University of Chicago Lectures.

Lecture (1): introduction to K-theory.

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Topological K-theory.

1. Group completion.
2. K-theory via group completion and equivalence relations.
3. Representable K-theory.
4. K-theory as a cohomology theory.
5. Examples.
6. Chern classes.

1. Group completion.

M abelian monoid, so it has commutative, associative $+$ with 0.

Ex. (i) $\mathbb{Z}_{\geq 0}$ under addition.

(ii) $\mathbb{Z}_{>0}$ under multiplication.

(iii) Any abelian group.

What's wrong with $\mathbb{Z}_{\geq 0}$? No inverses. Add them to get an abelian group $\mathbb{Z}$.
Similarly for $\mathbb{Z}_{>0}$. Add them to get $\mathbb{Q}_{>0}$.

Def. The group completion of an abelian monoid M is an abelian group $M^{-1}M$ and $M \to M^{-1}M$ s.t. for every $f: M \to A$ where A is an abelian group, there is a unique factorization through $M \to M^{-1}M$.
In a diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A \\ M^{-1}M \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$$\mathrm{Hom}_{\mathrm{AbMon}}(M, A) \cong \mathrm{Hom}_{\mathrm{AbGp}}(M^{-1}M, A) \quad \text{where } A \text{ is an abelian group,}$$

$$\begin{tikzcd} \mathrm{AbMon} \arrow[r, shift right, dotted, "F"'] & \mathrm{AbGp} \arrow[l, shift right] \end{tikzcd} : \text{Forgetful functor}$$

$F(M) = M^{-1}M$.

Prop. $M^{-1}M$ exists.

proof. Either use the adjoint functor theorem in some form to prove the existence of F, or do the following.

$\mathcal{F}(M)$ = free abelian group on symbols $[m]$, $m \in M$.

$M^{-1}M = \mathcal{F}(M)/R(M)$, $R(M)$ = subgroup generated by $[m+n]-[m]-[n]$.

(Weibel)

<u>Exercise</u>. (i) Every element of $M^{-1}M$ is of the form $[m]-[n]$ for some $m,n\in M$.

(ii) $[m]=[n]$ in $M^{-1}M \iff m+p=n+p$ for some $p\in M$.

[redacted]

This proves that $M^{-1}M$ is a quotient of the monoid $M\times M$ by the equivalence relation $(m,n)\sim(m+p,n+p)$.

<u>Exercise</u>. Prove that

(i) the group completion of $\mathbb{Z}_{\geq 0}$ is $\mathbb{Z}$,

(ii) ... $\mathbb{Z}_{>0}$ is $\mathbb{Q}_{>0}$, and

(iii) $A^{-1}A\cong A$ when A is an abelian group.

<u>Rem</u>. When M is a <u>semiring</u> (an object satisfying the axioms of a ring except for additive inverses), then $M^{-1}M$ is a ring. (EXERCISE).

2. K-theory via group completion.

X paracompact. $\mathrm{Vect}_{\mathbb{C}}(X)$ = abelian monoid of complex vector bundles on X.

$KU(X)$ = group completion of $\mathrm{Vect}_{\mathbb{C}}(X)$.

Grothendieck group of complex vector bundles

—or—

complex K-group.

Ring via tensor product.

In other words, $KU(X)$ is the free abelian group on iso classes of $\mathbb{C}$-vector bundles modulo the relation

$$[\eta \oplus \xi] = [\eta] + [\xi].$$

<u>Remark</u>. Real and quaternionic vector bundles lead to $KO(X)$ and $KSp(X)$.

The reduced complex K-theory of [redacted] a pointed space $(X,*)$ is

$$\widetilde{KU}(X) = \ker\left(KU(X) \longrightarrow KU(*)\right).$$

<u>Exs</u> (i) $KU(*) \cong \mathbb{Z}$.

(ii) $\widetilde{KU}(*) = 0$.

(iii) $\widetilde{KU}(X,*)$ is the Grothendieck group of virtual vector bundles of rank zero <u>at x</u>.

<u>Lemma</u>. If $(X,*)$ is connected, compact, Hausdorff, then $\widetilde{KU}(X,*)$ is the set of equivalence classes of vbs over X where $\eta \sim \xi$ if and only if $\eta \oplus \mathbb{1}^m \cong \xi \oplus \mathbb{1}^n$ for some $m,n \geq 0$.

<u>proof</u>. Use $\eta \longmapsto ([\eta] - [\mathbb{1}^{\mathrm{rank}\,\eta}])$.

3. Representable K-theory.

Thm. X compact Hausdorff.

$$KU(X) \cong [X, \mathbb{Z} \times BU] \quad \text{and} \quad \widetilde{KU}(X) \cong [X, \mathbb{Z} \times BU]_*.$$

proof. We can assume X is (path) connected, when it is enough to prove the second statement. In that case, $[X, \mathbb{Z} \times BU]_* \cong [X, BU]$. There is a natural abelian monoid map

$$\begin{aligned} \mathrm{Vect}_{\mathbb{C}}(X) &\longrightarrow [X, BU] \\ \eta^m &\longmapsto (X \longrightarrow BU_m \longrightarrow BU). \end{aligned}$$

Compactness implies surjectivity. Two bundles η^m and ξ^n map to the same element of $[X, BU]$ iff they are stably isomorphic, i.e., if

$$\eta \oplus \mathbb{1}^M \cong \xi \oplus \mathbb{1}^N$$

for $M, N \geq 0$. We conclude by the lemma.

4. K-theory as a cohomology theory.

$$KU^0(X) := [X, \mathbb{Z} \times BU],$$

$$\widetilde{KU}^0(X) := [X, \mathbb{Z} \times BU]_+ .$$

$$\widetilde{KU}^{-n}(X) := [\Sigma^n X, \mathbb{Z} \times BU]_+, \quad n \geq 0.$$

$$KU^{-n}(X) := [X, \Omega^n(\mathbb{Z} \times BU)]_+, \quad n \geq 0.$$
$$\Omega^n(\mathbb{Z} \times BU) \simeq \Omega^n BU .$$

<u>Bott Periodicity</u>. $\widetilde{KU}^{-2}(X) \cong \widetilde{KU}^0(\Sigma^2 X) \cong \widetilde{KU}^0(X)$.

More generally, for $n \geq 0$,

$$\begin{aligned} \widetilde{KU}^{-n-2}(X) &\cong [\Sigma^{n+2} X, \mathbb{Z} \times BU]_+ \\ &\cong [\Sigma^n X, \Omega^2 BU]_+ \\ &\underset{\text{Bott}}{\cong} [\Sigma^n X, \mathbb{Z} \times BU]_+ \\ &\cong \widetilde{KU}^{-n}(X). \end{aligned}$$

We can define now $\widetilde{KU}^n(X) := \widetilde{KU}^{-|n|}(X)$ for any n.

<u>Thm</u>. $\widetilde{KU}^*$ admits the structure of a reduced cohomology theory.

The unreduced theory for a CW pair is given by

$$KU^*(X, A) \cong \widetilde{KU}^*(X/A, *).$$

<u>Consequences</u>. (i) $A \subseteq X$ a subcomplex, $\cdots \to \widetilde{KU}^n(X/A) \to \widetilde{KU}^n(X) \to \widetilde{KU}^n(A) \to \widetilde{KU}^{n+1}(X/A) \to \cdots$.

(ii) homotopy invariant.

(iii) $\widetilde{KU}^n(X) \cong \widetilde{KU}^{n+1}(\Sigma X)$.

5. Examples.

(i) $\widetilde{KU}^0(S^n) \cong \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$

~~This can be proved also without Bott periodicity~~

$n \geq 2$.

(ii) $f : S^{2n-1} \to S^n$ a map, $X = S^n \cup_{S^{2n-1}} D^{2n}$ the pushout.

Then, (X, S^n) is a CW pair with quotient S^{2n}. The LES for the pair gives

$$\cdots \to \widetilde{KU}^m(S^{2n}) \to \widetilde{KU}^m(X) \to \widetilde{KU}^m(S^n) \to \widetilde{KU}^{m+1}(S^{2n}) \to \cdots$$

This splits into two cases.

n odd.

$$0 \to \widetilde{KU}^1(X) \to \mathbb{Z} \to \mathbb{Z} \to \widetilde{KU}^0(X) \to 0.$$

n even. $\widetilde{KU}^{-1}(X) = 0$,

$\widetilde{KU}^0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ (noncanonical splitting).

6. Chern classes.

$$c_t(x) = 1 + c_1(x)t + c_2(x)t^2 + \cdots$$

in $H^*(X,\mathbb{Z})[t]_0$

$|t| = -2$.

Since $c_t(x \oplus \xi) = c_t(x)\, c_t(\xi)$, we get

$$\widetilde{KU}(X) \longrightarrow H^*(X,\mathbb{Z})[t]_0 \, ,$$

for X connected.

Ex. $\underset{\underset{\mathbb{Z}}{\cong}}{\widetilde{KU}(S^2)} \xrightarrow{c_1} \underset{\underset{\mathbb{Z}}{\cong}}{H^2(S^2,\mathbb{Z})}$ is an isomorphism.