), d_n\}$ is a chain complex, called the **complex of CW chains** on X.

Definition 1.4.3 (CW homology). Let X be a CW complex, and let $\mathcal{C}^{\mathrm{CW}}(X)$ denote the complex of CW chains on X. The **CW homology** of X, denoted by $H_n^{\mathrm{CW}}(X)$, is the homology of this chain complex:

$$H_n^{\mathrm{CW}}(X) := H_n(\mathcal{C}^{\mathrm{CW}}(X))$$

The main usefulness of CW homology comes from the following.

Theorem 1.4.4. *Let X be a CW complex. Then $H_n^{\mathrm{CW}}(X) \approx H_n(X)$ for all $n \geq 0$.*

Theorem 1.4.4 has a number of useful consequences. First, it implies that $H_n^{\mathrm{CW}}(X)$ is a topological invariant; indeed an invariant of homotopy equivalence. Second, if X has no n-cells for any $n > N$, then clearly $H_n(X) = H_n^{\mathrm{CW}}(X) = 0$ for all $n > N$. Finally, since the set of n-cells forms a basis for $C_n(X)$, we obtain the trivial bound that $H_n(X) = H_n^{\mathrm{CW}}(X)$ has rank at most the number of n-cells of X.

Remark 1.4.5 (Euler characteristic). By the topological invariance of CW-homology, we can use CW-chains to compute Euler characteristic. This makes computations quite easy. For example, it is possible simply by inspection to compute that $\chi(S^n) = 1 + (-1)^{n+1}$, that $\chi(S_g) = 2 - 2g$, and that $\chi(\mathbb{RP}^n) = 0$ for n odd and $\chi(\mathbb{RP}^n) = 1$ for n even.

Similarly we can use CW-chains when computing Lefschetz numbers.

In order to compute the CW-homology groups we need a better handle on the boundary maps. This is given in terms of degrees of maps between spheres. We have for each $n \geq 0$ that $C_n^{\mathrm{CW}}(X)$ is generated by the set of n-cells of X. Let $\{\sigma_\alpha : D^n \to X\}$ be the set of n-cells of X, and let $\{\tau_\beta : D^{n-1} \to X\}$ be the set of $(n-1)$-cells. We then have, for each fixed α and β, a map

$$\psi_{\alpha\beta} : S^{n-1} \to S^{n-1}$$

defined as the composition of σ_α restricted to $\partial D_n = S^{n-1}$, which has image lying in $X^{(n-1)}$, followed by the quotient map $X^{(n-1)}/[X^{(n-1)} - \tau_\beta(D^{n-1})]$. We set

$$d_{\alpha\beta} := \deg(\psi_{\alpha\beta}) \in \mathbb{Z}$$

Thus $d_{\alpha\beta}$ measures how many times the boundary of the n-cell σ_α wraps around the $(n-1)$-cell τ_β. Since degree is computable in practice, so are the numbers $d_{\alpha\beta}$. The key to computing CW-homology is the following lemma.

Lemma 1.4.6 (Computing d_n). *Let notation be as above. Then for each α*

$$d_n(\sigma_\alpha) = \sum_\beta d_{\alpha\beta}\tau_\beta$$

Proof. This is easy from the definitions. Try it yourself. Otherwise see [Ha], page 141. □

Example 1.4.7. We begin with some examples that show how easy it is to compute CW homology. The CW structure of some of the examples is given above in Example 1.4.1 above.

1. For $n > 0$ each boundary map on CW-chains $C_i(S^n)$ on S^n is the zero map, so that $\tilde{H}_i(S^n) = 0$ for $i \neq n$ and $\tilde{H}_n(S^n) = C_n(S^n) = \mathbb{Z}$, generated by the n-cell.

2. For the genus $g \geq 1$ surface the complex of CW-chains is

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^{2g} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

where $\partial_1 = 0$ and $\partial_2(z) = a_1 + b_1 - a_1 + b_1 + \cdots + a_g + b_g - a_g - b_g = 0$ where z is the generator of $C_2(S_g)$. It follows that

$$H_i(S_g) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & i > 2 \end{cases}$$

3. Let $X = S^n \times S^n$. Then X can be given as a CW complex with one 0-cell, two n-cells, and one $2n$-cell. When $n = 1$ then $X = S^1 \times S^1$ is the torus, which we worked out above. For $n > 2$, the dimensions are such that all boundary maps are the zero map, so

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ \mathbb{Z}^2 & i = n \\ 0 & i \neq 0, n, 2n \end{cases}$$

4. Let $X = \mathbb{RP}^n$.

Example 1.4.8 (**Complex projective space).** The n-dimensional **complex projective space** $\mathbb{CP}^n$ is the space of (complex) lines in the $(n+1)$-dimensional complex vector space $\mathbb{C}^{n+1}$ passing through the origin. One cell in each dimension $2i$ with $0 \leq i \leq n$. Now a line in $\mathbb{C}^{n+1}$ is just the $\mathbb{C}$-span of a single nonzero vector $v \in \mathbb{C}^{n+1}$. Two vectors u, v determine the same line precisely when $u = \lambda v$ for some $\lambda \in \mathbb{C}^*$. Thus $\mathbb{CP}^n$ is the quotient $\mathbb{C}^{n+1} - \{0\}/\sim$ where $u\tilde{v}$ if $u = \lambda v$ for some $\lambda \in \mathbb{C}^*$. Each equivalence class $[v] \in \mathbb{CP}^n$ clearly has a representative v with $||v|| = 1$. Thus if we denote by S^{2n+1} the unit sphere in $\mathbb{C}^{n+1}$, we have that $\mathbb{CP}^n \approx S^{2n+1}/\sim$ where $u \sim v$ if $u = \lambda v$ with $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$. The set of such complex numbers is the unit circle in $\mathbb{C}$, where λ acts on and $(n+1)$-tuples in $\mathbb{C}^{n+1}$ by rotating each coordinate by the argument of λ.

For $n = 1$, we have that $\mathbb{CP}^1$ is the quotient of S^3 by a free S^1 action, with quotient $\mathbb{CP}^1 = S^2$, giving the famous **Hopf fibration** $S^1 \to S^3 \to S^2$, which we will study later. Now let $n \geq 1$. The last coordinate v_{n+1} of $v \in S^{2n+1} \subset \mathbb{C}^{n+1}$ is just a complex number. By multiplying v_{n+1} by λ with $|\lambda| = 1$, in other words by rotating the vector

v_{n+1} in $\mathbb{C}$, we can assume that v_{n+1} has zero imaginary part and positive real part. Since $\sum_{i=1}^{n+1} v_i = ||v|| = 1$, another way to look at this is that v can be taken to be of the form $v = (w, \sqrt{1 - |w|^2})$ with $w \in \mathbb{C}^n$ and $||w|| \leq 1$. The set of such w is just the closed unit ball $D^{2n} \subset \mathbb{C}^n$.

We have used up all of our freedom in finding an element for each equivalence class $[v] \in \mathbb{CP}^n$, except that in the special case when $v_{n+1} = 0$, that is when $w \in \partial D^{2n}$, we can again multiply the $(n+1)$-tuple V by any $\lambda \in \mathbb{C}^*$ without changing the last coordinate V_{n+1}. We are thus left with $\mathbb{CP}^n = D^{2n}/\sim$ where $\sim$ is given by $w \sim \lambda w$ for each $w \in \partial D^{2n}$ and $\lambda \in \mathbb{C}^*$, where $\partial D^n \approx S^{2n-1}$ is thought of as the unit sphere in $\mathbb{C}^n$. But the discussion above proves that the quotient of S^{2n-1} by the equivalence relation $\sim$ is precisely $\mathbb{CP}^{n-1}$.

When $n = 1$ the above discussion specializes to give that $\mathbb{CP}^1$ is obtained from D^2 by identifying its boundary $\partial D^2 \approx S^1$ to a point (since any unit vector can be rotated to any other), giving that $\mathbb{CP}^1$ is a CW-complex with one 0-cell and one 2-cell. With this as base case, the last paragraph gives by induction that $\mathbb{CP}^n$ is a CW complex with one cell in each dimension $0, 2, \ldots, 2n$, with the attaching map defined inductively by $D^{2n} \to \mathbb{CP}^{n-1}$ equal to the quotient map $\partial D^{2n} \approx S^{2n-1} \to S^{2n-1}/\sim \approx \mathbb{CP}^{n-1}$ described above.

The above description gives that $\mathbb{CP}^n$ is a CW complex with complex of CW-chains given by $C_2 = C_4 = \cdots = C_{2n} = \mathbb{Z}$, and with $C_i = 0$ otherwise. Of course this implies that all boundary maps ∂_i are zero, so that

$$H_i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & i = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

This computation extends immediately to give that $H_i(\mathbb{CP}^\infty) = \mathbb{Z}$ for all even $i \geq 0$ and $H_i(\mathbb{CP}^n) = 0$ for all odd $i \geq 0$.

1.5 Mayer-Vietoris

The Mayer-Vietoris sequence is one of the most use tools in the computation of homology.

Theorem 1.5.1 (Mayer-Vietoris). *Let X be any space, and let $i : A \to X$ and j :*

$B \to X$ be inclusions of subspaces. Suppose that the union of the interiors $\mathrm{int}(A) \cup \mathrm{int}(B)$ *is all of X. Then there is a long exact sequence in homology:*

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{i_* \oplus j_*} H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow H_0(X) \to 0$$

When $A \neq \emptyset$, such a long exact sequence exists for reduced homology. Both sequences are natural with respect to homomorphisms induced by continuous maps of triples $f : (X, A, B) \to (X', A', B')$.

Proof. Define $C_n(A+B)$ to be the subgroup of $C_n(X)$ consisting of those elements that can be written as the sum of an element of $C_n(A)$ and an element of $C_n(B)$. Note that when $A \cap B = \emptyset$ then $C_n(A+B) = C_n(A) \oplus C_n(B)$, but when $A \cap B \neq \emptyset$ this is typically no longer true. Let $\phi : C_n(A) \oplus C_n(B) \to C_n(A+B)$ be defined by $\phi(a, b) := i(a) + j(b)$. Then there is a short exact sequence of chain complexes

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{i_\# \oplus j_\#} C_n(A) \oplus C_n(B) \xrightarrow{\phi} C_n(A+B) \longrightarrow 0 \tag{1.13}$$

is exact. Here we are implicitly using that the image of ϕ lies in $C_n(A+B)$. By the Fundamental Theorem of Homological Algebra, (1.13) induces a long exact sequence

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{i_* \oplus j_*} H_n(A) \oplus H_n(B) \xrightarrow{\phi_*} H_n(A+B) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow H_0(A+B) \to 0$$

where of course $H_*(A+B)$ denotes the homology of the chain complex $\{C_n(A+B)\}$. The case of reduced homology is similar, and naturality is straightforward to check.

We claim that the inclusion $h : C_n(A+B) \to C_n(X)$ induces an isomorphism on homology (which one can check is natural, and also holds for reduced homology). With the above, this claim finishes the proof.

When X is a Δ complex with finitely many cells, the idea behind the claim is that since the interiors of A and B cover X, one can find a Δ-complex structure on X, each of whose cells lies in A or B (or both).

One can use the simplicial approximation theorem to directly define a chain homotopy inverse of h, giving that it is a chain homotopy equivalence. A slightly simpler method is to note that the short exact sequence

$$0 \to C_n(A+B) \to C_n(X) \to C_n(X)/C_n(A+B) \to 0$$

gives a long exact sequence, from which we see that it is enough to prove that the chain complex $\{C_n(X)/C_n(A+B)\}$ has vanishing homology in each degree. Unravelling the definitions, we must prove that for any $\sigma \in C_n(X)$ with $\partial\sigma \in C_{n-1}(A+B)$, there exists $\tau \in C_{n+1}(X)$ so that $\sigma + \partial_{n+1}\tau$ lies in $C_n(A+B)$. The construction of τ can be done by using simplicial approximation, repeatedly subdiving X. □

The Mayer-Vietoris Theorem gives us a powerful tool to compute homology.

Example 1.5.2 (Computing using Mayer-Vietoris). We begin with some easy examples.

1. $X = S^n, A = D^n_+, B = D^n_-$.
2. Connect sums $M\#N$ of n-manifolds.
   (a) Can compute $H_i(S_g)$ by induction, using $S_1 = T^2$ and $S_{g+1} \approx S_g\#T^2$.
   (b) 3-manifolds. Mention Kneser-Milnor Prime Decomposition Theorem.
   (c) Recall the following problem from the midterm: Let S_g be a closed, connected (oriented) genus $g \geq 2$ surface. Let $f : S_g \to S_g$ be a homeomorphism. Let M^3_f be the **mapping torus of** f:

$$M^3_f := \frac{S_g \times [0,1]}{(x,0) \sim (f(x),1) \ \ \forall x \in S_g}$$

   Assume that $f_* : H_1(S_g) \to H_1(S_g)$ is the identity. [Note: there exist incredibly complicated f, far from being homotopic to Id, with this property.]

   (a) Prove that $H_2(M^3_f) \approx \mathbf{Z}^{2g+1}$.
   (b) Even if you can't do part (a), find/guess $2g+1$ maps of surfaces into M^3_f that represent a basis for $H_2(M^3_f)$.

Chapter 2

Cohomology

Cohomology is a functor that is morally a kind of "dual" to homology. As abelian groups each can be constructed from the other. However, moving to this dual point of view exposes a remarkably rich structure that is ubiquitous in mathematics. Cohomology appears naturally in areas as diverse as algebraic geometry, dynamical systems, the theory of group extensions, the theory of foliations, and more. The fundamental nature of cohomology is not surprising when one realizes that the act of carrying when adding numbers is the same thing as evaluating a certain cocycle.

2.1 Cohomology of a chain complex

Just as with homology theory, one constructs the cohomology of a chain complex, and then applies this construction to simplicial chains, singular chains, CW chains, etc. This section is purely homological-algebraic. There is no topology here. We will apply this homological algebra to topology in the sections that follow.

2.1.1 Definition and basic properties

To construct cohomology groups we first have to understand how to "dualize" an abelian group A with respect a fixed "coefficient group" G. To this end, fix an abelian group

G, for example $G = \mathbb{Z}, \mathbb{Z}/d\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. For any abelian group A, let $\mathrm{Hom}(A, G)$ denote the set of homomorphisms $\phi : A \to G$. The set $\mathrm{Hom}(A, G)$ is itself an abelian group under addition:

$$(\phi + \psi)(a) := \phi(a) + \psi(a) \tag{2.1}$$

where the + on the left-hand side of (2.1) denotes the group operation in $\mathrm{Hom}(A, G)$ and the + on the right-hand side of (2.1) denotes the group operation in G. Any homomorphism $\Psi : A \to A'$ of abelian groups induces a homomorphism

$$\Psi^* : \mathrm{Hom}(A', G) \to \mathrm{Hom}(A, G)$$

defined by

$$\Psi^*(\phi)(a) := \phi \circ \Psi(a) \quad \text{for all} \quad \phi \in \mathrm{Hom}(A', G)$$

The operation $\phi \mapsto \phi^*$ is *natural* in the sense that $\mathrm{Id}^* = \mathrm{Id}$ and $\phi\psi)^* = \psi^* \circ \phi^*$. What we have just proved is that $\mathrm{Hom}(-, G)$ is a *contravariant* functor from the category of abelian groups and homomorphisms to itself.

Now let $\mathcal{C} := \{C_n, \partial_n\}$ be a chain complex of free abelian groups. Applying the functor $\mathrm{Hom}(-, G)$ to this setup gives a collection of abelian groups $C_n^* := \mathrm{Hom}(C_n, G)$, called the **cochain groups**, and homomorphisms $\partial_n^* : C_{n-1}^* \to C_n^*$, called **coboundary homomorphisms**. For historic reasons the dual homomorphism ∂_n^* is denoted by δ^n. We think of elements $f \in C_n$ as G-valued functions on C_n. Since C_n is free-abelian, elements of C_n^* are in bijective correspondence with labelings of the generators of C_n by elements of G.

Since $\partial_n \circ \partial_{n+1} = 0$ for each $n \geq 0$, it follows from functoriality that

$$\delta^{n+1} \circ \delta^n = \partial_{n+1}^* \circ \partial_n * = (\partial_n \circ \partial_{n+1})^* = 0^* = 0$$

The collection $\mathcal{C}^* := \{C_n^*, \delta^n\}$ is called the **cochain complex with coefficients in** G associated to the chain complex $\mathcal{C}$.

Definition 2.1.1 (**Cohomology with coefficients in** G**).** Let the terminology be as above. For any $n \geq 0$, the n^{th} **group of** $\mathcal{C}$ **with coefficients in** G, denoted by $H^n(\mathcal{C}; G)$ is defined to be

$$H^n(\mathcal{C}; G) := \mathrm{kernel}(\delta^n)/\mathrm{image}(\delta^{n-1})$$

Basic properties. Just as with homology theory, it is straightforward to check that any chain map $f : \mathcal{C} \to \mathcal{C}'$ induces for each $n \geq 0$ homomorphisms $f_n^* : H^n(\mathcal{C}'; G) \to H^n(\mathcal{C}; G)$, and the association $f \mapsto f^*$ is *natural* in the sense that

$$\mathrm{Id}_{\mathcal{C}}^* = \mathrm{Id}_{H^*(\mathcal{C};G)} \quad \text{and} \quad (f \circ g)^* = g^* \circ f^*$$

for chain maps $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{C}' \to \mathcal{C}''$. These facts follow from the corresponding properties of $\mathrm{Hom}(-, G)$ and the fact that

$$f \circ \partial = \delta \circ f$$

We have just shown that:

$H^n(-; G)$ is a contravariant *functor from the category of chain complexes and chain maps to the category of abelian groups and homomorphisms.*

Note the contravariance; it comes from the contravariance of the functor $\mathrm{Hom}(-, G)$.

Just as with homology, any chain homotopy between chain maps $f, g : \mathcal{C} \to \mathcal{C}'$ gives for each n the equality $f^* = g^* : H^n(\mathcal{C}'; G) \to H^n(\mathcal{C}; G)$. In particular any chain homotopy equivalence induces an isomorphism of cohomology groups.

For each statement about homology there is typically a corresponding statement for cohomology (although, as we will see, the converse is not true!). The proofs are usually the same. A particularly important example is the cohomology version of the Fundamental Theorem of Homological Algebra.

Theorem 2.1.2 (FTHA, cohomology version). *Let*

$$0 \to \mathcal{A} \stackrel{\phi}{\to} \mathcal{B} \stackrel{\psi}{\to} \mathcal{C} \to 0$$

be a short exact sequence of chain complexes. Then for each $n \geq 0$ there is a "connecting homomorphism" ∂ and a "long exact sequence"

$$\cdots \stackrel{(\phi_{n+1})^*}{\longleftarrow} H^{n+1}(\mathcal{C}) \stackrel{\partial}{\longleftarrow} H^n(\mathcal{A}) \stackrel{(\psi_n)^*}{\longleftarrow} H^n(\mathcal{B}) \stackrel{(\phi_n)^*}{\longleftarrow} H^n(\mathcal{C}) \stackrel{\partial}{\longleftarrow} \cdots$$

2.1.2 The Universal Coefficients Theorem

It is definitely not always true that the abelian groups $H^n(\mathcal{C}; G)$ and $H_n(\mathcal{C}; G)$ are equal, or somehow "duals" of each other, even when $G = \mathbb{Z}$. For example, consider the chain complex

$$\cdots \longrightarrow C_3 \xrightarrow{0} C_2 \xrightarrow{\times 2} C_1 \xrightarrow{0} C_0 \longrightarrow 0$$

$$\cdots \longrightarrow 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

whose corresponding dual chain complex $\mathcal{C}^*$ is given by

$$\cdots \longleftarrow 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

We then have

$$H_0(\mathcal{C}) = \mathbb{Z} \text{ and } H_1(\mathcal{C}) = \mathbb{Z}/2\mathbb{Z} \text{ and } H_2(\mathcal{C}) = \mathbb{Z}$$

but

$$H^0(\mathcal{C};\mathbb{Z}) = \mathbb{Z} \text{ and } H^1(\mathcal{C};\mathbb{Z}) = \mathbb{Z} \text{ and } H^2(\mathcal{C};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

No, the truth is more subtle. It turns out that if we know $H_i(\mathcal{C})$ for every $i \geq 0$, then the groups $H^i(\mathcal{C};G)$ are indeed determined.

How can we measure the difference between homology and cohomology? Well, let $\mathcal{C} := \{C_n, \partial_n\}$ be any chain complex. We claim there is a map

$$\Psi : H^n(\mathcal{C};G) \to \operatorname{Hom}(H_n(\mathcal{C}), G)$$

defined as follows. For any element $[\phi] \in H^n(\mathcal{C}, G)$ pick a cycle $\phi \in \operatorname{Hom}(C_n, G)$ with $\delta \circ \phi = 0$ representing the equivalence class $[\phi]$. Since

$$\phi(B_n) = \phi \circ \partial(C_n) = \delta \circ \phi(C_n) = 0$$

and so the restriction of ϕ to Z_n induces a homomorphism $\overline{\phi} : Z_n/B_n \to G$. Another choice of representative $\psi \in \operatorname{Hom}(C_n, G)$ of $[\phi] \in H^n(\mathcal{C}, G)$ is of the form $\psi = \phi + \delta\tau$ for some τ, and so on any $\sigma \in Z_n$ we have $\delta\tau(z) = \tau\partial z = 0$, so that

$$\psi(z) = \phi(z) + \delta\tau(z) = \phi(z).$$

This proves that the map

$$\Psi([\phi]) := \overline{\phi}$$

is well-defined. It is straightforward to check that Ψ is surjective, so that we have a short exact sequence

$$0 \longrightarrow \ker(\Psi) \longrightarrow H^n(\mathcal{C}, G) \xrightarrow{\Psi} \operatorname{Hom}(H_n(\mathcal{C}, G)) \longrightarrow 0 \tag{2.2}$$

$\ker(\Psi)$ is not always trivial. The subtlety of the problem in determining $\ker(\Psi)$ is that it does not just depend on $H_n(\mathcal{C})$. in order to describe $\ker(\Psi)$ precisely we need to use a certain functor called Ext, which we now describe.

Fix an abelian group G. To any abelian group A we define an abelian group $\mathrm{Ext}(A, G)$ via the following procedure: Let I be any set of generators for A. By the universal property of free abelian groups, the free abelian group V_I on the set I surjects onto A; let K be the kernel of this surjection. Note that since K is a subgroup of the free abelian group V_I, it is also free abelian. We thus have a chain complex $\mathcal{C}$ of free abelian groups

$$\cdots \to 0 \to C_2 \to C_1 \to C_0 \to 0$$

$$\cdots \to 0 \to K \to V_I \to A \to 0$$

which gives a dual chain complex

$$\cdots \leftarrow 0 \leftarrow K^* \leftarrow V_I^* \leftarrow A^* \leftarrow 0$$

By construction we see that $H^0(\mathcal{C}; G) = H^2(\mathcal{C}; G) = 0$. it is not hard to check (do it!) that the group $H^1(\mathcal{C}; G)$, which in general may not vanish, does not depend on the choice of generating set I for A. We then define

$$\mathrm{Ext}(A; G) := H^1(\mathcal{C}; G)$$

Ext is actually a functor from the category of abelian groups to itself, but we won't need this. By the way, the name "$\mathrm{Ext}(A, G)$" comes from the fact that it is a key object in classifying (isomorphism classes of) group extensions of the group G by the group A.

The main thing we will need are the following elementary computations.

- $\mathrm{Ext}(A \oplus B, G) = \mathrm{Ext}(A, G) \oplus \mathrm{Ext}(B, G)$.
- $\mathrm{Ext}(A, G) = 0$ for free abelian A.
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$, where nG is the subgroup of G consisting of n^{th} powers $g^n \in G$ (or, written additively, elements ng with $g \in G$).

These three computations easily determine $\mathrm{Ext}(A, G)$ for any finitely generated abelian group A.

We can now describe $\ker(\Psi)$ from Equation (2.2) above; namely

$$\ker(\Psi) = \mathrm{Ext}(H_{n-1}(\mathcal{C}, G)) \tag{2.3}$$

Thus it is the $(n-1)^{\text{st}}$ homology group that determines, via $\mathrm{Ext}(-, G)$, the difference between $H_n(\mathcal{C})$ and $H^n(\mathcal{C}, G)$. We record this as the following.

Theorem 2.1.3 (Universal Coefficients Theorem). *Let $\mathcal{C}$ be a chain complex of free abelian groups. Then there exists for each $n \geq 0$ a short exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \longrightarrow H^n(\mathcal{C}; G) \xrightarrow{\Psi} \mathrm{Hom}(H_n(\mathcal{C}), G) \longrightarrow 0$$

The proof of Equation (2.3) is straightforward but somewhat involved; the interested reader can consult §3.1 of [Ha]. By the above discussion, it implies Theorem 2.1.3. Note that Theorem 2.1.3 implies that the collection of homology groups $\{H_n(\mathcal{C})$ determines the collection of cohomology groups $\{H^n(\mathcal{C}; G)\}$. An important special case of the Universal Coefficients Theorem is when G is a field, which gives the following.

Corollary 2.1.4. *Let $\mathcal{C}$ be a chain complex of free abelian groups. Then for any field $\mathbb{F}$ and any $n \geq 0$:*

$$H^n(\mathcal{C}; \mathbb{F}) \approx \mathrm{Hom}(H_n(\mathcal{C}), \mathbb{F})$$

Another direct corollary of Theorem 2.1.3 is the following.

Corollary 2.1.5. *Let $\mathcal{C}$ be a chain complex of free abelian groups. Suppose that $H_i(\mathcal{C})$ is finitely generated for $i = n, n-1$. Let $T_i(\mathcal{C})$ be the torsion subgroup of $H_i(\mathcal{C})$. Then*

$$H^n(\mathcal{C}; \mathbb{Z}) = H_n(\mathcal{C})/T_n(\mathcal{C}) \oplus T_{n-1}(\mathcal{C}).$$

2.2 Cohomology of spaces

With the above algebraic setup, we can plug in the simplicial, singular, and CW chain groups $C_n(X)$ of Δ-complexes (or spaces, or CW complexes) X to define the corresponding cohomology groups with coefficients in G via

$$H^n(X;G) := H^n(\{C_n(X)\})$$

In this case we can work out what the coboundary operator $\delta^n : C^n(X) \to C^{n+1}(X)$ looks like, namely for any $\sigma \in C_{n+1}(X)$ - GIVE.

With this setup we now have all of the theorems and computational tools for cohomology that we had for homology, such as: homotopy invariance, relative groups and the LES of pairs, excision, Mayer-Vietoris, etc.

2.2.1 Cup product

Cohomology theory is so useful because it has a lot more structure; in particular it allows us to associate to any space X not just groups $H^n(X)$, but a ring, as we now explain.

In order to define the extra structure we want, we need to fix once and for all not just an abelian group of coefficients, but a *ring* R of coefficients. Of course a ring is just an abelian group under addition, but it has the additional structure of multiplication. Here you should think of $\mathbb{R} = \mathbb{Z}, \mathbb{Z}/d\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Given a space X we can combine the cohomology groups into one group by declaring

$$H^*(X;R) := \bigoplus_{n\geq 0} H^n(X;R)$$

We say that an element $c \in H^*(X;R)$ has **degree** d if $c \in H^d(X;R)$. We define a product on the abelian $H^*(X;R)$, as follows: Given any cochains $a \in C^i(X;R), b \in C^j(X;R)$, define the **cup product** $\sigma \cup \tau C^{i+j}(X;R)$ to be the unique linear extension of the homomorphism defined on a basis element $\sigma : \Delta^{i+j} = [v_0 v_1 \cdots v_{i+j}] \to R$:

$$(a \cup b)(\sigma) := a(\sigma|_{[v_0 \cdots v_i]}) \cdot b(\tau|[v_i \cdots v_{i+j}]) \tag{2.4}$$

Here the symbol $\cdot$ on the right-hand side of (2.4) is multiplication in the ring R. This is exactly where we use that R is a ring, not just a group. It is easy to verify directly from the definition that the operation $\cup$ satisfies a "graded product rule" under the coboundary operator: for each $a \in C^i(X;R), b \in C^j(X;R)$

$$\delta(a \cup b) = \delta a \cup b + (-1)^i a \cup \delta b$$

Equation (2.4) implies that the cup product of cocycles is again a cocycle, and the cup product $a \cup \delta b = \delta a \cup b = 0$, so that cup product induces a homomorphism

$$\cup : H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R)$$

In particular we have defined a multiplication map

$$\cup : H^*(X;R) \otimes H^*(X;R) \to H^*(X;R)$$

This multiplication is both associative and distributive, since this is already true at the level of chains. The zero element $0 \in H^0(X;R)$, namely the zero map, is the identity element of $H^*(X;R)$ as an abelian group under addition. Let $1 \in H^0(X;R)$ denote the unique linear extension of the homomorphism $C_0(X) \to R$ that assigns the value 1 to each 0-simplex. It is clear that $a \cup 1 = 1 \cup a = a$ for all $a \in H^*(X;R)$. Note here that we have just used the fact that R is a ring with 0 and 1.

We have just proved the following.

Proposition 2.2.1 ($H * (X)$ **is a ring).** *Let R be any ring and let X be any space. Then $H^*(X;R)$ is a ring under the operation of addition of cocycles and cup product.*

It follows from the construction that this ring structure is natural: any continuous map $f : X \to Y$ of spaces induces a (grading preserving) homomorphism of graded rings $f^* : H^*(Y;G) \to H^*(X;G)$, and if f is homotopic to g then $f^* = g^*$. It follows that any homotopy equivalence induces an isomorphism of cohomology rings.

Examples 2.2.2.

1. We begin with an example of two spaces with isomorphic (co)homology groups, but non-isomorphic cohomology rings. Let $X = S^1 \vee S^1 \vee S^2$, and let T^2 denote the torus. Then $H_i(X) \approx H_i(T^2)$ for each $i \geq 0$. Thus $H^i(X;G) \approx H^i(T^2;G)$ for any coefficient group G. However, the two rings $H^*(X;\mathbb{Z})$ and $H^*(T^2;G)$ are not isomorphic, so that X is not homotopy equivalent to T^2.

2. Compute the cohomology ring of $S_g, g \geq 2$.

3. Compute $H^*(\mathbb{RP}^n; \mathbb{Z})$ as a ring. Same for $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$.

4. Let X be a space, and let $U, V \subset X$ be open subspaces with $X = U \cup V$. Suppose that U and V are acyclic (i.e. all reduced homology groups vanish).

   (a) Prove that $a \cup b = 0$ for all $a, b \in H^*(X)$ of positive degree. [Hint: Look up the **relative cup product** in Hatcher, and use it.]

   (b) Conclude that $\mathbb{RP}^2$ and T^2 cannot be written as a union of two open, acyclic subspaces. Also conclude that the suspension of any space has trivial cup product in positive degree.

Bibliography

[Ha] A. Hatcher, Algebraic Topology.

[Mu] J. Munkres, Elements of Algebraic Topology.