**A History of Algebraic Topology**

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**Abstract:** The development of algebraic topology in the twentieth century is one of the most important advances of its time. However, how did this development get under way? What forces worked to advance the subject, and perhaps more interestingly, to retard the development in its early days? In this talk I will present some historical approaches to these questions and the answers they produce.

*Toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'*ANALYSIS SITUS.

HENRI POINCARÉ

The history of algebraic topology is not so easy to describe. As a hybrid subject, its roots lie in different places and the synthesis of these diverse ideas gave birth to the activity. It is also the case that the field of study, algebraic topology, only came to be in a time when branches of mathematics separated themselves into different communities. This separation was not a feature of earlier times. It is also the case that the development of algebraic topology can be said to be one of the most impressive features of twentieth century mathematics.

To frame the history, let's consider some of the ways in which twentieth century mathematics was different than other times.

1) The *rise of abstraction*, pioneered by Hilbert and his coworkers in Göttingen. This approach to developing mathematics was much admired and emulated. Eventually topological notions were organized in this manner.

2) New *centers of activity* grew up into what might be called "schools," although this term must be used with caution. The spread of pioneering ideas often came from these centers outward, and so examining them is important.

3) Most history of mathematics is presented as a series of successes. What about the *failures*? Topology did not simply spring forth in its present form, and some of the paths that were less successful were abandoned, sometimes to be taken up again later. A deeper presentation of history must include these failures and the contexts in which they occurred.

In his recent book on the development of contemporary mathematics ("*Plato's Ghost*" [**?**]) Jeremy Gray identified a feature of mathematics early in the twentieth century: The end of the nineteenth century is marked by a "growing appreciation of error," especially in the developments in analysis and in the foundations of mathematics. There was a sense of 'anxiety' that was evident and the role of this anxiety was to foster, for example, a deeper desire for rigor. Anxiety about methods and results was not the only expression of fear. There was also anxiety about the status of this emerging field of topological investigation.

We recount three incidents that reveal a kind of anxiety over algebraic topology. In a recent article [?], Beno Eckmann, a student of Heinz Hopf, recalled an encounter with Hermann Weyl (1885–1955). Eckmann asked Weyl why he had published his 1923/24 papers [?], *Analysis Situs Combinatorio*, in the *Revista Matematica Hispano-Americana*, and *in Spanish*! Weyl replied that he did not want to draw attention to the publication, that his colleagues should not read them. The subject matter was not serious mathematics!

Another remark on topology from the decade of the 20's is contained in a 1929 address [?] to the Deutsche Mathematische Vereinigung (DMV) by B. L. van der Waerden (1903–1996) who described combinatorial topology as "a battlefield of differing methods. . . ." The lack of a rigorous definition of manifold was the key issue here, and van der Waerden, the quintessential Göttinger mathematician, wanted more clarity in this endeavor.

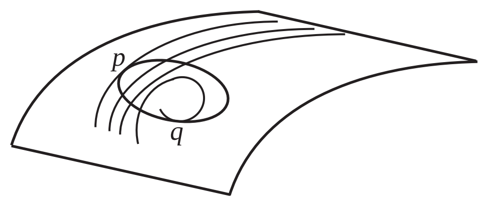
A key figure in the emergence of algebraic topology is Brouwer, whose charismatic nature and leadership ability made him a guru to young topologists as well as a threat to other leadership figures. Though Brouwer's work in topology was limited to the remarkable years 1909–1912, he kept a hand in the field by encouraging others, and through his prominence as cooperating editor of *Mathematische Annalen* from 1915. The *Annalen* was based in Göttingen and had become the most prestigious journal in mathematics in the years around the First World War. Its principal editors were Hilbert, Einstein, Blumenthal and Carathéodory; Brouwer was listed among the cooperating editors. When Brouwer's intuitionistic stance on mathematics threatened Hilbert's leadership on foundational questions, Hilbert requested the removal of Brouwer from the position of editor for the *Annalen*. Brouwer reacted with a flurry of irate letters, but eventually he withdrew from the *Annalen*. He then founded a new journal, *Compositio Mathematicae*. In their correspondance, the young topologists at the time, Hopf and Alexandroff, discussed this conflict with considerable interest. Hopf felt that Brouwer's absence from the editorial board would make it less likely that his papers, and papers in topology more generally, would appear in this journal which drew the most attention in the mathematical community. Alexandroff felt that Hopf's work and reputation had reached such a stage in 1929 that he could publish anywhere. However, Hopf's next paper was submitted to *Crelle* (Hs 160.99), and after his arrival in Zürich in 1931, his work appeared primarily in *Commentarii Mathematici Helvetici* or Brouwer's *Compositio Mathematicae*.

In what follows, we will relate these instances to the emergence of a new field in the twentieth century. Along the way we will present some of the context that made the path to the study of algebraic topology possible.

This talk is based on various accounts of the history of these ideas (notably by Scholz [?] and by Epple [?]) and on archival research in Zürich, Paris, and elsewhere.

## §1. All roads lead from Poincaré

In a series of memoirs [?] on the global properties of solution curves to differential equations on orientable surfaces, Poincaré introduced some topological notions relating the Euler characteristic of a surface ($V-E+F$) and the behaviour of singularities of a flow on the surface. To understand his definition of the index of a singularity of a flow on a surface, consider a simple closed curve on the surface which encloses a simply-connected region, a cycle.



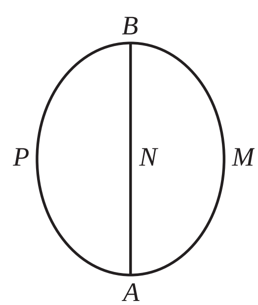

The flow passes through this cycle at the points along it and a flow line might pass through transversely or meet the curve tangentially and remain either outside the region (an external point on the cycle) or inside the region (an internal point on the cycle). The index of the cycle is defined to be the integer

$$J = \frac{e - i - 2}{2},$$

where $e$ is the number of external points on the cycle and $i$ the number of internal points. A region without singularities has index zero. The index of a singularity is the index of a cycle that encloses a region containing only that singularity. The local pictures of a flow near a singularity had been worked out by Poincaré and the various cases determine the index for a cycle enclosing the singular point.

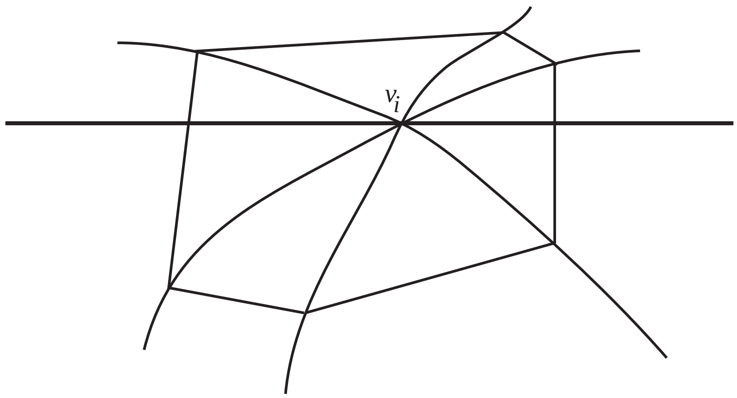
The main result of this development is the *Poincaré index theorem* [**?**]: if the number of singular points of a flow on an orientable surface is finite, the sum of the indices at the singular points is minus the Euler characteristic of the surface, $V - E + F = 2 - 2p$, where $p$ denotes the genus (the number of handles) of the surface.

To prove this Poincaré first proves that the index is additive with respect to a subdivision of a cycle into two pieces: If $C = APBMA = ANBMA + APBNA$, then




$$\text{ind.}APBMA = \text{ind.}ANBMA + \text{ind.}APBNA.$$

The step to the global result comes by triangulating the surface into simply-connected regions and comparing the contribution along edges or through the vertices. Since an exterior point along an edge is an interior point for the adjacent cycle, these contributions cancel and the sum $\sum_\alpha E(\alpha) - I(\alpha)$ is determined by the contributions through vertices:




Each vertex contributes its valence less 2 and so

$$\sum_{\textit{cycles, } \alpha} E(\alpha) - I(\alpha) = \sum_{\textit{vertices}} \text{valence } - 2 = \sum_{\textit{vertices}} \text{valence } - 2\#\text{vertices}$$
$$= 2\#\text{edges} - 2\#\text{vertices}.$$

It follows that

$$\sum_{\textit{cycles, } \alpha} E(\alpha) - I(\alpha) - 2 = 2\#\text{edges} - 2\#\text{vertices} - 2\#\text{faces} = -2\chi(S) = 2(2p-2).$$

Poincaré reaped the immediate consequences of this result—for example, on a two-dimensional sphere, every flow must have a singular point (the theorem affectionately called the *Hairy Ball Theorem*; the wind is not blowing somewhere on the globe); the only closed compact orientable surface possessing a singularity-free flow is the torus. He had developed in earlier papers of this series a local classification of singularities as *cols* (passes), *nœuds* (nodes), and *foyers* (foci) and by computing the contribution of each singularity we find

$$\#\textit{nœuds} - \#\textit{cols} + \#\textit{foyers} = \chi(S).$$

## §2. Manifolds

In the celebrated paper *Analysis situs* and its supplements, Poincaré [**?**] initiated the topological study of manifolds. He gave examples arising in various ways—as the inverse image of a regular value of a differentiable function from an open subset of $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$; as a set with a finite atlas of differentiable parametrizations; as a geometric cell complex assembled out of simplices and satisfying the local manifold condition; and more generally as a cell complex made by identifying handle bodies along their boundaries. Poincaré did not unify these examples with a single definition (as was his style). However, he did introduce new topological methods of study, including the notions of cobordism, homology, the fundamental group, etc. (see Scholz [?]). Poincaré defined a notion of equivalence of manifolds, *'homéomorphismes'*, given by changes of coordinates (the present-day diffeomorphism). By collecting all such *homéomorphismes* together into a 'group', Poincaré related this group implicitly to the generalized notion of geometry found in the *Erlangen Programm* of Felix Klein (1849–1925): Thus analysis situs, or topology, was a branch of Geometry.

He went on to prove the homological property of Poincaré duality for compact, closed, and oriented manifolds, and he posed the problem of generalizing the success of nineteenth century geometers in classifying surfaces to higher-dimensional manifolds.

From the outset, the importance of having a sharp definition of manifold was clear. David Hilbert (1862–1943) sought an axiomatic characterization of the plane as a manifold in his researches on the foundations of geometry [?]. Hilbert's basic notion was that of neighborhoods, and this idea was refined by Hermann Weyl (1885–1955) in his celebrated 1913 book on Riemann surfaces [?]. Weyl's definition of a two-dimensional manifold is based on a system of neighborhoods, at least one for each point, with each neighborhood being homeomorphic to an open subset of the plane. After the development of general topological spaces in 1914 [?] by Felix Hausdorff (1868–1942), the role of neighborhood systems and separation assumptions was made precise and Weyl later added a separation condition to his axioms [?].

The combinatorial description of a manifold was exposed in the 1907 article [?] of Max Dehn (1878–1952) and Heegaard for Klein's *Enzyklopädie der Mathematischen Wissenschaften* on *Analysis Situs*. They took as basic the abstract data that describe a triangulation, the cells and their incidence data, which were called a *schéma d'un polyèdre* by Poincaré. They discussed the questions of Poincaré whether given a scheme, was it realized by a manifold, and whether two manifolds with the same abstract scheme need be homeomorphic, for which they introduced a notion of combinatorial equivalence via mappings between schemes to substitute for homeomorphisms. Later, Steinitz [?] and Tietze (1880–1964) [?] independently posed the *Hauptvermutung* for manifolds: Do two triangulations of a manifold have a common refinement? More generally it was asked if a compact manifold always has a triangulation. In higher dimensions, without a common definition, the study of manifolds was fraught with difficulties, expressed by van der Waerden as a "battleground of different methods."

The development of a theory of manifolds may be characterized as a response to two impulses after Poincaré. The first impulse was the computation of the new invariants and this favored the combinatorial description of manifolds. The second impulse sought new examples, especially of three-manifolds, in the hope of the resolution of the Poincaré conjecture.

An axiomatic description of manifolds was achieved by Veblen and J.H.C. Whitehead [?]. In short order, Whitney [?] showed that differentiable manifolds were identifiable with subsets of Euclidean spaces and so inherited notions like tangent and normal bundles and, when needed, a Riemannian metric.

### §3. On Brouwer

In the beginning of the twentieth century certain basic topological questions remained unsolved, among which the most important were Hilbert's Fifth problem (on continuous groups of transformations of manifolds), the question of the topological invariance of dimension, and the Jordan curve theorem. Motivated by his philosophical interests in the foundations of geometry, Brouwer worked on Hilbert's fifth problem which led him into a study of methods in topology. In particular, he immersed himself in the work of Arthur Schoenfliess (1853–1928) on the topology of the plane [?]. In short order, Brouwer's penetrating critical faculties spotted flaws in Schoenfliess's work, leading him away from the fifth problem to questions about the foundation of topology. His investigations of mappings of surfaces led him to his first fixed point theorem—a continuous orientation-preserving mapping of the two-sphere to itself must have a fixed point [?]. Around this time (1909) he also proved, using Schoenfliess's methods, that a continuous vector field on a two-sphere must have a singular point (where it is zero or infinite), an improvement of the differentiable result of Poincaré. He states of these results [?]: *At first sight one might even suppose that they can be directly deduced out of each other.*

The conundrum of their connection prompted Brouwer to write to Jacques Hadamard (1865–1963) who suggested that Brouwer study Poincaré's memoirs on flows on surfaces [?]. During a visit to Paris over Christmas 1909, Brouwer kept a notebook in which he sketched a definition of Poincaré's index of a mapping in the combinatorial setting. This led to a proof of the invariance of dimension and with that Brouwer opened up a new landscape for combinatorial topology. In particular, up to this point, focus was on the combinatorial representation of objects like manifolds, representatives for homology classes in a manifold, and relations that determine the fundamental group from the combinatorial structure. Brouwer introduced methods that made any continuous mapping between such objects representable up to a deformation by combinatorial data, allowing the focus to shift from objects to the mappings between them.

A characteristic example is Brouwer's proof of the topological invariance of dimension [?]. Suppose $K$ and $L$ are geometric complexes of cells in some Euclidean spaces, and $f\colon K \to L$ is a continuous mapping that takes vertices in $K$ to vertices of $L$. We can approximate $f$ by the mapping $\beta\colon K \to L$ defined by representing a point in $K$ by its barycentric coordinates and extending $f$ linearly from the vertices according to the coordinates. Brouwer showed that a fine enough subdivision of $K$ yielded an approximation homotopic to $f$, that is, there is a continuous deformation between $f$ and the combinatorial mapping $\beta$.

In this context Brouwer defined the degree of a mapping (*Abbildungsgrad*) to be the integer $p-q$ where $p$ is the number of points in the preimage of a generic point for which the orientation is preserved by the mapping, and $q$ is the number of points in the preimage for which the orientation is reversed by the mapping. Brouwer showed how this difference is unchanged by deformations of the mapping and so the simplicial approximation of the mapping, a combinatorial tool, was sufficient to describe an invariant of the mapping.

Brouwer developed his mapping degree [?] further to prove a fixed point theorem about mappings of spheres: if $f\colon S^n \to S^n$ is a continuous function, then $f$ has a fixed point (a point $P$ in $S^n$ with $f(P) = P$) whenever the mapping degree $\deg(f) \neq (-1)^{n+1}$. Brouwer then proved his celebrated fixed point theorem as a consequence:

A continuous function $F\colon e^n \to e^n$ has a fixed point. Here $e^n$ is the collection of $n$-vectors in $\mathbb{R}^n$ with length less than or equal to 1, the unit ball in $\mathbb{R}^n$. The new tool of the mapping degree and new concepts like simplicial approximation and the relation of homotopy between maps are key to the development of topology and Brouwer's contributions stand as a gate to a new chapter in the subject.

Though Brouwer's contributions were viewed with respect, they were acknowledged as difficult to understand and so did not attract an immediate following.

### §4. On Noether, Hopf, and Vietoris

Brouwer published his new approach to topology in the years 1910–1913 after which he contributed little to the subject. His work on invariance of dimension led to an interest in a topological theory of dimension that was eventually developed by Pavel Urysohn (1898–1924) and Karl Menger (1902–1985). Urysohn's work in the early 1920's interested Brouwer who soon invited reseachers in topology to visit him in Amsterdam. Among the visitors were Paul Alexandroff (1896–1982), Leopold Vietoris (1891–2002), Urysohn, Menger, and later, Witold Hurewicz (1904–1956) and Hans Freudenthal (1905–1990) were his assistants.

In December 1925, the eminent Göttingen algebraist Emmy Noether (1882–1935) visited Blaricum, Brouwer's vacation home, and the group of mathematicians around Brouwer. At a dinner in her honor at Brouwer's (recalled in [**?**]) she explained how the numerical invariants of combinatorial topology were better organized as the invariants of groups, to be called Betti groups. The dinner party included Alexandroff and Vietoris, and the first papers written on homology *groups* were by Vietoris and Heinz Hopf (1894–1971).

In his work, Vietoris extended the simplicial homology theory of Dehn and Heegaard and Poincaré in two novel ways [**?**]. The first was in applying Noether's suggestion to use groups instead of numerical invariants. Working mod 2, he defined an addition on cycles (sums of simplices with mod 2 zero boundary), calling the group of dimension $n$ cycles the *n-te Zusammenhangsgruppe* and the maximum number of independent cycles the *n-te Zusammenszahl*. This permitted Vietoris to argue with matrices with coefficients in the integers mod 2.

The second extension was to a wider class of spaces, namely compact metric spaces. The main objects of study, manifolds, are often compact metric spaces, and the *Hauptvermutung* pointed out a possible gap between the combinatorial representation of a manifold as a complex and its topological nature as a space. In [**?**] the notion of an $\epsilon$-complex was introduced for all $\epsilon > 0$. If $X$ is a compact metric space, then a combinatorial $p$-simplex in $X$ is a choice of a $(p+1)$-tuple of points in $X$ and if such a choice has diameter $\leq \epsilon$, then it is a generator for the $p$-chains on $X$. Vietoris showed that Alexander and Veblen's mod 2 homology groups for a finite Euclidean complex coincided with the groups obtained using the $\epsilon$-complex.

Hopf learned of Noether's algebraic suggestion through his friendship with Alexandroff and his time spent in Göttingen. Hopf had mastered Brouwer's methods in his thesis and *Habilitationsschrift* in which, among other things, he extended Brouwer's results on homotopy classes of mappings between manifolds. In his 1928 paper giving an extension of the Euler-Poincaré formula, Hopf presented the formalism for homology groups in the framework of modules over a ring. The simplices of a geometric complex form the generators of a free module and the kernel of the boundary homomorphism module its image gives the homology group as a quotient group. Using this formalism Hopf gave an elegant proof of the Lefschetz fixed point theorem [**?**]. Hopf also extended the range of homology theory by considering any geometric cell complex, without the assumption of being a manifold.

In Hopf's reformulation of homology as groups, the mapping degree of Brouwer has a particularly simple statement [**?**]. If $M$ and $N$ are oriented $n$-dimensional manifolds, then, by Poincaré duality, $H_n(M)$ and $H_n(N)$, the $n$th homology groups, are both isomorphic to the free abelian group $\mathbb{Z}$. If $f\colon M \to N$ is a continuous mapping, then $f$ induces a mapping $f_*\colon H_n(M) \to H_n(N)$ which is a homomorphism $f_*\colon \mathbb{Z} \to \mathbb{Z}$. Since any such homomorphism is multiplication by some integer, the mapping degree can be seen to be that integer. Thus $f_*(1) =$ the mapping degree.

The extent to which homology might be used as a tool to study spaces other than manifolds was a theme partially motivated by the lack of success in avoiding the use of triangulations (the *Hauptvermutung*) and because the properties of general topological spaces were being developed in parallel to the development of combinatorial ideas. Alexandroff advanced a general theory for compact spaces that lay between the ideas of Vietoris for compact metric spaces and of Hopf for geometric complexes. He defined the notion of the *nerve of a covering* which consists of an abstract cell complex whose vertices consisted of the open sets $U_\alpha$ in the covering and whose $p$-simplices $[U_0, U_1, \ldots, U_p]$ satisfied $U_0 \cap \cdots \cap U_p \neq \emptyset$. By taking finer coverings of a space, the homology of this abstract cell complex is seen to converge to a common set of generators, giving a notion of the Betti numbers of the space [**?**]. For compact manifolds, these Betti numbers coincide with the usual Betti numbers.

To extend Alexandroff's ideas to any topological space, the Czech mathematician Eduard Čech (1893–1960) considered the collection of all finite open coverings [**?**]. This collection is ordered by the relation of inclusion, one cover being finer than another. The relations between the homology groups of the nerve of each cover is encoded in the homomorphisms between the associated homology groups. Čech introduced the notion of the inverse limit of such a system of groups which could be taken as the homology group of the space. In this framework he was able to give new proofs of the Poincaré and Alexander duality theorems.

It is interesting to contrast the fate of these two papers that are the first in the study of homology groups. Vietoris's paper is abstract, self-contained, and extends methods for internal goals. Hopf's paper looks out to other areas of mathematics, algebra and geometry, for its goals, connecting with tried and true results from a new viewpoint. Vietoris's paper did not obtain much interest in the intervening years, but lately there is considerable interest in $\epsilon$-homology in several efforts to apply topology (see the work of G. Carlsson and R. Ghrist).

## §5. Homotopy

In §12 of *Analysis Situs* Poincaré associated a 'group of substitutions' (the term that described an abstract group most closely in Poincaré's time) to a manifold $V$.

The idea is based on the fact that line integrals give the same value on small loops in $V$. The elements of the group were permutations of the values of a multiply valued function on $V$ generated by the loops in $V$. A path in $V$ beginning and ending at the some fixed point of $V$ can be broken up into small loops and connecting paths. We can sum the effect on branches of the function by integrating along this path. We add loops $C_1$ and $C_2$ by following first $C_1$, then $C_2$. Poincaré denotes this by $C_1 + C_2$ but stipulates that the addition need not be commutative. If loops are equivalent, that is, if $A$ can be deformed in $V$ to $B$, then he writes $A \equiv B$, distinguishing this equivalence relation from homologies.

To compute the group associated to a manifold, Poincaré identified certain fundamental loops $C_1, \ldots, C_p$ for which any other loop is equivalent to a sum of multiples of these loops. To identify the relations among such fundamental loops, Poincaré used the representation of a manifold as a cell complex, constructing relations from what happens in a single cell, and from the relations among cells. By ignoring the order of terms in an expression in the fundamental loops one has fundamental homologies among generators for the 1-cycles. This is now given by the theorem of Poincaré that the fundamental group made abelian is the first homology group, $\pi_1(V)/[\pi_1(V), \pi_1(V)] \cong H_1(V)$. However, Poincaré did not use this terminology.

The primary application of the fundamental group for Poincaré was to act as an invariant of manifolds. In the 1892 *Comptes Rendus* announcement preceding *Analysis Situs* [**?**], he posed the problem of classifying manifolds of dimension greater than two through their Betti numbers, and showed that these invariants were insufficient by producing two 3-manifolds with the same Betti numbers but different fundamental groups.

The construction of three-manifolds given by Poincaré was inspired by his interest in complex function theory and the method of Riemann surfaces led Poincaré to the use of the universal covering space of a manifold for computations. This space is simply-connected and any loop in the manifold can be lifted to it. Any null-homotopic loop lifts to a closed loop in the universal cover and equivalent curves lift to curves sharing the same endpoint. The permutations of the points corresponding to a single point correspond to lifts of loops and hence give a permutation presentation of the fundamental group. Since abstract groups were not part of the mathematical canon at the time, this permutation presentation was important to transmission of the idea. It made the universal cover a tool in the development.

The combinatorial description of a manifold as a cell complex was developed between 1900 and 1920 as a means of computing Betti numbers, torsion coefficients, and the fundamental group. In 1908 Tietze [**?**] developed an algorithm for computing the fundamental group from the assumption that any loop in a cell complex could be deformed to a path along the 1-cells of the complex and so the fundamental group is a quotient of the edge loops under identifications determined by the higher-dimensional cells. This presentation moved the notion of group toward abstraction. The fundamental group, defined by Poincaré as a group of substitutions, became more abstractly presented in the work of Tietze and Wirtinger. Using his algorithm Tietze posed the problem of the dependence of the presentation upon the triangulation. In particular, if we are given two abstractly presented groups, can we recognize them as the same group? This problem, called the *isomorphy problem*, was considered by Tietze to be exceptionally difficult. Much later the theory of computation was developed and the isomorphy problem shown to be unsolvable.

The term "homotopy" first appears in the 1907 *Enzyklopädie* article of Dehn and Heegaard [**?**]. The meaning is different than the present usage, however, because their term *homotop* implies homeomorphic. The rigid structure of a cell complex is transformed in Dehn's and Heegaard's sense of homotopy by moving vertices to vertices, edges to edges, etc. The transformation is the focus of the definition. It is Brouwer who made explicit the idea of homotopy. In this framework, the role of simplicial approximation can be further appreciated, and concepts like the mapping degree shown to be a homotopy invariant of a mapping.

Hopf showed that the mapping degree of a continuous mapping of the $n$-dimensional sphere to itself completely characterized its class, that is, two mappings $S^n \to S^n$ with the same degree are homotopic. Mappings between manifolds of different dimension were the next step in this development. The simplest case to examine would be among the maps $S^3 \to S^2$ for which there is a particularly well-known map, taking a pair of complex numbers to the complex line through the origin and that point in $\mathbb{C}^2$, that is,

$$\eta\colon (z_1, z_2) \in S^3 \subset \mathbb{C}^2 \mapsto [z_1, z_2] \in \mathbb{C}P^1 \cong S^2,$$

where $\mathbb{C}P^1$ is the collection of all such lines, known as the complex projective line. In his earlier work Hopf developed the mapping degree of Brouwer and so he sought a generalization for this case. Hopf associated to any continuous mapping $f\colon S^3 \to S^2$ a integer $\gamma(f)$, computed by choosing two points in $S^2$ and considering their preimages, which are closed curves in general; $\gamma(f)$ is given by their linking number and is called now the *Hopf invariant*. Hopf showed that if $f$ and $f'$ were homotopic, then their Hopf invariants agreed. Furthermore, the mapping $\eta$ has Hopf invariant one. Finally, by analyzing how the Hopf invariant changes under composition with mappings $F\colon S^3 \to S^3$ and $G\colon S^2 \to S^2$, he showed that there are infinitely many mappings from $S^3$ to $S^2$ that were not homotopic to one another.

The importance of Hopf's paper [**?**] cannot be underestimated. It opened up a new class of topological problems. At the 1932 International Congress of Mathematicians in Zürich, Čech gave the definition of the higher homotopy groups of a space, $\pi_n(X)$ for $n \geq 2$, as a generalization of Poincaré's fundamental group. In the context of homotopy as a relation between mappings, we can express the fundamental group as a quotient of the set of all loops where we identify together two loops if they are homotopic to one another. Loops are mappings of a circle to the space with a given point of the circle going to a chosen basepoint in the space. To generalize this, perform the same quotient on the mappings of any sphere $S^n$ to the given space where a given point of the sphere is taken to map to a given point of the space. Hopf's theorem detecting homotopy classes of mappings from $S^3$ to $S^2$ showed that $\pi_3(S^2)$ contained a copy of $\mathbb{Z}$ as a group. Čech's contribution [**?**] was unappreciated at the time: Hopf and Alexandroff advised that since these new groups were commutative and homology already posed enough problems the new groups were uninteresting; furthermore, Čech presented these new invariants without applications or relations to other topological ideas.

In a series of papers [**?**] in 1935, rich in ideas, Hurewicz independently introduced the higher homotopy groups of a space. His definition differed from Čech's because it came with connections to the successful past of topology. Hurewicz, like most topologists of the time, worked on topological questions of a point-set and combinatorial nature. His $n$th homotopy group ($n > 1$) of a space $X$ was the fundamental group of the topological space of mappings, $\mathrm{map}((S^{n-1}, e_1), (X, x_0))$, where $X$ is a topological space with nice properties. From this definition Hurewicz could apply theorems of a point-set nature on the extension of continuous mappings, from which he deduced the relations between the groups $\pi_n(G)$, $\pi_n(G/H)$, and $\pi_n(H)$ for $H$ a connected subgroup of a connected Lie group $G$. With these relations he put Hopf's computation $\pi_3(S^2)$ into the context of the study of Lie groups, and so constituted one of the first steps in the development of fibre spaces. Hurewicz also related the higher homotopy groups to homology groups; there is a group homomorphism $h\colon \pi_n(X) \to H_n(X)$ for all $n$; if a space $X$ has homotopy groups $\pi_n(X) = \{0\}$ for $1 \leq n \leq N$ for some $N > 1$, then $H_n(X) = \{0\}$ for $1 \leq n \leq N$ and $\pi_{N+1}(X) \cong H_{N+1}(X)$. If a space $X$ has $\pi_n(X) = \{0\}$ for $2 \leq n \leq N$ for some $N > 2$, then the homology groups of $X$, $H_n(X)$, for $2 \leq n < N$ are invariants of the fundamental group $\pi_1(X)$ alone.

Finally, Hurewicz introduced the notion of homotopy equivalence of spaces. Spaces $X$ and $Y$ are *homotopy equivalent* if there are continuous mappings $f\colon X \to Y$ and $g\colon Y \to X$ with $g \circ f$ homotopic to the identity mapping on $X$ and $f \circ g$ homotopic to the identity mapping on $Y$. For example, the Möbius band is homotopy equivalent to a circle. This relation is much cruder than the relation of homeomorphism and hence more accessible to classification. The higher homotopy groups and the homology groups are invariants of the homotopy equivalence class of a space. This notion provided a new foundation for the development of combinatorial invariants of spaces and manifolds.

## 6. Final Remarks

Hurewicz had spread the news of his researches in a series of sketchy notes in the Dutch Academy Proceedings. He also lectured on them at an important event in the history of algebraic topology—the International Conference on Topology in Moscow, 4–10 September 1935. This conference was organized by Alexandroff and brought together "a large proportion of the active topologists in the world," according to Alan Tucker (1905–1995), a student of Lefschetz who attended the conference. Tucker wrote of the meeting [**?**]:

*The International Topological Conference held at Moscow last September showed that the subject has attained a definite measure of maturity and a wide range of influence on other branches of mathematics, but that it is still undergoing rapid growth and flux.*

What marks this conference as significant is the sense of a research community of "active topologists" that it represented. It was international with participants from the Soviet Union, France, Germany, Holland, Switzerland, Czechoslovakia, Poland, and the United States. The independent discoveries of similar research paths brought mathematicians together, incited a flurry of papers, and set an agenda for progress in the subject. Leadership in the field was also clear with the attendance of Heegaard, Lefschetz, Alexander, Alexandroff, and Hopf.

1935 was the year in which two important texts in topology made their appearance. The first [**?**] was written by H. Seifert and W. Threlfall (1888–1949) and appeared in late 1934. It focused on the methods suited for the study of manifolds, especially of dimensions two and three. Particular attention is given to the fundamental group and covering spaces, which had not had a thorough treatment previously. They also treat homology groups with applications to manifolds.

The second text [**?**] was *Topologie I* written by Alexandroff and Hopf who spent time together after the Moscow conference putting the final touches on it. This volume was the first of three proposed volumes intended to give a view of *"Topologie als ein Ganzes,"* and not, as Reidemeister wrote in his Zentralblatt review, *"eine Darstellung der ganzen Topologie."* Courant had suggested the project during one of the authors' visits to Göttingen for his *Grundlehren der Mathematischen Wissenschaften* series with Springer-Verlag. The writing was based on lecture courses the authors were giving at the time and took place over the years 1928–1935. The book formed the focus of much of their correspondence during these years. The main topics are point-set topology, homology theory with applications to polyhedra, and then links to geometric questions especially in the study of continuuous mappings between polyhedra. The future volumes would treat manifolds, the fundamental group, dimension theory and further topics in point-set topology. The rapidly developing state of topology was particularly evident at the Moscow conference and they decided to abandon the later volumes. The comprehensive nature of the book and its thorough exploration of the topics give it a finality that is consistent with its dedication to Brouwer.

Thus, by 1935, a subdiscipline of mathematics, algebraic topology, had matured to the point where there was an international group of researchers working on recognized problems using shared methods. There were elementary accounts of the main ideas to introduce new researchers to the activity, and active leaders, like Hopf, Lefschetz, and Alexandroff, to attract students into the field.

The success of algebraic topology is due in part to the development it fostered in related parts of mathematics. Answers to questions in the foundations of analysis were found with new methods and results like fixed point theorems led to new analytic ideas. The study of the fundamental group of knots and three-manifolds led to insights about abstract groups, while homology groups presented a novel set of algebraic notions that later developed into homological algebra in which topological notions are imported to classify algebraic objects. However, this development was not smooth. There were false starts, difficult expositions, paths that looked to lead nowhere, and an atmosphere of change around the practice of mathematics. The anxiety of van der Waerden was overcome with the advances of Veblen and Whitehead, of Hopf and Alexandroff, overcome by their own successes, and of Weyl, misplaced as the field came to dominate the twentieth century.

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