INTRODUCTION

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Welcome all. This workshop is an experiment, both mathematical and social. We hope to introduce some of the fundamental ideas of algebraic topology, using the actual historical development as a partial guide to the organization of the material. Socially, this is both a one week summer school and one week of an eight week REU. I hope people will find a way to get to know each other a little. There will be a get together starting 7:30 tomorrow evening around a fire or two on the point, the little peninsula beyond 55th Street.

I am not a mathematical historian, but I want to start by giving some idea of the beginnings of the subject in the mid 1930's. The introduction of homology groups as cycles modulo boundaries is due to Emmy Noether. The story is that that happened at a dinner at Brouwer's house in 1925. Hopf first explicitly used homology groups in a 1928 paper. Cohomology and something like a modern understanding of Poincaré duality came a few years later. How many people do NOT know what a homotopy between continuous maps is? Don't be shy, raise your hands if you don't know.

Hopf proved that the Hopf map $S^3 \to S^2$ is not null homotopic in 1930, introducing the Hopf invariant for that purpose. Hurewicz introduced homotopy groups in 1935. Hopf defined H-spaces and, implicitly, Hopf algebras in 1939 (published 1941). At that time $\pi_4(S^3)$ was zero. In 1945 or so it became $\mathbb{Z}/2\mathbb{Z}$. So it goes with homotopy groups, then and now.

Homotopy groups are very easy to define. For a based space X, $\pi_n(X)$ is the set of based homotopy classes of based maps $S^n \longrightarrow X$. When $n = 0$, this is the set of path components of X. When $n = 1$, it is the fundamental group of X. When $n \geq 2$, it is an abelian group. While these groups are easy to define, they are fiendishly difficult to compute and the history is riddled with mistakes. In fact they were actually first defined by Cech, in 1932, but he was persuaded not to publish, both because they were abelian, so perhaps uninteresting, and because he had no mechanism for computing them.

Incredibly, already in 1937, Freudenthal proved the Freudenthal suspension theorem. We say that a based space X is n-connected if $\pi_k(X) = 0$ for $k \leq n$.

Theorem 0.1. *If X is $(n-1)$-connected, then the suspension homomorphism*

$$\pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k < 2n - 1$ and a surjection for $k = 2n - 1$.

I must define this homomorphism. For based spaces X and Y, the wedge $X \vee Y$ is the one-point union (identify the two basepoints), and $X \wedge Y$ is the quotient space $X \times Y/X \vee Y$, where we think of $X \vee Y$ as $X \times * \cup * \times Y \subset X \times Y$. The suspension ΣX is $X \wedge S^1$ and $S^k \wedge S^1$ is homeomorphic to S^{k+1}. The suspension homomorphism, which Freudenthal denoted E for Einhängen, sends $f\colon S^k \longrightarrow X$ to

$$f \wedge \mathrm{id}\colon S^{k+1} \cong S^k \wedge S^1 \longrightarrow X \wedge S^1 = \Sigma X.$$

I will not prove anything today, but I will sketch ideas. For a pair of spaces (X, A), so $A \subset X$, and a basepoint $* \in A$, there are relative homotopy group $\pi_k(X, A)$. A quick definition is that

$$\pi_n(X, A) = \pi_{n-1}(P(X; *, A)$$

where $P(X, A)$ is the space of paths $p\colon I \longrightarrow X$ such that $p(0) = *$ and $p(1) \in A$. If $A = *$, this is just a redefinition of $\pi_n(X)$, hence the indexing, but that may not be obvious.

To see it, let ΩX denote the space of loops $S^1 \longrightarrow X$, namely the paths in X that start and end at the basepoint. We may identify S^1 with $I/\partial I$, so these are just based maps $S^1 \longrightarrow X$. We observe that

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

$f\colon \Sigma X \longrightarrow Y$ corresponds to $g\colon X \longrightarrow \Omega Y$ if $f(x, s) = g(x)(s)$. In particular,

$$\pi_n(X) = [S^n, X] \cong [S^{n-1}, \Omega X] \cong \cdots \cong \pi_1(\Omega^{n-1}X) \cong \pi_0(\Omega^n X).$$

This can be viewed as an axiomatization of homotopy groups, and I'll introduce language to say that later on.

Do you all know what an exact sequence of groups is? Please raise your hands if you don't! [Explanation if needed]. There is a long exact sequence

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \pi_{n-1}(X, A) \longrightarrow \cdots$$

The first map is induced by the inclusion $A \longrightarrow X$, the second by the inclusion $(X, *) \longrightarrow (X, A)$, and the third by the end point projection $p_1\colon P(X, A) \longrightarrow A$.

The Hopf bundle $p\colon S^3 \longrightarrow S^2$ is obtained as the orbit space S^3/S^1, where S^3 is the unit sphere in $\mathbb{C}^2$, S^1 is the unit sphere in $\mathbb{C}$ and S^1 acts on S^3 by $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$. The map p is an example of a fiber bundle. For such a bundle $p\colon E \longrightarrow B$ with fiber F, there is a long exact sequence (LES)

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots .$$

The point is that p induces a map of pairs $(E, F) \longrightarrow (B, *)$ that induces an isomorphism of homotopy groups. In the case of the Hopf bundle, this is

$$\cdots \longrightarrow \pi_n(S^1) \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^2) \longrightarrow \pi_{n-1}(S^1) \longrightarrow \cdots .$$

I assume you know that $\pi_1(S^1) = \mathbb{Z}$ and that $\pi_k(S^n) = 0$ for $k < n$. For the latter, any continuous map $S^k \longrightarrow S^n$ can be homotoped to one that misses a point, and then you can deform to a constant map since $S^n - pt$ is contractible. Since S^1 has universal cover $\mathbb{R}$, $\pi_k(S^1) = 0$ for $k \geq 2$. We conclude by the LES that $\pi_2(S^2) \cong \pi_1(S^1)$ and that $\pi_3(S^3) \cong \pi_2(S^2)$. But the suspension theorem gives that $\pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$ for $n \geq 2$. We conclude first that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$ and second that the map p has infinite order. That is an ahistorical derivation of Hopf's 1930 result. It was much harder before Freudenthal came along. In fact, Freudenthal was a student of Hopf, and the story goes that he was the easiest of students: he came to Hopf one day and said "I want to be your student and here is my thesis."

There is a wonderful result, called the homotopy excision theorem, due originally to Blakers and Massey in a series of papers 1951-53. It gives the real explanation of the Freudenthal suspension theorem.

A triad is a triple $(X; A, B)$, where A and B are subspaces of X. We let $C = A \cap B$ and assume that C is non-empty. We say that $(X; A, B)$ is excisive if X is the union of the interiors of A and B. Assume for simplicity that these spaces are all path connected. We say that the pair (A, C) is n-connected if $\pi_k(A, C) = 0$ for $k \leq n$.

Theorem 0.2. *If $(X; A, B)$ is an excisive triad such that (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected, where $m \geq 2$ and $n \geq 1$, then*

$$\pi_k(A, C) \longrightarrow \pi_k(X, B)$$

is an isomorphism if $k < m + n - 2$ and an epimorphism if $k = m + n - 2$.

The cone CX is $X \wedge I$, where I has basepoint 1 (it had basepoint 0 when defining PX). We can form an excisive triad $(\Sigma X; C_+X, C_-X)$ using two copies of CX with intersection X, fattened slightly so as to make the triad excisive. Freudenthal suspension follows via

$$\pi_k(X) \cong \pi_{k+1}(C_+X, X) \longrightarrow \pi_{k+1}(\Sigma X, C_-X) \cong \pi_{k+1}(\Sigma X).$$

The isomorphisms hold by the LES, because the cones are contractible and so have zero homotopy groups except for π_0. The homotopy excision theorem applies to the middle arrow.

The suspension homomorphism E fits into the EHP exact sequence

$$\cdots \longrightarrow \pi_q(S^n) \xrightarrow{E} \pi_{q+1}(S^{n+1}) \xrightarrow{H} \pi_{q+1}(S^{2n+1}) \xrightarrow{P} \pi_{q-1}(S^n) \xrightarrow{E} \pi_q(S^{n+1}) \longrightarrow \cdots$$

when $q \leq 3n - 2$. Here H is a generalization of the Hopf invariant that Hopf introduced. It is due to George Whitehead. P is a map called the Whitehead product. It is due to J.H.C. Whitehead, no relation. Working one prime at a time, there are extensions of this sequence valid for all values of q. This is the starting point for calculations in unstable homotopy theory. I won't say anything about the proof. This workshop will focus on stable homotopy theory, which starts from the stable homotopy groups $\pi_n^s = \pi_{n+q}(S^q)$ for q large.

Wouldn't it be nice to have invariants for which excision holds in all dimensions? Then these invariants would all be stable. In fact, we do. That is what homology is all about. It gives related invariants that are far simpler to compute. I am not going to construct homology, but I do want to explain what it is.

How many of you have not seen categories, functors, and natural transformations? Please do not be shy and raise your hands if you have NOT seen these. Briefly, in 1945, Eilenberg and MacLane introduced the language we now all use when comparing different subjects, and algebraic topology is about comparing topology to algebra. Their coining of the word "category" comes directly from Immanuel Kant's use of the same word, which in turn comes from Aristotle.

Definition 0.3. Category $\mathscr{C}$: class of objects, sets $\mathscr{C}(x, y)$ of morphisms, identity morphisms, composition

$$\mathscr{C}(y, z) \times \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z)$$

associative and unital.

Small versus large. Sets, spaces, groups, abelian groups, etc.

Definition 0.4. Functor $F \colon \mathscr{C} \longrightarrow \mathscr{D}$. Object $Fx \in \mathscr{D}$ for each object $x \in \mathscr{C}$. Functions

$$F \colon \mathscr{C}(x, y) \longrightarrow \mathscr{D}(Fx, Fy)$$
$$F(h \circ g) = F(h) \circ F(g), \quad F(\mathrm{id}_x) = \mathrm{id}_{Fx}$$

Free abelian group functor $Sets \longrightarrow Ab$. Vector space on a given basis $Sets \longrightarrow vectorspaces$. Forgetful functor $Spaces \longrightarrow Sets$.

Covariant versus contravariant.

Definition 0.5. Contravariant functor.

$$F\colon \mathscr{C}(x,y) \longrightarrow \mathscr{D}(Fy,Fx)$$

$$F(h\circ g) = F(g)\circ F(h), \quad F(\mathrm{id}_x) = \mathrm{id}_{Fx}$$

Vector space dual: vector spaces to vector spaces.

Eilenberg liked to say that they defined categories in order to be able to define functors and defined functors in order to be able to define natural transformations.

Definition 0.6. Natural transformation $\eta\colon F \longrightarrow G\colon \mathscr{C} \longrightarrow \mathscr{D}$. For each object x in $\mathscr{C}$, a morphism $\eta_x\colon Fx \longrightarrow Gx$ in $\mathscr{D}$ such that the following diagram commutes for each morphism $f\colon x \longrightarrow y$ in $\mathscr{C}$:

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ Gx & \xrightarrow[Gf]{} & Gy \end{array}$$

η is defined the same way for all x. A map in a category is an isomorphism if it has an inverse. A natural isomorphism is a natural transformation such that each η_x is an isomorphism. The inverses give a natural isomorphism η^{-1}. Let $\mathscr{T}$ denote the category of based spaces (well-based technically) and let $h\mathscr{T}$ denote its homotopy category, which is obtained by identifying two maps if they are homotopic. We are interested in functors defined on $h\mathscr{T}$.

Theorem 0.7. *Up to natural isomorphism, there is a unique sequence of functors π_n from based spaces to based sets if $n = 0$, to groups if $n = 1$ and to abelian groups if $n \geq 2$ together with natural isomorphisms $\lambda\colon \pi_n(X) \cong \pi_{n-1}(\Omega X)$ for $n \geq 1$ and λ_0 from $\pi_0(X)$ to the set of components of X, based at the basepoint component.*

Proof. We are assuming known that $\pi_0(\Omega X)$ has a natural group structure and that $\pi_0(\Omega^2 X)$ has a natural abelian group structure. The natural isomorphisms are required to preserve group structure. Given a second sequence (π'_n, λ_n), we get

$$\pi_n(X) \cong \pi_0(\Omega^n X) \cong \pi'_0(\Omega^n X) \cong \pi'_n(X).$$

□

It is not hard to derive the properties of homotopy groups of pairs form the axioms, but that requires the formal theory of fibration sequences, using that Ω^n preserves them.

What we really want to axiomatize is homology, since there are alternative constructions that look very different. Here it is not hard to define axioms for based spaces and axioms for pairs of spaces with no basepoints in sight and to prove that these axioms are equivalent. Let $\mathscr{U}$ and $\mathscr{U}^2$ denote the categories of unbased spaces and unbased pairs of spaces. We are interested in functors defined on $h\mathscr{U}$ and $h\mathscr{U}^2$.

Definition 0.8. A reduced homology theory $\tilde{H}_*$ on based spaces X consists of functors

$$\tilde{H}_q \colon h\mathscr{T} \longrightarrow \mathscr{A}b$$

together with natural "suspension isomorphisms"

$$\sigma \colon \tilde{H}_q(X) \longrightarrow \tilde{H}_{q+1}(X)$$

that satisfy the following axioms

(1) Exactness. For a nice inclusion (technically a cofibration) $A \longrightarrow X$, the sequence

$$\tilde{H}_q(A) \longrightarrow \tilde{H}_q(X) \longrightarrow \tilde{H}_q(X/A)$$

is exact.

(2) Additivity. For a set of based spaces X_i, the natural map

$$\oplus \tilde{H}_q(X_i) \longrightarrow \tilde{H}_q(\vee X_i)$$

is an isomorphism.

Additivity is an addendum to the Eilenberg-Steenrod axioms which was added by Milnor. It is essential as soon as one deals with spaces that have non-zero homology for arbitrarily large q. We will see a lot of those this week.

There is one more axiom that was introduced later and that distinguishes what is called singular homology from other homology theories. The axioms above implicitly assume that the given spaces are nice; technically, they must have the homotopy types of CW complexes, as Inna Zakharevich will clarify this afternoon. For general spaces one needs another axiom. A map $f \colon X \longrightarrow Y$ is said to be a weak equivalence, or weak homotopy equivalence, if $f_* \colon \pi_q(X, x) \longrightarrow \pi_q(Y, f(x))$ is an isomorphism for every q and every choice of basepoint x. Note that this definition does not assume that f preserves given basepoints.

(3) Weak equivalence. If $f \colon X \longrightarrow Y$ is a weak equivalence, then

$$f_* \colon \tilde{H}_*(X) \longrightarrow \tilde{H}_*(Y)$$

is an isomorphism.

In all of the axioms above, q could be any integer, but there is a last Eilenberg-Steenrod axiom that, among many other things, forces the negative homology groups to be zero.

(4) Dimension. $\tilde{H}_q(S^0) = 0$ for $q \neq 0$ and $\tilde{H}_0(S^0) = \pi$ for some abelian group π.

It is not true that $\tilde{H}$ is unique in general, but when the dimension axiom is satisfied the homology theory $(\tilde{H}, \sigma)$ is unique in the sense that any two are naturally isomorphic. The proof is not nearly as simple as for π_*, and there are constructions that are so different looking that the most efficient way to show they are equivalent is to verify that both satisfy the axioms.

The original Eilenberg-Steenrod axioms were for homology theories on pairs of unbased spaces and look rather different at first sight. They bring excision into play and express our desire to have excision hold in general. We think of a space X as the pair $(X, \emptyset)$.

Definition 0.9. A homology theory H_* on pairs of spaces (X, A) consists of functors

$$H_q \colon h\mathscr{U}^2 \longrightarrow \mathscr{A}b$$

together with natural transformations

$$\partial\colon H_q(X,A) \longrightarrow H_{q-1}(A)$$

that satisfy the following axioms.

(1) Exactness. For any pair (X,A), the following sequence

$$\cdots \longrightarrow H_q(A) \longrightarrow H_q(X) \longrightarrow H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \cdots$$

is exact.

(2) Excision. For an excisive triad $(X;A,B)$, the map

$$H_*(A, A\cap B) \longrightarrow H_*(X,B)$$

is an isomorphism.

(3) Additivity. For a set of pairs (X_i, A_i), the natural map

$$\oplus \tilde{H}_q(X_i, A_i) \longrightarrow \tilde{H}_q(\amalg(X_i, A_i))$$

is an isomorphism.

Again, we need another axiom to deal with general pairs. A map

$$f\colon (X,A) \longrightarrow (Y,B)$$

is a weak equivalence if $f\colon X \longrightarrow Y$ and $f\colon A \longrightarrow B$ are weak equivalences.

(4) Weak equivalence. If $f\colon (X,A) \longrightarrow (Y,B)$ is a weak equivalence, then

$$f_*\colon H_*(X,A) \longrightarrow H_*(Y,B)$$

is an isomorphism.

The original Eilenberg-Steenrod dimension axiom says

(5) Dimension. $H_q(pt) = 0$ for $q \neq 0$ and $H_0(pt) = \pi$ for some abelian group π.

For each Abelian group π, there is a unique homology theory satisfying the dimension axiom.

Given a theory H_* on pairs, we define the corresponding reduced theory on based spaces by

$$\tilde{H}_*(X) = H_*(X,*).$$

The suspension axiom follows from excision and exactness in exactly the same way that the Freudenthal suspension theorem followed from the homotopy excision theorem, and the rest of the axioms are deduced easily.

Given a theory $\tilde{H}_*$ on based spaces, we define $H_*(X) = \tilde{H}_*(X_+)$, where X_+ is the union of X and a disjoint basepoint. General pairs (X,A) are naturally homotopy equivalent to nice pairs. On a nice pair (X,A) we define

$$H_*(X,A) = \tilde{H}_*(X/A).$$

To define

$$\partial\colon H_q(X,A) \longrightarrow H_{q-1}(A)$$

on a nice pair, we notice that $X_+/A_+ = X/A$. We define the unreduced cone CA to be CA_+ and see that if $Y = X_+ \cup_A CA$, then collapsing the contractible space CA to a point gives a homotopy equivalence $Y \longrightarrow X/A$ while collapsing X to a point gives a natural map $Y \longrightarrow \Sigma A_+$. Then ∂ is the composite

$$H_q(X,A) = \tilde{H}_q(X_+/A_+) \xleftarrow{\cong} \tilde{H}_q(Y) \longrightarrow \tilde{H}(\Sigma A_+) \xrightarrow{\sigma^{-1}} \tilde{H}_{q-1}(A_+) = H_{q-1}(A)$$

From here it is not to hard to deduce the weak equivalence, exactness, additivity, and weak equivalence axioms for H. The dimension axiom, if assumed on $\tilde{H}$ is clear on H. It is not so obvious how to prove the excision axiom, and I hope that Inna will explain how to do that this afternoon.

Full details of everything I have said may be found in "A concise course in algebraic topology", [95] on my web page.

http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf