Math 312 - Homework 1

Problem 1

Is $\mathbb{Q}$ a G_δ set?

Problem 2

What is the σ-algebra generated by the half-open intervals $[a, b)$? How is it related to the Borel σ-algebra: smaller, bigger, not comparable?

Problem 3

If $f_1, f_2, \ldots : X \to \mathbb{R}$ are Borel measurable, prove that $g = \sup(f_n) : X \to \overline{\mathbb{R}}$ is Borel measurable.

Problem 4

Is there a σ-algebra which is countably infinite?

Problem 5 $(*)$

Give an example of a compact metric space X and distinct finite Borel measures μ_1, μ_2 on X which agree on the balls in the topology.

Problem 6

Give an example of measurable functions $f_1, f_2, \ldots : X \to \mathbb{R}$ for which the inequality in Fatou's lemma is strict, i.e.

$$\int \liminf f_n < \liminf \int f_n.$$

Problem 7

Give an example of measurable functions $f_1, f_2, \ldots : X \to \mathbb{R}$ converging pointwise to a measurable function $f : X \to \mathbb{R}$ that do not satisfy the conclusion of the dominated convergence theorem, i.e.

$$\int f \, d\mu \neq \lim_{n\to\infty} \int f_n \, d\mu.$$

Math 312 - Homework 2

Problem 1

Show that $\frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2} \cdot \frac{d\mu_2}{d\mu_3}$ for any σ-finite measures $\mu_1 \ll \mu_2 \ll \mu_3$.

Problem 2

If $\{B_\alpha\}_{\alpha \in A}$ are arbitrarily many balls in $\mathbb{R}^n$, whose radii are all bounded above by some constant, prove there exists a subset $\{B_\alpha \mid \alpha \in X\}$ of pairwise disjoint balls and a constant $c > 0$ such that

$$\bigcup_{\alpha \in X} (cB_\alpha) \supseteq \bigcup_{\alpha \in A} B_\alpha$$

where cB means the ball with the same center as B and c times the radius.

Problem 3

Is an arbitrary union of closed unit balls in $\mathbb{R}^n$ necessarily Lebesgue measurable?

Math 312 - Homework 3

Problem 1 ($*$)

We say that $\delta > 0$ is good if, for every measurable $A \subseteq \mathbb{R}$ such that $\lambda(A) > 0$ and $\lambda(\mathbb{R} \setminus A) > 0$, there is an $x \in \mathbb{R}$ such that

$$\delta \leq \underline{d}(x, A) \leq \overline{d}(x, A) \leq 1 - \delta.$$

Prove that $\delta = \frac{1}{4}$ is good.

Problem 2

Given measurable $A, B \subseteq \mathbb{R}^n$ with $\lambda(A), \lambda(B) > 0$, prove that

$$A + B = \{x + y \mid x \in A, y \in B\}$$

contains a ball, i.e. its interior is non-empty.

Problem 3

What are the compact sets in the d-topology?

Problem 4

Show that the d-topology in $\mathbb{R}^2$ is not the same as the product topology from two copies of $\mathbb{R}$ each with the d-topology.

Math 312 - Homework 4

Problem 1

Prove the $p = \infty$ case of the Riesz-Fisher theorem. That is, prove that $L^\infty(\mu)$ is complete for any measure space (X, μ).

Problem 2

Let (X, μ) be a finite measure space. If $f \in L^p(\mu)$ for all $1 < p < \infty$, must it be the case that $f \in L^\infty(\mu)$?

Problem 3

Let (X, μ) be a σ-finite measure space. Prove that any element of $(L^p(\mu))^*$ is integration against an $L^q(\mu)$ function. That is, for any bounded linear functional Λ on $L^p(\mu)$, prove there is some $g \in L^q(\mu)$ such that $\Lambda(f) = \int fg\, d\mu$.

(Recall that we did this in class in the case when μ is a finite measure.)

Problem 4

Let $X = Y = [0, 1]$, and define $f : X \times Y \to \mathbb{R}$ to be

$$f(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Let μ be the Lebesgue measure on X, and let ν be the counting measure on Y. Calculate

$$\int_X \left(\int_Y f(x, y)\, d\nu \right) d\mu$$

and

$$\int_Y \left(\int_X f(x, y)\, d\mu \right) d\nu,$$

and explain why Fubini fails.

Problem 5

On the midterm, you showed that for measurable $A, B \subseteq [0, 1]$, the function

$$f(t) = \lambda(A \cap (B + t))$$

is continuous. Now, find $\int f(t)$.

Math 312 - Homework 5

Problem 1

Find a basis for the vector space of harmonic polynomials in x and y of degree 6.

Problem 2

If a harmonic function u on $\mathbb{R}^2$ depends only on $r = x^2 + y^2$, what form must u have? What if u depends only on φ?

Math 312 - Homework 6

Problem 1

Does the function $\left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$ have a primitive on the following domains? If yes, find one.

1. The upper half plane
2. The lower half plane
3. The right half plane
4. The left half plane
5. $\mathbb{R}^2 \setminus \{0\}$

Problem 2

Prove Apollonius's theorem: for any $a, b \in \mathbb{R}^n$ and positive $c \in \mathbb{R}$, the set of x such that

$$\frac{|x-a|}{|x-b|} = c$$

is a sphere.

Problem 3

Prove that the Poisson kernel

$$P(x,y) = P(\rho\cos(\delta), \rho\sin(\delta)) = \frac{1-\rho^2}{1-2\rho\cos(\delta)+\rho^2}$$

is harmonic.

Problem 4

In class, we took a continuous function f on the unit circle and defined a function u by

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} f \cdot P(\rho, \varphi - \nu)\, d\varphi.$$

We showed that this was harmonic on the open unit disk. Now, prove that the function

$$u^*(a) = \begin{cases} u(a) & \text{if } |a| < 1, \\ f(a) & \text{if } |a| = 1 \end{cases}$$

is continuous.

Problem 5

If $f \in L^1$ and $g \in L^p$, does it follow that $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$?

Problem 6

If $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p$, $g \in L^q$, show that $f * g$ is continuous and tends to 0 as $|x| \to \infty$.

Problem 7

Prove that there is no ‘‘unit’’ function, i.e. that there is no $f \in L^1$ such that $f * g = g$ for every $g \in L^1$.

Math 312 - Homework 7

Problem 1

Show that for any Borel-measurable, non-decreasing $f : [0, \infty) \to [0, \infty)$ and any non-negative random variable X, we have, for all c,

$$P(X \geq c) \leq \frac{E(f(X))}{f(c)}.$$

Math 312 - Homework 8

Problem 1

We showed in class that independent random variables X and Y are orthogonal, i.e. they satisfy $E(XY) = E(X)E(Y)$. Is the converse true?

Problem 2

Suppose that $X_1, X_2, \ldots$ have the property that $E(X_n) \to \mu$ and $\mathrm{Var}(X_n) \to 0$. Show that $X_n \to \mu$ in probability, but not necessarily almost surely.