Glossary and Cheat-Sheet for the UChicago ATSS

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The following glossary of terms and notations is included for quick reference. This is not meant to replace a well-motivated explanation.

$(-)_+$
: Denotes the operation of adding a disjoint base-point to an unbased space. So $X_+ = X \amalg \{*\}$ with basepoint $*$.

$[-,-]$ and $[-,-]_*$
: A common notation for homotopy classes and pointed homotopy classes of maps between two spaces.

π_*^s, $\pi_* S$
: The stable homotopy groups of spheres. That is, $\pi_n(S)$ denotes the stable value of $\pi_{n+k}(S^k)$ as $k \to \infty$. That this stable value is attained at a finite stage is a theorem of Freudenthal.

$BO(k), BSO(k), BU(k)$
: These denote the classifying spaces for real, oriented, and complex rank k vector bundles, respectively. These are only well-defined as homotopy types, but here are explicit models for these spaces: $BO(k)$ is modeled by the Grassmanian of k-dimensional subspaces of $\mathbb{R}^\infty$, $BSO(k)$ is modeled by the space of pairs (V, ϵ) where V is a k-dimensional subspace of $\mathbb{R}^\infty$ and ϵ is a choice of orientation, and $BU(k)$ is modeled by the Grassmanian of subspaces of $\mathbb{C}^\infty$ with (complex) dimension k.

BO, BSO, BU
: There is an inclusion $BO(k) \hookrightarrow BO(k+1)$ given, using our Grassmanian model, by taking a subspace $V \subset \mathbb{R}^\infty$ and sending it to the subspace $\mathbb{R} \oplus V \subset \mathbb{R} \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$. The space BO is defined as the union $\bigcup_{k \geq 0} BO(k)$. Similar definitions hold for BSO and BU.

K, KO
: See **K-theory**.

$K\mathbf{R}$
: See **Atiyah's Real K-theory**.

K_G, KO_G
: See **equivariant K-theory**.

$MO(k), MSO(k), MU(k), ...$
: Sitting over $BO(k)$ is the tautological rank k vector bundle, call it γ_k. It is defined as the subspace $\gamma_k \subset BO(k) \times \mathbb{R}^\infty$ consisting of pairs $([V], x)$ such that $x \in V$. We let $MO(k)$ denote the Thom space $\mathrm{Th}(\gamma_k)$. Similar definitions hold for $MSO(k)$ and $MO(k)$.

$MO_*, MSO_*, MU_*, ...$
: The symbol MO_n denotes the stable value of $\pi_{n+k} MO(k)$ as $k \to \infty$. The symbol MO_* denotes the graded abelian group $\bigoplus MO_k$. Similar definitions apply to MSO_* and MU_*.

Ω
: See **loopspace**.

π_n
: See **homotopy groups**.

$\wedge$

See **smash product**.

Σ, S

See **suspension**.

S^V

See **one-point compactification**.

Sq^n

See **Steenrod operation**.

$\mathrm{Th}(E), \mathrm{Th}_X(E), X^E$

See **Thom space**.

$\vee$

See **wedge sum**.

$X \times_Z Y$

See **fiber product, pullback**.

Adem relations

These are identities between elements in the (mod 2) **Steenrod algebra** and provide a way of writing certain compositions of **Steenrod operations** in terms of other compositions of Steenrod operations. Explicitly, if $a < 2b$, the Adem relations state:

$$Sq^q Sq^b = \sum_{c \geq 0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c.$$

The binomial coefficients are taken mod 2.

Atiyah's Real K-theory

The same as K-theory, but with complex vector bundles replaced by **R**eal vector bundles.

category

A category is a pair of sets $(\mathrm{Obj}, \mathrm{Mor})$ together with the following structure:

(a) Two functions $s, t : \mathrm{Mor} \longrightarrow \mathrm{Obj}$ called *source* and *target*,

(b) A function $\iota : \mathrm{Obj} \longrightarrow \mathrm{Obj}$ called *identity* or *unit*,

(c) A function

$$\{(f, g) : s(f) = t(g)\} = \mathrm{Mor} \times_{\mathrm{Obj}} \mathrm{Mor} \xrightarrow{\circ} \mathrm{Mor}$$

called composition.

We denote $\iota(a)$ by id_a or 1_a, and $\circ(f, g)$ by $f \circ g$. Composition is required to be associative and unital. That is, whenever these formulae make sense they hold: $(f \circ g) \circ h = f \circ (g \circ h)$, and $f \circ 1_{s(f)} = f \circ 1_{t(f)} = f$.

Chern classes

Chern classes are certain characteristic classes of complex vector bundles. Explicitly, Chern classes are a sequence $\{c_i\}_{i \geq 0}$ of natural transformations

$$c_i : \mathrm{Vect}_{\mathbb{C}}(-) \longrightarrow H^{2i}(-, \mathbb{Z})$$

where $\mathrm{Vect}_{\mathbb{C}}(X)$ denotes the set of isomorphism classes of complex vector bundles on X. They are characterized by the following axioms:

C1. (Whitney sum formula) If E and F are complex vector bundles on X, then

$$c_n(E \oplus F) = \sum_{i+j=n} c_i(E) c_j(F).$$

C2. (Normalization) $c_0(E) = 1$ for all E, and if $\mathcal{O}(-1)$ denotes the tautological line bundle on $\mathbb{C}P^k$, then

$$c_j(\mathcal{O}(-1)) = \begin{cases} 1 & j = 0 \\ -x & j = 1 \\ 0 & \text{else} \end{cases}$$

where $x \in H^2(\mathbb{C}P^k)$ is Poincaré dual to the standard inclusion of $\mathbb{C}P^{k-1} \subset \mathbb{C}P^k$.

It is often useful to define the *total Chern class* as the formal sum $c = 1 + c_1 + c_2 + \cdots$. For example, the Whitney sum becomes $c(E \oplus F) = c(E) \cdot c(F)$.

cobordism

We say that two compact n-manifolds M and N are cobordant if there exists and $(n+1)$-dimensional manifold with boundary W together with an identification of its boundary $\partial W \cong M \amalg N$. There are similar definitions when the stable normal bundles of M and N have extra structure. Cobordism is an equivalence relation and cobordism classes form a graded ring under disjoint union and Cartesian product. This graded ring is sometimes denoted Ω_*^O with oriented, complex, and framed variants Ω_*^{SO}, Ω_*^U, and Ω_*^{fr}. It is a theorem of Thom (and Pontryagin for the framed case) that $MO_* \cong \Omega_*^O$, $MSO_* \cong \Omega_*^{SO}$, $MU_* \cong \Omega_*^U$, and $\pi_*(S) = \Omega_*^{\mathrm{fr}}$.

Eilenberg-Steenrod axioms

Let **hPair** denote the category whose objects are pairs of spaces (X, A) and whose morphisms are homotopy classes of maps of pairs $[(X, A), (Y, B)]$. An *(ordinary) homology theory* on **hPair** is a sequence of functors for $i \geq 0$

$$h_i : \mathbf{hPair} \longrightarrow \mathbf{Ab}$$

satisfying the following axioms:

(i) (Excision) If (X, A) is a pair and $U \subset X$ is such that $\overline{U}$ is contained in the interior of A, then the map

$$(X - U, A - U) \longrightarrow (X, A)$$

induces an isomorphism on h_i for all i.

(ii) (Additivity) If $X = \coprod_\alpha X_\alpha$ is a disjoint union of spaces, then the natural map

$$\bigoplus_\alpha h_i(X_{\alpha)} \longrightarrow h_i(X)$$

is an isomorphism.

(iii) (Exactness) Associated to each pair (X, A) is a long exact sequence, natural in the pair:

$$\cdots \longrightarrow h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \longrightarrow \cdots$$

(iv) (Dimension) Let $*$ be the one point space. Then $h_n(*, \varnothing) = 0$ for $n \neq 0$.

equivariant K-theory

Much the same as K-theory, but with vector bundles everywhere replaced by G-equivariant vector bundles.

equivariant vector bundle

An equivariant vector bundle over a G-space, X, is a vector bundle E over X equipped with an action of G which commutes with the projection map and is linear on each fiber.

fiber product, pullback

Given maps of sets or spaces, $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, we define the fiber product $X \times_Z Y$ to be the subset (or subspace) of $X \times Y$ consisting of pairs (x, y) with $f(x) = g(y)$. This is also sometimes denoted by f^*Y (or g^*X, depending on whether we're laying down or standing up), and depicted in a diagram:

$$\begin{array}{ccc} f^*Y = X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

framed manifold

Technically, a framed manifold is a manifold with a specified trivialization of its tangent bundle. However, we often say 'framed' when we mean 'stably framed'. A stably framed manifold is a manifold equipped with a trivialization of its stable normal bundle. Explicitly, this means the data of an embedding $M \hookrightarrow \mathbb{R}^N$ for $N \gg 0$ together with a trivialization of the normal bundle ν for this embedding.

functor

Given two categories $\mathcal{C}, \mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ consists of two functions $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ respecting all the structure: so $s(F(f)) = F(s(f))$, $t(F(f)) = F(t(f))$, $F(1_a) = 1_{F(a)}$, and $F(f \circ g) = F(f) \circ F(g)$.

homotopy groups

Given a pointed space X, we denote by $\pi_n(X)$ the set of homotopy classes of pointed maps $[S^n, X]_*$. Equivalently, it is the set of homotopy classes of maps of pairs $[(D^n, \partial D^n), (X, x_0)]$.

K-theory

Given a compact space X, we denote by $K(X)$ or $K^0(X)$ (resp. $KO(X)$ or $KO^0(X)$) the group completion of the monoid of complex (resp. real) vector bundles on X. Explicitly, let $\mathrm{Vect}(X)$ denote the monoid of isomorphism classes of vector bundles on X. The group completion (or *Grothendieck construction*) is obtained by taking the free abelian group on the set of symbols $[E]$ for $E \in \mathrm{Vect}(X)$ modulo the relation $[E] + [F] \sim [E \oplus F]$.

If X is not compact we define $K(X)$ to be $[X, BU \times \mathbb{Z}]$ and $KO(X)$ to be $[X, BO \times \mathbb{Z}]$.

For a pointed space, X, define reduced K-theory by $\widetilde{K}(X) := \ker(K(X) \to K(x_0) = \mathbb{Z})$. Similarly one defines $\widetilde{KO}(X)$.

When $n > 0$, we can define $K^{-n}(X) := \widetilde{K}(\Sigma^n(X_+))$. Bott periodicity allows us to extend this definition for all $n \in \mathbb{Z}$. There's a similar story for KO.

loopspace

Given a based space, X, the loopspace ΩX is the space $\mathrm{Map}_*(S^1, X)$ of based maps with the compact-open topology.

natural transformation

A natural transformation $\eta : F \longrightarrow G$ between two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ is a collection of morphisms $\eta_a : F(a) \longrightarrow G(a)$ for each $a \in \mathcal{C}$ subject to the condition of *naturality*: for every morphism $f : a \longrightarrow b$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ {\scriptstyle \eta_a}\downarrow & & \downarrow{\scriptstyle \eta_b} \\ G(a) & \xrightarrow[G(f)]{} & G(b) \end{array}$$

normal bundle

Let $f : M \longrightarrow N$ be a map of smooth manifolds with the property that $Df_x : T_xM \longrightarrow T_{f(x)}N$ is an injection for all $x \in M$. (Such a map is called an immersion). The normal bundle, ν_f, is defined as the quotient of f^*TN by the sub-bundle TM.

one-point compactification

Given a space X, the one-point compactification of X is the space X^+ whose underlying set is $X \amalg \{\infty\}$ with topology given by declaring the inclusion $X \hookrightarrow X^+$ to be continuous and a set U containing ∞ to be open if $X^+ \setminus U \subset X$ is closed and compact. When $X = V$ is a vector space, we denote the one-point compactification by S^V.

orientation

An orientation of a real vector bundle E of rank k over a space X is a choice of equivalence class of nonzero section of the top exterior power bundle $\Lambda^k E$. Two such sections are considered equivalent if they differ by a positive constant. A vector bundle is called *orientable* if it admits an orientation. A manifold is called *oriented* or *orientable* if its tangent bundle is so.

Poincaré duality

If W is a compact, oriented n-manifold with boundary (possibly empty), then Poincaré duality states that there is an element $[W, \partial W] \in H_n(W, \partial W)$ with the property that the cap product gives an isomorphism:

$$\cap[W, \partial W] : H^i(W) \cong H_{n-i}(W, \partial W).$$

For cohomology with coefficients in a field k, this isomorphism is compatible with the cup product in the sense that the cup product gives an isomorphism

$$H^j(W; k) \xrightarrow{\cong} \mathrm{Hom}_k(H^{n-j}(W, \partial W; k), k).$$

(This also holds with $\mathbb{Z}$-coefficients if the cohomology is torsion-free.)

Pontryagin classes

If E is an oriented, real vector bundle on X it is unfortunately standard to define

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(X, \mathbb{Z}).$$

If E happens to already be a complex vector bundle then there is a relationship between the Chern classes and the Pontryagin classes given by:

$$(1 - p_1 + p_2 - \cdots) = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots).$$

Pontryagin-Thom collapse

If $U \subset X$ is an open subspace of X, the Pontryagin-Thom collapse map is the map $f : X^+ \longrightarrow U/\partial U$ given by $f(x) = x$ if $x \in U$ and $f(x) = \{\partial U\}$ otherwise.

(Atiyah) Real vector bundles

Given a C_2-space, X, a **R**eal vector bundle over X is a complex vector bundle E over X and a C_2-action on E such that the projection map is equivariant and the C_2-action is conjugate-linear on each fiber.

smash product

Given pointed spaces X and Y, the smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$. Explicitly, we quotient by the equivalence relation $(x_0, y) \sim (x_0, y')$ and $(x, y_0) \sim (x', y_0)$. If this means anything to you, the smash product is *not* the product in the category of pointed spaces, so beware.

smooth manifold

A topological manifold is a Hausdorff, second-countable, locally Euclidean space. (You can safely ignore "second-countable" and just think "partitions of unity exist.") A smooth atlas on a topological manifold is a chosen cover $\{U_\alpha\}$ together with open embeddings $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ is smooth wherever it is defined. Two smooth atlases are said to be equivalent if there is a third smooth atlas containing both of them. A smooth structure is an equivalence class of smooth atlases, or, equivalently, the choice of a maximal smooth atlas. A smooth manifold is a manifold with a smooth structure. It is important to never think about this definition again after reading it once.

space

When we say space we probably mean a *compactly generated, weak Hausdorff space*. But you can safely think 'manifold' or 'CW-complex' and you'll never be lead astray. Just in case, here is the definition of that mouthful: a space X is compactly generated if a subset $U \subset X$ is open if and only if $U \cap K$ is open for all compact $K \subset X$. Given any topological space X one can define a new topological space $k(X)$ with the same underlying set by declaring $U \subset k(X)$ to be open if and only if $U \cap K$ is open in X for all compact subspaces $K \subset X$. One can prove $k(X)$ is compactly generated. If X and Y are compactly generated spaces, we (somewhat abusively) denote by $X \times Y$ the space $k(X \times Y)$. A weak Hausdorff space is one where the diagonal map $X \longrightarrow X \times X$ is a closed map. If X and Y are compactly generated, weak Hausdorff spaces then the space $\mathrm{Map}(X, Y)$ with the compact-open topology has all the properties you want.

stable normal bundle

Let M be a compact n-manifold. Choose any embedding $M \hookrightarrow \mathbb{R}^{n+k}$ into a Euclidean space and let $M \longrightarrow BO(k)$ be a homotopy class representing the normal bundle of this embedding. Then the stable normal bundle is the homotopy class of the composite $M \longrightarrow BO(k) \longrightarrow BO$. This can be shown to be independent of the embedding. In practice, by 'stable normal bundle' we mean 'choose an embedding of M into a really big Euclidean space and consider its normal bundle.'

If $B \longrightarrow BO$ is a map then a B-structure on M is a choice of homotopy class of map $M \longrightarrow B$ such that $M \longrightarrow B \longrightarrow BO$ is homotopic to the stable normal bundle. Common choices of B include BSO, BU, and a point. These lead to the notion of an oriented manifold, a stably complex manifold, and a stably framed manifold, respectively.

Steenrod operations

The Steenrod operations are certain additive natural transformations $Sq^n : H^*(-, \mathbb{F}_2) \longrightarrow H^{*+n}(-, \mathbb{F}_2)$. While there are odd primary versions, we will only consider the mod 2 case here. They are characterized by the following axioms:

(i) (Degree) If $i > p$, then $Sq^i(x) = 0$ for all $x \in H^p(X)$, and $Sq^p(x) = x^2$,

(ii) (Cartan) $Sq^n(xy) = \sum_{i+j=n}(Sq^i x)(Sq^j y)$,

(iii) (Normalization) Sq^0 is the identity map.

Steenrod algebra

The Steenrod algebra is a graded, non-commutative, associative $\mathbb{F}_2$-algebra generated by the symbols Sq^i subject only to the **Adem relations**. Alternatively, it is the algebra of stable, degree-shifting, additive natural endomorphisms of $H^*(-, \mathbb{F}_2)$ under composition. (It is a non-trivial theorem that these two definitions are equivalent, and not at all obvious.)

Stiefel-Whitney classes

These are defined exactly as Chern classes with the following modifications: (i) replace complex vector bundles with real vector bundles, (ii) replace $H^{2i}(-, \mathbb{Z})$ with $H^i(-, \mathbb{Z}/2)$, and (iii) replace $\mathbb{C}P^n$ with $\mathbb{R}P^n$.

suspension

If X is a space, denote by SX the space $I \times X/\sim$ where the equivalence relation is given by declaring $(0, x) \sim (0, x')$ and $(1, x) \sim (1, x')$ for all $x, x' \in X$. This is sometimes called the unreduced suspension. If X is a pointed space, we denote by ΣX the space $SX/I \times \{x_0\}$.

Thom class

If E is a vector bundle of rank k on X a *Thom class* is an element $U_E \in H^k(X^E, R)$ which restricts to the usual generator of $H^k(S^k)$ under the map $S^k \cong \{x\}^{E_x} \longrightarrow X^E$ induced by the inclusion $\{x\} \hookrightarrow X$, for any point $x \in X$. If $R = \mathbb{Z}/2$ such a class always exists. If $R = \mathbb{Z}$ such a class is equivalent to a choice of *orientation* of E.

Thom space

If E is a vector bundle over a space X, the Thom space is the one-point compactification E^+ of E. This is usually denoted by X^E or $\mathrm{Th}(E)$.

Thom isomorphism

If E is an *oriented* vector bundle of rank k over a space X, then the exterior cup product with the Thom class induces an isomorphism $U_E \cdot (-) : H^*(X, \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{*+k}(X^E, \mathbb{Z})$. The same is true for $\mathbb{Z}/2$-coefficients but we no longer need to assume that E is oriented.

vector bundle

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A family of $\mathbb{F}$-vector spaces over a space X is a map $p : E \longrightarrow X$ together with continuous maps $\mathbb{F} \times E \longrightarrow E$ and $E \times_X E \longrightarrow E$ commuting with the projections down to X such that these maps restrict to $\mathbb{F}$-vector space structures on each fiber.

The trivial family, denoted $\underline{\mathbb{F}}$ is the space $\mathbb{F} \times X$ equipped with the evident projection and structure maps. An isomorphism of families of vector spaces over X is a continuous homeomorphism $E \to E'$ commuting with all the structure maps (this implies it is linear on each fiber.) A vector bundle is a family of vector spaces with the property that, for each $x \in X$ there is a neighborhood $U \subset X$ of x such that $E|_U$ is isomorphic to the trivial family over U.

wedge sum

Given pointed spaces X and Y, the wedge sum $X \vee Y$ is the subspace of $X \times Y$ given by pairs (x, y) where $x = x_0$ or $y = y_0$. Equivalently (up to homeomorphism), it is the quotient of $X \amalg Y$ by the equivalence relation $x_0 \sim y_0$. The wedge sum is the coproduct in the category of pointed spaces (if that means anything to you.)