MONDAY - TALK 4
ALGEBRAIC STRUCTURE ON COHOMOLOGY

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1. Cohomology

Let R be a commutative ring. You can think of $R = \mathbb{Z}$ or $R = \mathbb{Z}/2$ and I'll write $H^*(X) = H^*(X, R)$ unless the statement depends on the choice of R. Recall that for each n, there is a group $H^n(X)$ called the n'th cohomology group of X. It an be computed using cellular cochains for a CW approximation of X or the singular chains on X. We package this information together into one graded abelian group as:

$$H^*(X) = \bigoplus_{n \geq 0} H^n(X).$$

This is a graded abelian group. An element $\alpha \in H^n(X)$ is has *degree* n, written $|\alpha| = n$. The goal of today is to describe the algebraic structure of $H^*(X)$.

Remember that cohomology $H^n(X, R)$ is computed using the chain complex whose n'th term is

$$C^n(X, R) = \mathrm{Hom}_R(C_n(X, R), R),$$

where $C_n(X, R)$ is the cellular chain complex of X. The *coboundary* is obtain by precomposition with the boundary of $C_n(X, R)$:

$$\begin{array}{ccc} C_n(X,R) & \xrightarrow{\delta} & C_{n-1}(X,R) \\ & & \downarrow{\scriptstyle \alpha} \\ & & R \end{array}$$

So $\partial(\alpha) = \alpha \circ \delta$.

Example 1.1.
- *$H^0(pt, R) = R$ and $H^k(*, R) = 0$ if $k \neq 0$ since the cellular chain complex has one cell and*

$$\mathrm{Hom}_R(R, R) \cong R$$

(just decide where 1 goes and the rest is determined since the map must respect R–multiplication). So

$$H^*(pt, R) \cong R.$$

- *The n–sphere has cellular chain complex with one cell in degree 0 and n and no cells otherwise. So $H^k(S^n, R) = R$ if $k = 0, n$ and is zero otherwise. So*

$$H^*(S^n, R) \cong R \oplus R\epsilon$$

where ϵ is a generator of H^n.

- $\mathbb{R}P^n$ *has a cellular structure with one cell in each dimension. The chain complex has a* $\mathbb{Z}$ *in each degree* $0 \leq k \leq n$ *and is zero otherwise.*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1+(-1)^n} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Again, $\mathrm{Hom}_R(\mathbb{Z}, R) \cong R$ *and precomposition with multiplication by* 2 *is the same as multiplication by* 2*, so*

$$0 \longleftarrow R \xleftarrow[1+(-1)^n]{} R \longleftarrow \ldots \longleftarrow R \xleftarrow[2]{} R \longleftarrow R \longleftarrow 0$$

computes the cohomology. In particular,

$$H^k(\mathbb{R}P^n, R) = \begin{cases} R & k = 0 \text{ or } k = n \text{ and } n \text{ is odd} \\ R/2 & k > 0 \text{ is even.} \end{cases}$$

Note, if $R = \mathbb{Z}/2$*, then* $H^k(\mathbb{R}P^n, \mathbb{Z}/2) = \mathbb{Z}/2$ *for all* $0 \leq k \leq n$ *and zero otherwise, so*

$$H^*(\mathbb{R}P^n, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2w_1 \oplus \ldots \oplus \mathbb{Z}/2w_n$$

where w_k *is the unique non-zero element of* H^k.

2. The ring structure and cup product

2.1. **Idea and example.** $H^*(X) = H^*(X, R)$ has the structure of a graded ring. Given an element $\alpha \in H^n(X)$ and $\beta \in H^m(X)$, we want to define a product

$$\alpha\beta = \alpha \cup \beta \in H^{n+m}(X).$$

That is, an R–linear maps

$$H^n(X) \otimes H^m(X) \to H^{n+m}(X).$$

The unit of the ring is gotten as follows. Every space has a map $X \to pt$. This gives a map

$$R = H^*(pt) \xrightarrow{p^*} H^*(X)$$

The unit of $H^*(X)$ is $p^*(1_R)$ which we just denote by $1 \in H^0(X)$. The product is not commutative on the nose, but it is what we call *graded commutative*

$$\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha.$$

Exercise 2.1. Prove that if $\alpha \in H^q(X, R)$ has odd degree q, then $2\alpha^2 = 0$ in $H^{2q}(X, R)$.

Example 2.2.

- $H^*(pt, \mathbb{Z}) = \mathbb{Z}$ *as a ring.*
- $H*(S^n, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}\epsilon_n$. *Since* $1 \cup \epsilon = \epsilon \cup 1$*, we just need to specify* $\epsilon \cup \epsilon$*. However,* $\epsilon \cup \epsilon \in H^{2n}(S^n, \mathbb{Z}) = 0$*. So,* $H^*(S^n, \mathbb{Z}) \cong \mathbb{Z}[\epsilon]/(\epsilon^2)$*. This is called an* exterior algebra.
- *We will see later that* $H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]/(w^n)$ *for* $w = w_1$ *the non-zero element of* $H^1(\mathbb{R}P^n, \mathbb{Z}/2)$.

Remark 2.3. There is also a cup product in relative homology:

$$H^n(X, A) \otimes H^m(X) \to H^{n+m}(X, A).$$

3. Tensor product of Chain complexes

If C_* and C'_* are chain complexes, then

$$(C_* \otimes C'_*)_n = \bigoplus_{i+j=n} C_i \otimes C'_j.$$

and differential

$$\delta^{C_* \times C'_*}(x \otimes y) = \delta^{C_*}(x) \otimes y + (-1)^{|x|} x \otimes \delta^{C'_*}(y).$$

4. Kunneth formula and the cup product

Our goal is to define a maps

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(X) \to H^n(X)$$

which will give the multiplication.

We will do this in two steps,

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(X) \to H^n(X \times X) \xrightarrow{\Delta^*} H^n(X)$$

where the second map is is just the map induced by the diagonal

$$X \to X \times X, \quad x \mapsto (x, x).$$

We fist look at two CW complexes X and Y. Then $X \times Y$ is also a CW complex with cells n–cells

$$\{e_i^X \times e_j^Y \mid i + j = n\}.$$

One can check that

$$\delta(e_i^X \times e_j^Y) = \delta(e_i^X) \times e_j^Y + (-1)^i e_i^X \times \delta(e_j^Y).$$

(Example: $D^1 \times D^1$.)

In fact:

$$C_*(X) \otimes C_*(Y) \cong C_*(X \times Y).$$

From this, we can do some homological algebra and get a map

$$k : H^*(X) \otimes H^*(Y) \to H^*(X \times Y).$$

Theorem 4.1 (Künneth Isomorphism)**.** *If R is nice enough, if one of X or Y has R–torsion free homology and the CW complexes Y has finitely many cells in each dimensions, the map k is an isomorphism. In particular, $H^*(X)$ is always torsion free when R is a field.*

Definition 4.2. The composite

$$H^*(X) \otimes H^*(X) \xrightarrow{k} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

defines the cup product:

$$x \cup y = \Delta^* k(x \otimes y).$$

Exercise 4.3. Compute the cohomology ring of the n–torus $\mathbb{T}^n = (S^1)^{\times n}$ with coefficients in $\mathbb{Z}$.

5. Pairing between homology and cohomology

There is an evaluation map

$$C^n(X, R) \otimes C_n(X, R) = \mathrm{Hom}_R(C_n(X, R), R) \otimes C_n(X, R) \to R$$

which takes a function α and a chain a and maps it to the value of α on a, that is, $\alpha(a)$.

Once can check that this gives a pairing

$$H^n(X, R) \otimes H_n(X, R) \to R.$$

We will denote its value by

$$\langle \alpha, a \rangle .$$

In fact, if $R = F$ is a field, this is even better. The map $\alpha \mapsto \langle \alpha, - \rangle$ is an isomorphism

$$H^n(X, F) \xrightarrow{\cong} \mathrm{Hom}_F(H_n(X, F), F),$$

where the right hand side is the vector space dual of $H_n(X, F)$.

In the more general case, we always get a relationship between homology and cohomology which is called the Universal Coefficient Theorem.

6. Poincaré Duality

Let M be a compact n-manifold. We can see what orientable and non-orientable means in terms of the example S^2 and $\mathbb{R}P^2$.

Theorem 6.1. *If M is a compact n–dimensional manifold without boundary, then $H_i(M,\mathbb{Z}) = 0$ for $i > n$. Further,*
(a) If M is orientable, then $H_n(M,\mathbb{Z}) \cong \mathbb{Z}$.
(b) If M is not orientable, then $H_n(M,\mathbb{Z}) = 0$.

This agrees with our computations for S^2 and $\mathbb{R}P^2$. There's also a notion or R–orientability for different coefficients R. For now, let's just say that if $R = \mathbb{Z}/2$, every compact n–dimensional manifold is $\mathbb{Z}/2$–orientable so that (a) always holds in that case. Intuitively, this is because orientation is determined by reflections and since $-1 = 1$, these are invisible to cohomology with coefficients in $\mathbb{Z}/2$.

Choose a generator

$$[M] \in H_n(M,\mathbb{Z}) \cong \mathbb{Z}.$$

Then $[M]$ is called the *fundamental class*. You can think of $[M]$ as being represented by a sum of all the n–cells in $C_n(M,\mathbb{Z})$ (which is finite since M is compact). The cells have a natural orientation coming from $\mathbb{R}^n$. Since M is oriented, one can choose the CW structure so that the boundaries align in a way that makes $[M]$ a cycle.

Theorem 6.2 (Poincaré Duality)**.** *Let M be a closed R–oriented n–dimensional topological manifold. Consider the pairing the pairing*

$$H^p(M,R) \otimes H^{n-p}(M,R) \xrightarrow{\langle - \cup -, [M]\rangle} R \tag{6.0.1}$$

where

$$\alpha \otimes \beta \mapsto \langle \alpha \cup \beta, [M]\rangle .$$

If $R = \mathbb{Z}/2$, this pairing is non-degenerate. That is, for each $\alpha \in H^p(M,\mathbb{Z}/2)$ with $\alpha \neq 0$, there exists $\beta \in H^{n-p}(M,\mathbb{Z}/2)$ such that $\langle \alpha \cup \beta, [M]\rangle \neq 0$ and vice versa.

Further, for $R = \mathbb{Z}$ and $H^n(M,\mathbb{Z})$ and $H^{n-p}(M,\mathbb{Z})$ are torsion free, the pairing is also non-degenerate.

7. Ring structure of $H^*(\mathbb{R}P^n, \mathbb{Z}/2)$

Proposition 7.1. *There is a ring homomorphism*

$$H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]/(w^n)$$

for w the non-zero element of H^1.

Proof. Since $\mathbb{R}P^1 \simeq S^1$, the claim is clear for $n = 1$. Suppose that the claim holds for $\mathbb{R}P^{n-1}$. Note that the natural inclusion $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$ induces a surjective ring homomorphism

$$H^*(\mathbb{R}P^n, \mathbb{Z}/2) \to H^*(\mathbb{R}P^{n-1}, \mathbb{Z}/2)$$

(so in particular, an isomorphism for $0 \leq k \leq n-1$). Therefore, w^{n-1} is the non-zero element in $H^{n-1}(\mathbb{R}P^n, \mathbb{Z}/2)$. However, since

$$H^1(\mathbb{R}P^n, \mathbb{Z}/2) \otimes H^{n-1}(\mathbb{R}P^n, \mathbb{Z}/2) \xrightarrow{\langle - \cup -, [\mathbb{R}P^n]\rangle} \mathbb{Z}/2$$

is non-singular, $\left\langle w \cup w^{n-1}, [\mathbb{R}P^n] \right\rangle \neq 0$, so $w \cup w^{n-1} \neq 0$. □

Remark 7.2. Note that $H_n(\mathbb{R}P_n, \mathbb{Z}/2) = \mathbb{Z}/2$, the fundamental class $[\mathbb{R}P^n]$ is the unique non-zero element.

Remark 7.3. There are inclusions

$$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \ldots \subset \mathbb{R}P^n \subset \mathbb{R}P^{n+1} \ldots$$

and

$$\mathbb{R}P^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}P^n.$$

You should convince yourself that this implies that

$$H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$$

for $\alpha \in H^1$.