TUESDAY - TALK 7
VECTOR BUNDLES 3

Contents

1. Higher Structure: Steenrod Squares

In this section, $H^*(X) = H^*(X, \mathbb{Z}/2)$. We describe operations on cohomology that generalize the cup product. For $i \geq 0$, the *i'th Steenrod square* is a group homomorphism

$$Sq^i : H^n(X) \to H^{n+i}(X).$$

They have the following properties:

(1) $Sq^0(\alpha) = \alpha$ (i.e. Sq^0 is the identity morphism.)
(2) If $\alpha \in H^k(X)$, then $Sq^k(\alpha) = \alpha^2$.
(3) If $\alpha \in H^k(X)$, and $i > k$, then $Sq^k(\alpha) = 0$.
(4) Sq^1 is the Bockstein homomorphism, i.e., the connecting homomorphism for the long exact sequence on cohomology induced from the short exact sequence on coefficients

$$0 \to \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \to \mathbb{Z} \to 0.$$

(In particular, $Sq^1 Sq^1 = 0$).
(5) There is a commutative diagram

$$\begin{array}{ccc} H^n(X) & \xrightarrow{\cong} & H^{n+1}(\Sigma X) \\ \downarrow{\scriptstyle Sq^i} & & \downarrow{\scriptstyle Sq^i} \\ H^{n+i}(X) & \xrightarrow{\cong} & H^{n+i+1}(\Sigma X). \end{array}$$

(6) (Cartan Formula) $Sq^i(x \cup y) = \sum_{j+k=i} Sq^j(x) \cup Sq^k(x)$.
(7) (Adem Relations) If $a < 2b$, then

$$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c.$$

(8) (Naturality) If $f : X \to Y$ is a continuous map, then $f^* Sq^i = Sq^i f^*$.

Remark 1.1. There are also relative Steenrod operations $Sq^i : H^n(X, A) \to H^{n+i}(X, A)$.

Exercise 1.2. For $x \in H^n(X)$, write

$$Sq(x) = Sq^0(x) + Sq^1(x) + \ldots + Sq^n(x).$$

Prove that the Cartan formula implies that

$$Sq(xy) = Sq(x)Sq(y).$$

Example 1.3. Let's compute what these do to the ring $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2)$. We have

$$\begin{aligned} Sq(w) &= Sq^0(w) + Sq^1(w) \\ &= w + w^2. \end{aligned}$$

Therefore,

$$\begin{aligned} Sq(w^i) &= (w + w^2)^i \\ &= w^i(1+w)^i \\ &= w^i \sum_{k=0}^{i} \binom{i}{k} w^k \\ &= \sum_{k=0}^{i} \binom{i}{k} w^{i+k}. \end{aligned}$$

Therefore, we read off

$$Sq^k(w^i) = \binom{i}{k} w^{i+k}.$$

Remark 1.4. Note that binomial coefficients are easy to compute modulo 2. Let $i = i_0 + 2i_1 + \ldots + 2^r i_r + \ldots$ for $i_r = 0$ or 1 (note that this sum is finite). Similarly, let $k = k_0 + 2k_1 + \ldots + 2^r k_r + \ldots$. Then

$$\binom{i}{k} \equiv \prod_r \binom{i_r}{k_r} \mod (2).$$

Definition 1.5. The Steenrod algebra $\mathcal{A}$ is the $\mathbb{Z}/2$ algebra on the symbols Sq^i modulo the Adem relations and the relation $Sq^0 = 1$.

Definition 1.6. A Steenrod square Sq^r is *decomposable* if it can be written as a sum of products of Sq^i's with $i < r$. It is called *indecomposable* if it is not decomposable.

For example, you can verify that

$$Sq^3 = Sq^1 Sq^2$$

so Sq^3 is decomposable.

Exercise 1.7. Prove that Sq^r is indecomposable if and only if $r = 2^l$. (Hint: Compute its effect on $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2)$ for one direction and use the Adem relations for the other).

Proposition 1.8. *As an algebra, $\mathcal{A}$ is generated by the symbols Sq^{2^i}.*

2. Characteristic Classes

Recall that there are isomorphisms

$$Vect_n^{\mathbb{R}}(X) \cong [X, Gr_n(\mathbb{R}P^\infty)], \qquad Vect_n^{\mathbb{C}}(X) \cong [X, Gr_n(\mathbb{C}P^\infty)].$$

Now, let $B_n = Gr_n(\mathbb{R}P^\infty)$ and $R_n = \mathbb{Z}/2$ or $B_n = Gr_n(\mathbb{C}P^\infty)$ and $R_n = \mathbb{Z}$. Then given a vector bundle $\xi : E \to X$ with classifying map $f_\xi : X \to B_n$, we get a map

$$H^*(B_n, R_n) \xrightarrow{f_\xi^*} H^*(X, R_n).$$

Given $c \in H^*(B_n, R_n)$,

$$c(\xi) := f_\xi^*(c)$$

is an algebraic invariant for the vector bundle ξ called a *characteristic class*. These are interesting because they can allow us to distinguish between vector bundles, i.e., if there exists c such that $c(\xi) \neq c(\eta)$, the $\xi \ncong \eta$.

For M a smooth manifold, let $TM \to M$ be the tangent bundle of M. Then the characteristic classes

$$c(M) = c(TM)$$

are of particular interest since they give invariants of smooth manifolds.

Computing characteristic classes is closely tied to computing $H^*(B_n, R_n)$. When $B_n = Gr_n(\mathbb{R}P^\infty)$, the characteristic classes are called *Siefel-Whitney* classes. When $B_n = Gr_n(\mathbb{C}P^\infty)$, they are called *Chern* classes. I will only talk about Siefel-Whitney (SW) classes in this talk.

3. Cohomology of $Gr_n(\mathbb{R}P^\infty)$

Theorem 3.1. *$H^*(Gr_n(\mathbb{R}P^\infty), \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$ for w_i of degree i called the i'th SW class.*

In fact, one call $(\mathbb{R}P^\infty)^n \to Gr_n(\mathbb{R}P^\infty)$ classifying the n–plane bundle $\gamma_1 \times \ldots \times \gamma_1$ and prove that this gives an isomorphism

$$H^*(Gr_n(\mathbb{R}P^\infty), \mathbb{Z}/2) \to H^*((\mathbb{R}P^\infty)^n, \mathbb{Z}/2)^{\Sigma_n} \subset \mathbb{Z}/2[w_1]^{\otimes n} \cong \mathbb{Z}/2[w_{1,1}, \ldots w_{1,n}]$$

where Σ_n is the symmetric group on n–letters and acts by permuting the copies of $\mathbb{R}P^\infty$. The class w_i goes to the i'th elementary symmetric polynomial on the $w_{1,i}$'s.

4. Relation to Steenrod Squares

For any n-plane bundle ξ on X, there are elements

$$w_i(\xi) \in H^i(X, \mathbb{Z}/2).$$

A construction of the SW classes goes as follows. Any n-plane bundle ξ on X is $\mathbb{Z}/2$ orientable. and a Thom isomorphism

$$\varphi : H^i(X) \to H^{n+i}(E^+, \infty),$$

where E^+ is the one point compactification of E. We can define

$$w_i(\xi) = \varphi^{-1}(Sq^i(\varphi(1))).$$

Rather than being so formal about it, will study the SW classes by looking at the properties that they satisfy.

5. Axioms for Stiefel–Whitney classes

(1) $w_0(\xi) = 1$

(2) (Naturality) Given a pull back diagram

$$\begin{array}{ccc} f^*(\xi) & \longrightarrow & \xi \\ \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

where

$$f^*(\xi) = \{(y, v)) \mid y \in Y,\ v \in E,\ p(v) = f(y)\}$$

we have

$$f^* w_i(\xi) = w_i(f^*(\xi)).$$

(3) If ξ and η are bundles over X, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)$$

(4) The tautological line bundle $\gamma_1 \to \mathbb{R}P^1$ (aka, the Möbius band on S^1) satisfies

$$w_1(\gamma_1) \neq 0$$

We can package (3) by defining the *total Stiefel-Whitney class*

$$w(\xi) = w_0(\xi) + w_1(\xi) + \ldots + w_n(\xi) + \ldots$$

(the sum stops at $w_{dim(\xi)}(\xi)$). Then

$$w(\xi \oplus \eta) = w(\xi) \cup w(\eta).$$

6. The SW classes of the trivial bundle

Example 6.1. For any X, let $\epsilon_n = \epsilon_n(X)$ be the trivial n–plane bundle on X. Then, $\epsilon_n(X) = f^*(\epsilon_n(pt))$ for $f : X \to pt$. So in particular, $w(\epsilon_n(X)) = 1$. This implies that

$$w(\xi \oplus \epsilon_n) = w(\xi)$$

for any n.

7. Normal bundles and the Whitney product formula

Now, let $M \xrightarrow{f} \mathbb{R}^{n+k}$ be the immersion of a smooth manifold M. (An immersion means that the map on tangent spaces $TM_x \to \tau\mathbb{R}^{n+k}_{f(x)}$ is an injection. I.e., locally, f looks like an injection.) Then we can define a k–plane bundle $\nu(M)$ which is the orthogonal complement of TM in $\mathbb{R}^{n+k}$. Note that

$$\nu(M) \oplus TM = \tau(\mathbb{R}^{n+k}) = \epsilon_{n+k}(\mathbb{R}^k).$$

Therefore, letting $w(M) = w(TM)$, we have the *Whitney product formula*

$$w(\nu(M))w(M) = 1.$$

Note that $w(\nu(M))$ does not depend on ν, so we think of $w(\nu(M))$ as the formal inverse of $w(M)$ and write

$$\overline{w}(M) = w(\nu(M)).$$

Example 7.1. The natural inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ has a trivial normal bundle, so

$$\overline{w}(S^n) = 1$$

This forces $w(S^n) = 1$, so that the SW do not see the tangent bundle of S^n. However, S^n has a non–trivial tangent bundle for $n > 1$ (you will see this for $n = 2$ in the exercises).