ONWARDS AND UPWARDS, OR STREAM OF CONSCIOUSNESS

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This is the last talk, and there are so many things left to say, so many loose ends to loose ends to loose ends. This will descend into a mere outline shortly. I will begin with some of the most basic foundations everyone should see written down once. Then move on to a tiny bit of an introduction to stable homotopy and the stable homotopy category.

Following up Mark Behrens, I will then say a bit about what foundations went into the recent solution of the Kervaire invariant one problem and some directions ripe for further development.

1. Cohomology theories

These come in two equivalent forms, reduced $\tilde{E}$ and unreduced E.

(1) Functors $\tilde{E}^q\colon h\mathscr{T} \longrightarrow Ab$ for all integers q
(2) Natural isomorphisms $\sigma\colon \tilde{E}^q(X) \longrightarrow \tilde{E}^{q+1}(\Sigma X)$ for all q.
(3) Exactness: Exact sequences

$$\tilde{E}^q(X/A) \longrightarrow \tilde{E}^q(X) \longrightarrow \tilde{E}^q(A)$$

for all CW pairs.
(4) Wedge: Isomorphisms

$$\tilde{E}^q(\vee_i X_i) \longrightarrow \times_i \tilde{E}^q(X_i)$$

(5) Weak Equivalence: Weak equivalences induces isomorphisms on $\tilde{E}^*$.

Unreduced theory from reduced: $E^q(X) = \tilde{E}^q(X_+)$ and $E^q(X,A) = \tilde{E}(X/A)$

Reduced theory from unreduced: $\tilde{E}^q(X) = E^q(X,*)$, X based.

Via CW approximation, the Weak Equivalence axiom defines $\tilde{E}^*$ on general X.

Observation: For any based Y, the functor $[X,Y]$ of X satisfies the Exactness and Wedge axioms on CW complexes and pairs.

Theorem 1.1 (Brown representability)**.** *Any contravariant functor $k\colon h\mathscr{T} \longrightarrow Ab$ on CW complexes that satisfies the exactness and wedge axioms is representable as $k(X) = [X,Y]$ for some Y.*

Definition 1.2. An Ω-prespectrum is a prespectrum E such that each adjoint map

$$E_n \longrightarrow \Omega E_{n+1}$$

is an equivalence.

Corollary 1.3. *An Ω-spectrum determines a cohomology theory $\tilde{E}^*$ by*

$$\tilde{E}^q(X) = [X, E_q]$$

and

$$\tilde{E}^{-q}(X) = [\Sigma^q X, E_0],$$

both for $q \geq 0$, and every cohomology theory $\tilde{E}$ is of this form.

Maps of cohomology theories are represented by maps $E_q \longrightarrow E'_q$ such that the following diagrams commute *up to homotopy.*

$$\begin{tikzcd} E_q \arrow[r] \arrow[d] & E'_q \arrow[d] \\ \Omega E_{q+1} \arrow[r] & \Omega E'_{q+1} \end{tikzcd}$$

That is a starting point, but one wants to be able to treat spectra as objects in a good category just like the category of based spaces, but better. Stable homotopy theory deals with structures invariant under suspension. They should live in a category that has also the structure we know and love, including smash products and adjoint function spectra, so that

$$\mathscr{S}(X \wedge Y, Z) \cong \mathscr{S}(X, F(Y, Z)).$$

There should be sphere spectra S^q for all integers q such that $S^q \wedge S^{-q} \cong S^0 \equiv S$ and the functors Σ and Ω should be adjoint equivalences. Up to homotopy we had nearly this much already in 1964. We only arrived a fully satisfactory point-set level category three decades later, starting here in 1993.

I'll come back to that, but I want to say a bit more about what the stable homotopy category already allows us to do. I talked about cohomology above. The first thing the stable category allows one to do is conveniently define the cohomology of spectra rather than just spaces. We have homotopy groups of spectra

$$\pi_q(E) = [S^q, E]$$

for all integers q. There is no such thing as an unreduced cohomology theory on spectra, so we generally write E^* rather than $\tilde{E}^*$. We set

$$E^q(X) = \pi_{-q} F(X, E)$$

and check the specrum level axioms. What about homology? For this, having a good smash product of spectra is wonderful. We define

$$E_q(X) = \pi_q(E \wedge X)$$

These are analogues of *Ext* and *Tor* if you know what they are. There is a suspension spectrum functor $\Sigma^\infty \colon h\mathscr{T} \longrightarrow h\mathscr{S}$, and the reduced cohomology and homology of spaces are those of their suspension spectra. This is enormously more effective than the classical theory. Only ordinary homology and cohomology are computable by chains and cochains. The homotopy category $h\mathscr{S}$ is closed symmetric monoidal under $\wedge$ and F, and these behave formally just like tensor and hom in the category of modules over a commutative ring. Ring spectra up to homotopy are monoids in $h\mathscr{S}$, with products $\phi \colon E \wedge E \longrightarrow E$ and unit $S \longrightarrow E$, and they quite trivially give cup products when the input is a space. Smash products give

$$\pi_q(X) \otimes \pi_r(Y) \longrightarrow \pi_{q+r}(X \wedge Y).$$

For spaces X, we have $X_+ \wedge X_+ = (X \times X)_+$, and we have the composite map

$$\begin{tikzcd} F(X_+, E) \wedge F(X_+, E) \arrow[d, "\wedge"] \\ F((X \times X)_+, E \wedge E) \arrow[d, "F(\Delta,\phi)"] \\ F(X_+, E) \end{tikzcd}$$

Putting these together gives cup products. Spectra don't have diagonal maps.

This gives a more modern setting than we discussed earlier. K-theory is represented by K, the Ω-spectrum with $K_{2n} = BU \times \mathbb{Z}$ and $K_{2n+1} = U$. When we have a prespectrum T_n such that each $T_n \longrightarrow \Omega T_{n+1}$ is an inclusion, we define

$$(LT)_n = \operatorname{colim} \Omega^k T_{n+k}$$

and see that $(LT)_n$ is actually homeomorphic to $\Omega(LT)_{n+1}$. Now that's what I call a *spectrum*! I have done so since 1969. We construct Thom spectra MG from Thom prespectra TG that way, for your favorite classical family of Lie groups G: O, SO, $Spin$, U, Sp. The functor Σ^∞ is another example:

$$(\Sigma^\infty X)_n = \operatorname{colim} \Omega^k \Sigma^{n+k} X.$$

Everything we have talked about this week led up to this homotopical world of spectra, where we now work so freely.

Mark Behrens talked about the Kervaire invariant one problem, a truly spectacular theorem of Hill, Hopkins, and Ravenel dating from 2009. Haynes Miller wrote a lovely survey paper about it for the Bourbaki seminar in 2010. With total lack of modesty, I will quote from his paper:

Hill, Hopkins, and Ravenel marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

— The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers;

— The theory of structured ring spectra, implemented by May and many others; and

— Equivariant stable homotopy theory, as developed by May and collaborators.

To give some idea of where the subject stands now and where it might go, I will say a bit out each of these three.

After Frank Adams' death in 1989, 30 years after Bott's bombshell, I found an unpublished draft paper on his desk. It began "The work I shall report has the following significance. At one time it seemed as if homotopy theory was utterly without system; now it is almost proved that systematic effects predominate." He was referring to the relationship between periodic phenomena in algebra and periodicity phenomena in homotopy theory, as encoded in what is called chromatic theory. It is all about calculations with periodic spectra connected with formal group laws. But it all started with Bott's K-theory spectra, which are still the most important ones geometrically. There are certain periodic ring spectra E_n which encode the chromatic filtration through the nth level and $K(n)$ which encode the difference between the $(n-1)$st level and the nth. The J theory Mark talked about is central to understanding the first layer, which is now complete. The second filtration is now largely understood calculationally. TMF is central here. To avoid embarrassing myself in front of real experts, like Mark, Agnes, and Zhouli, I'll just say that there is lots more work to be done to obtain a full conceptual understanding, but calculational understanding beyond the second level seems beyond understanding by mere humans.

$h\mathscr{S}$ is a wonderful place to work, and in stable homotopy theory we worked there and only there until 1993. With Ray and Quinn, I defined highly structured commutative ring spectra, called E_∞ ring spectra, way back in 1972. I finally understood what they really are in work with Elmendorf, Kriz, and Mandell, starting in 1993. There is a perfectly good symmetric monoidal category $\mathscr{S}$ of spectra, before passage to homotopy, and E_∞ ring spectra are essentially nothing but commutative monoids in that category. This allows us to finally develop a good theory of module spectra, allowing us to seriously mimic commutative algebra in the world of spectra. MU is an example of a spectrum that one sees in nature as a commutative ring spectrum. Using the new theory of modules, we constructed the $K(n)$ as MU-modules, making the old construction in terms of cobordism of manifolds with singularities obsolete. Goerss, Hopkins, and Miller proved that the E_n are commutative ring spectra, and use of this fact pervades modern work in chromatic homotopy theory.

One question from the audience was about putting T, for topological, in front of some algebraic term. We have Topological Hochschild homology and Topological cyclic homology, for example. Here, but not always, what we mean is that we have mimicked classical algebraic constructions by replacing rings, modules, and algebra, by ring, module, and algebra spectra, as our point-set level understanding of spectra now allows. Specialized to algebra via application to Eilenberg-MacLane spectra, like $HR = K(R,0)$, this gives strengthened versions of the algebraic invariants we knew before. Now the sphere spectrum S plays the role of $\mathbb{Z}$ in algebra, since S is the unit for the smash product and therefore maps into any commutative ring spectrum. Passage to homotopy groups gives the Hurewicz homomorphism, but we now see it in a much more highly structured context. There is a new subject of derived algebraic geometry, spearheaded by Lurie and others, which essentially starts from commutative ring spectra.

Equivariant algebraic topology, stable and unstable, is ubiquitous. All over mathematics we study objects together with their symmetries. Starting with a group G, usually finite or compact Lie but sometimes profinite or infinite discrete, one studies spaces and spectra with G actions. This is an incredibly rich field that has fascinated me since 1966. I keep coming back to it. It plays a big role in many classical areas, and in topological cyclic homology, for example, but it has only recently become fashionable, thanks to its use in the solution of the Kervaire invariant one problem.

Talking to an audience with many present and potential graduate students, it seems fitting to end with this area, because we know so little about it. It is absolutely full of open questions, at various levels. Hill, Hopkins, and Ravenel lucked out since they needed just a small amount of not very difficult calculation in their amazing work. We still know almost nothing about calculations in this area, and we don't even understand some very basic foundations.

Question 1. What is ordinary equivariant homology homotopically? The point is that ordinary theory, called Bredon homology, is characterized by a dimension axiom, which says that $H_q^G(G/H) = 0$ for $q \neq 0$, while $H_0^G(G/H) = N(G/H)$ for a prescribed functor N from the orbit category of G to Ab, called a homological coefficient system. *In general, we only know how to construct these theories via suitable chains, not via homotopy groups or anything else more topological.*

Question 2. How can we compute ordinary equivariant homology and cohomology? The list of known calculations is embarrassingly small. One key problem is that the Serre spectral sequece is like Leray's pre-fibration level version, with non-trivial local coefficients intrinsically present.

Some homology and cohomology theories, those whose coefficient systems extend to Mackey functors, are genuinely stable, admitting suspension isomorphisms for representation spheres, and these can be constructed via homotopy groups of genuine G-spectra, but these homology theories are best understood as graded not over the integers but over the real representation ring $RO(G)$.

Question 3: Even rationally, can we compute equivariant characteristic classes? We have equivariant classifying spaces $B_G(\Pi)$, where Π is the structural group of some class of bundles, for example $\Pi = U(n)$ or $\Pi = O(n)$, and G is the ambient group of symmetries. For line bundles and cyclic groups G of prime order, we know how to compute, but for almost nothing else.

Question 4: Can we compute the $RO(G)$-graded cohomology of a point for some nonabelian G? We (sort of) know the answer for cyclic groups of prime power order, but for nothing else.

Question 5: What is the relationship between geometric and cohomological orientation theory? These are equivalent nonequivariantly, but they are not equivalent equivariantly. There are even two papers of my own, one with Costenoble and Waner, on orientation theory. One is geometric, with Costenoble and Waner, and the other is cohomological, and I do not know how to relate the two.

Here are some more advanced questions.

Question 6: There is an analogue of the Atiyah-Segal completion theory with K_G replaced by MU_G or any module over it, due to Greenlees and myself, but we only proved it for finite extensions of a torus. Does it still hold for general compact Lie groups? This raises many questions.

Question 7: Is there a geometric reason that $\pi_*(MU)$ is concentrated in even degrees? Are the homotopy groups of the fixed point spectra of MU_G concentrated in even degrees?

Question 8: Is there a reasonable equivariant chromatic theory?

Question 9: As a start, can we obtain a complete understanding of equivariant J-theory, which should give understanding of chromatic level one?

Question 10: We understand several versions of global Mackey functors, defined compatibly for all finite groups G. Is there a corresponding theory of global Tambara functors? You probably have no idea what Tambara functors are, but the $\pi_0(E^H)$ of a commutative ring G-spectrum together have such a structure.

General topic: Fully develop various relationships between geometric and algebraic topology equivariantly.

Wild speculation: Is there an algebraic geometry of schemes over Mackey (and Tambara?) functors.

I could keep on going forever, but most of you are probably thinking about going home. Thank you for coming.