BOTT PERIODICITY

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Bott's 1957 announcement of his periodicity theorem transformed algebraic topology forever. To quote Atiyah in his obituary for Bott "This paper was a bombshell. The results were beautiful, far-reaching and totally unexpected." In briefest form, the theorem says

$$BU \times \mathbb{Z} \simeq \Omega^2(BU \times \mathbb{Z}) \quad \text{and} \quad BO \times \mathbb{Z} \simeq \Omega^8(BO \times \mathbb{Z})$$

When Bott started work on this, $\pi_{10}(U(n))$ was claimed to be $\mathbb{Z}/3\mathbb{Z}$ in one paper and to be cyclic of order a power of two in another, as yet unpublished. He showed the unpublished claim was correct.

I'll begin with a brief sketch of his original proof, restricting to the complex case, next place it in his original context of differential geometry, and then give a sketch proof by the methods of algebraic topology, still restricting to the complex case.

We have

$$S^{2n-1} \cong U(n)/U(n-1)$$

since $U(n)$ acts transitively with isotropy group of a point $U(n-1)$. By the long exact sequence of the bundle

$$U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1}$$

and the fact that $\pi_k(S^q) = 0$ for $k < q$, $\pi_k(U(n)) = \pi_k(U)$ for $k/2 < n$. Let $Gr_n(\mathbb{C}^{2n})$ be the Grassmannian manifold of complex n-planes in $\mathbb{C}^{2n}$ and let $V_n(\mathbb{C}^{2n})$ be the Stiefel manifold of n-frames in $\mathbb{C}^{2n}$. We have a principal $U(n)$-bundle

$$p\colon V_n(\mathbb{C}^{2n}) \longrightarrow Gr_n(\mathbb{C}^{2n}),$$

which can also be seen via

$$V_n(\mathbb{C}^{2n}) \cong U(2n)/U(n) \quad \text{and} \quad Gr_n(\mathbb{C}^{2n}) \cong U(2n)/U(n) \times U(n).$$

By the long exact sequence of homotopy groups, if $1 \le k << n$,

$$\pi_{k-1}(U(n)) \cong \pi_k(Gr_n(\mathbb{C}^{2n})).$$

In a sense I'll sketch, Bott showed that $Gr_n(\mathbb{C}^{2n})$ is the "manifold of minimal geodesics" in $\Omega SU(2n)$. By Morse theory, it follows that $\Omega SU(2n)$ is homotopy equivalent to a space obtained from $Gr_n(\mathbb{C}^{2n})$ by attaching cells of dimension at least $2n+2$. Therefore, for $1 \le k << n$,

$$\pi_k(Gr_n(\mathbb{C}^{2n})) \cong \pi_k(\Omega SU(2n)).$$

Clearly

$$\pi_k(\Omega SU(2n)) = \pi_{k+1}(SU(2n)) = \pi_{k+1}(U(2n)).$$

Therefore

$$\pi_{k-1}(U) \cong \pi_{k+1}(U).$$

and the proof gives a map $BU \longrightarrow \Omega SU$ realizing the isomorphism.

A direct proof just constructing the map and showing that it is an isomorphism on homology is possible and is in the book "More concise algebraic topology". I'll sketch that proof. The argument for BO is analogous but much less intuitive from the point of view of algebraic topology. Two proofs by algebraic topology were given just a few years after Bott's original proof, by Moore in a Cartan seminar and by Dyer and Lashof. Combining the easier parts of the two gives a relatively easy proof by direct homological calculation that will appear in a book on characteristic classes I'm writing now.

However, it is Bott's original proof that really explains why the theorem is true and gives the identification of the intermediate spaces in the real 8-fold periodic case. In fact, his proof is a specialization of a very general theorem using Morse theory to go directly from differential geometry to homotopical conclusions. Its simplest special case gives a Morse theoretic proof of the Freudenthal suspension theorem in the case of spheres.

Bott's paper "The stable homotopy of the classical groups" starts with a general symmetric space M. That means that M is a compact connected Riemann manifold such that for every point $p \in M$, there is an isometry ι_p, called an involution, that fixes p and reverses geodesics through p.

$$\iota_p(p) = p \quad \text{and} \quad d\iota_p = -\operatorname{id}_{T_pM}.$$

This implies that if γ is a geodesic starting at p, then $\iota_p(\gamma(t)) = \gamma(-t)$. It also implies that any two points of M can be connected by a geodesic. Therefore translation along geodesics shows that the action of the isometry group G on M is transitive. If H is the isotropy group at a point $p \in M$, then M can be identified with G/H. Not all homogeneous spaces are symmetric, but one can characterize which ones are.

Let $\nu = (P, Q; h)$ be a triple consisting of two points P and Q on M together with a homotopy class h of curves joining P to Q. Bott thinks of ν as a basepoint on M. He defines M^ν to be the set of all geodesics of minimal length which join P to Q and are in the homotopy class h. Thinking of a geodesic as a path, define a map from the unreduced suspension ΣM^ν to M by sending (s,t) to $s(t)$. For a fixed small $t > 0$, this map is 1 to 1 on M^ν and serves to define a topology on it. Write ν_* for the induced homomorphism $\pi_k(M^\nu) \longrightarrow \pi_{k+1}(M)$.

Let s be an arbitrary geodesic on M from P to Q. The index of s, denoted by λ_s, is the properly counted sum of the conjugate points of P in the interior of s. (To define conjugate points requires defining Jacobi fields; p and q are conjugate along s from p to q if there exists a non-zero Jacobi field J along s which vanishes at two points t. The multiplicity of p and q as conjugate points is equal to the dimension of the vector space consisting of all such Jacobi fields.) Bott writes $|\nu|$ for the first positive integer which occurs as the index of some geodesic from P to Q in the class h. In terms of these notions Bott's principal result is the following theorem.

Theorem 0.1. *Let M be a symmetric space. Then for any base point ν on M, M^ν is again a symmetric space. Further, ν_* is surjective in positive dimensions less than $|\nu|$ and is bijective in positive dimensions less than $|\nu| - 1$. Thus:*

$$\pi_k(M^\nu) = \pi_{k+1}(M) \quad \text{for} \quad 0 < k < |\nu| - 1.$$

Example 0.2. Let $M = S^n$, $n > 2$, and let $\nu = (P, Q)$ consist of two antipodes; because S^n is simply connected the class h is unique. Then M^ν is S^{n-1}, viewed as geodesics through equatorial points, and $\nu*\colon \pi_k(S^{n_1}) \longrightarrow \pi_k(S^n)$ coincides with the usual suspension homomorphism. The integers which occur as indexes of geodesics joining P to Q, form the set $0,\ 2(n-1),\ 4(n-1), \cdots$. Hence $|\nu| = 2(n-1)$, and the theorem yields the Freudenthal suspension theorem. If $\nu = (P, Q)$ with Q not the antipode of P, then M^ν is a single point and $|\nu| = n - 1$. Then the theorem merely implies that $\pi_k(S^n) = 0$ for $0 < k \le n - 2$.

Bott writes "At first glance the evaluation of $|\nu|$ may seem a formidable task". But he explains why it is really routine, at least to him. Since M^ν is another symmetric space, the construction can be iterated, and it leads to ν-sequences. Some are identified, with the relevant $|\nu|$, as

$$U(2n)/U(n) \times U(n) \xrightarrow{2n+2} U(2n)$$

$$0(2n)/O(n) \times O(n) \xrightarrow{n+1} U(2n)/0(2n) \xrightarrow{2n+1} sp(2n)/U(2n) \xrightarrow{4n+2} Sp(2n)$$

$$Sp(2n)/Sp(n) \times Sp(n) \xrightarrow{4n+1} U(4n)/Sp(2n) \xrightarrow{8n-2} SO(8n)/U(4n) \xrightarrow{8n-2} SO(8n)$$

Therefore

$$\pi_k(U) \cong \pi_{k+1}(BU) \cong \pi_{k+2}(U)$$

$$\pi_k(O) \cong \pi_{k+1}(BO) \cong \pi_{k+2}(U/O) \cong \pi_{k+3}(Sp/U) \cong \pi_{k+4}(Sp)$$

$$\pi_k(Sp) \cong \pi_{k+1}(BSp) \cong \pi_{k+2}(U/Sp) \cong \pi_{k+3}(O/U) \cong \pi_{k+4}(O)$$

$$\mathbb{Z}, 0 \qquad \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}.$$

In retrospect we can write down maps, do homological calculations, and reprove the required equivalences of spaces and loop spaces. But the insight from differential geometry is central to knowing what the relevant spaces are. Homology and cohomology are mentioned and some homological consequences of his homotopical arguments are stated, but Bott's work is all Morse theoretic identification of homotopical connectivity estimates of the maps ν_*.

A leisurely treatment of Bott's original proof is in Milnor's book "Morse theory", but he only gets to symmetric spaces on page 109. There are many other proofs of Bott periodicity. In the complex case, the same year as Bott's paper, Atiyah saw that the Riemann-Roch theorem led to a beautifully simple reformulation that gives the form always used nowadays. That form requires that you know what K-theory is. As Dylan told us yesterday, $K(X)$ is the Grothendieck construction on $Vect(X)$. There is a Künneth type product induced by tensor products of vector bundles.

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y).$$

Atiyah's verson of Bott periodicity says this is an isomorphism when X is compact and $Y = S^2$. There is a reduced version

$$\tilde{K}(X) \otimes \tilde{K}(S^2) \xrightarrow{\cong} \tilde{K}(\Sigma^2 X),$$

which amounts to an equivalence $BU \longrightarrow \Omega^2 BU$ on the represented level, which I expect Ben will say more about. He later gave a proof using an analysis of clutching functions to compare trivializations on $X \times D^+$ and $X \times D^-$. Bott himself, in another 1959 paper, one that is almost never referred to and that was solicited by Atiyah, first showed that his original proof leads to the conclusion in the form wanted by Atiyah and Hirzebruch. He also showed that the product reformulation was also true in real K-theory, denoted KO, with S^2 replaced by S^8. Briefly, he showed by explicit comparison that the adjoints of his original equivalences

$$BU \longrightarrow \Omega_0^2 BU \quad \text{and} \quad BO \longrightarrow \Omega^8 BO$$

are homotopic to maps

$$\Sigma^2 BU \longrightarrow BU \quad \text{and} \quad \Sigma^8 BO \longrightarrow BO$$

that represent the isomorphisms given by multiplication by the the respective Bott classes.

There are many other proofs of Bott periodicity. The algebraic source of periodicity is most clearly seen in modules over Clifford algebras, explained in a fundamental paper of Atiyah, Bott, and (posthumously) Shapiro. That algebraic periodicity was turned into a new proof of Bott periodicity by Wood and Karoubi, independently. They used Banach algebras to realize the Clifford module periodicity geometrically. Another quite concrete and homotopically elementary proof was given quite recently by Aguilar and Prieto in the complex case and extended to the real case by Mark Behrens. But I will sketch a calculational proof. It has the virtue of calculating the homology and cohomology of all relevant spaces on the way.

Zhouli mentioned that the dual of an algebra is a coalgebra. In the example of BO that he gave you, BO has a product and a diagonal map, and $H^*(BO)$ has both a product and a coproduct; its product is a map of coalgebras, which a diagram chase shows is the same as saying its coproduct is a map of algebras. Such a structure is called a Hopf algebra. Hopf implicitly introduced the idea around 1940.

The homology Hopf algebras $H_*(BU;\mathbb{Z})$ and $H_*(BO;\mathbb{F}_2)$ enjoy a very special property: they are self-dual, so that they are isomorphic to the cohomology Hopf algebras $H^*(BU;\mathbb{Z})$ and $H^*(BO;\mathbb{F}_2)$. The proof of this basic result is due to my adviser John Moore. It is purely algebraic and explicitly determines the homology Hopf algebras from the cohomology Hopf algebras. Zhouli used this in his sketch proof of Thom's calculation of $\pi_*(TO)$. We focus on the complex case. Homology and cohomology are always to be taken with coefficients in $\mathbb{Z}$. I assume you know that the cohomology Hopf algebra is given by

$$H^*(BU) = P\{c_i \mid i \geq 1\} \quad \text{with} \quad \psi(c_n) = \sum_{i+j=n} c_i \otimes c_j \tag{0.3}$$

Determination of the homology algebra is a purely algebraic problem in dualization and a full proof is in More Concise Section 21.6.

The dual coalgebra of a polynomial algebra $P[x]$ is written $\Gamma[x]$; when $P[x]$ is regarded as a Hopf algebra with x primitive, meaning that

$$\psi(x) = x \otimes 1 + 1 \otimes x,$$

$\Gamma[x]$ is called a divided polynomial Hopf algebra.

Clearly $H^*(BU(1);\mathbb{Z}) = P[c_1]$ is a quotient algebra of $H^*(BU;\mathbb{Z})$. Write

$$H_*(BU(1)) = \Gamma[\gamma_1]$$

It has basis $\{\gamma_i \mid i \geq 0\}$ and coproduct $\psi(\gamma_n) = \sum_{i+j=n} \gamma_i \otimes \gamma_j$, where $\gamma_0 = 1$ and γ_i is dual to c_1^i. The inclusion $BU(1) \longrightarrow BU$ induces an identification of this as a sub coalgebra of $H_*(BU)$. This sub coalgebra freely generates $H_*(BU)$.

Theorem 0.4. *$H_*(BU) = P\{\gamma_i \mid i \geq 1\}$, where $\gamma_i \in H_*(BU(1))$ is dual to c_1^i.*

The self duality of $H^*(BU)$ plays a central role in a quick proof of (complex) Bott periodicity. We describe how that works. The essential point is to prove the following result.

Theorem 0.5. *There is a map $\beta\colon BU \longrightarrow \Omega SU$ of H-spaces which induces an isomorphism on homology.*

It follows from the dual Whitehead theorem that β must be an equivalence. In that rarely cited sequel, Bott wrote down explicit maps coming from the general theory of symmetric spaces. I will write down the key map, but then I will only say what it does, leaving out proofs. Full details are in More Concise Section 21.6.

We take $BU = U/U \times U$ to be the union of the Grassmannians

$$U(2n)/U(n) \times U(n).$$

We let U be the union of the $U(2n)$ and SU be the union of its subgroups $SU(2n)$ of unitary transformations with determinant one.

Abbreviate notation by writing $\mathbb{V}^n$ to be the sum of n copies of $\mathbb{C}^\infty$.

It is convenient to use paths and loops of length π. Taking $0 \leq \theta \leq \pi$, define $\nu(\theta) \in U(\mathbb{V}^2)$ by

$$\nu(\theta)(z', z'') = (e^{i\theta} z', e^{-i\theta} z'').$$

Then $\nu(0)$ is multiplication by 1, $\nu(\pi)$ is multiplication by -1, and $\nu(\theta)^{-1} = \nu(-\theta)$. The genesis in a symmetric isometry might be visible. Define

$$\beta\colon U(\mathbb{V}^2) \longrightarrow \Omega SU(\mathbb{V}^2)$$

by letting

$$\beta(T)(\theta) = [T, \nu(\theta)] = T\nu(\theta)T^{-1}\nu(-\theta)$$

where $T \in U(\mathbb{V}^2)$. Clearly $[T, \nu(\theta)]$ has determinant one and $\beta(T)$ is a loop at the identity element e of the group $SU(\mathbb{V}^2)$. Since $\nu(\theta)$ is a scalar multiplication on each summand $\mathbb{V}$, if $T = T' \times T'' \in U(\mathbb{V}) \times U(\mathbb{V})$, then $\beta(T)(\theta) = e$. Therefore β passes to orbits to give a well-defined map

$$\beta\colon BU = U/U \times U \longrightarrow \Omega SU.$$

To define the H-space structure on BU, choose a linear isometric isomorphism $\xi\colon \mathbb{V}^2 \longrightarrow \mathbb{V}$ and let the product T_1T_2 be the composite

$$\mathbb{V}^2 \xrightarrow{(\xi^{-1})^2} \mathbb{V}^4 \xrightarrow{T_1 \oplus T_2} \mathbb{V}^4 \xrightarrow{\gamma} \mathbb{V}^4 \xrightarrow{\xi^2} \mathbb{V}^2,$$

where γ interchanges the middle two summands. Up to homotopy, the product is independent of the choice of ξ. The H-space structure we use on ΩSU is the pointwise product, $(\omega_1\omega_2)(\theta) = \omega_1(\theta)\omega_2\theta$. It is an exercise to verify that β is an H-map.[1]

Let $\{e_i'\}$ and $\{e_i''\}$ denote the standard bases of two copies of $\mathbb{V}$ and let $\mathbb{C}_1^n$ and $\mathbb{C}_2^n$ be spanned by the first n vectors in each of these bases. Let

$$j\colon U(\mathbb{C}_1^n \oplus \mathbb{C}_2^1) \longrightarrow U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n)$$

be the inclusion. Restrictions of β give a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}P^n = U(\mathbb{C}_1^n \oplus \mathbb{C}_2^1)/U(\mathbb{C}_1^n)\times U(\mathbb{C}_2^1) & \xrightarrow{\ \alpha\ } & \Omega SU(\mathbb{C}_1^n \oplus \mathbb{C}_2^1) = \Omega SU(n+1) \\
\Big\downarrow j & & \Big\downarrow \Omega j \\
U(2n)/U(n)\times U(n) = U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n)/U(\mathbb{C}_1^n)\times U(\mathbb{C}_2^n) & \xrightarrow{\ \beta\ } & \Omega SU(\mathbb{C}_1^n \oplus \mathbb{C}_2^n) = \Omega SU(2n).
\end{array}$$

Passing to colimits over n, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}P^\infty & \xrightarrow{\ \alpha\ } & \Omega SU \\
\Big\downarrow j & & \simeq \Big\downarrow \Omega j \\
BU & \xrightarrow{\ \beta\ } & \Omega SU.
\end{array}$$

The right arrow is an equivalence, as we see from a quick check of homology or homotopy groups.

We claim that $H_*(\Omega SU)$ is a polynomial algebra on generators δ_i of degree $2i$, $i \geq 1$, and that $\alpha_*\colon H_*(\mathbb{C}P^\infty) \longrightarrow H_*(\Omega SU)$ is a monomorphism onto the free abelian group spanned by suitably chosen polynomial generators δ_i. The algebra proving the self-duality implies the topological statement that

$$j_*\colon H_*(\mathbb{C}P^\infty) \longrightarrow H_*(BU)$$

is a monomorphism onto the free abelian group generated by a set $\{\gamma_i\}$ of polynomial generators for $H_*(BU)$, hence the claim will complete the proof.

Think of S^1 as the quotient of $[0,\pi]$ obtained by setting $0 = \pi$. Let

$$i\colon U(\mathbb{C}_1^{n-1} \oplus \mathbb{C}_2^1) \longrightarrow U(\mathbb{C}_1^n \oplus \mathbb{C}_2^1)$$

be the inclusion. It induces a map $i\colon \mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n$ that leads to the left diagram below, and the right diagram is its adjoint.

(0.6)
$$\begin{array}{ccc}
\mathbb{C}P^{n-1} & \xrightarrow{\ \alpha\ } & \Omega SU(n) \\
\Big\downarrow i & & \Big\downarrow \Omega i \\
\mathbb{C}P^n & \xrightarrow{\ \alpha\ } & \Omega SU(n+1) \\
\Big\downarrow \rho & & \Big\downarrow \Omega\pi \\
S^{2n} & \xrightarrow[\ h\ ]{} & \Omega S^{2n+1}
\end{array}
\qquad
\begin{array}{ccc}
\Sigma\mathbb{C}P^{n-1} & \xrightarrow{\ \hat{\alpha}\ } & SU(n) \\
\Big\downarrow \Sigma i & & \Big\downarrow i \\
\Sigma\mathbb{C}P^n & \xrightarrow{\ \hat{\alpha}\ } & SU(n+1) \\
\Big\downarrow \Sigma\rho & & \Big\downarrow \pi \\
\Sigma S^{2n} & \xrightarrow[\ \hat{h}\ ]{} & S^{2n+1}
\end{array}$$

Here $\rho\colon \mathbb{C}P^n \longrightarrow \mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}$ is the quotient map and $\pi(T) = T(e_n')$.

[1] This is also part of the 1970's infinite loop space story; details generalizing these H-space structures and maps to the context of actions by an E_∞ operad may be found at the beginning of E_∞ ring spaces and E_∞ ring spectra.

Lemma 0.7. *The composite $\Omega\pi \circ \alpha \circ i$ is trivial, so that $\Omega\pi \circ \alpha$ factors as the composite $h\rho$ for a map h. Moreover, the adjoint $\hat{h}$ of h is a homeomorphism.*

Proof. Details of the first statement are elementary and it is also elementary to check that $\hat{h}$ is injective. Details are in More Concise. Then the image of $\hat{h}$ is open by invariance of domain and closed by the compactness of ΣS^{2n}, hence is all of S^{2n+1} since S^{2n+1} is connected. □

Armed with this elementary geometry, we return to homology. The rightmost column in the second diagram is a fibration, and it is standard to use it to compute $H_*(\Omega SU(n+1))$ by induction on n. We have $SU(2) \cong S^3$, and, inductively the cohomology Serre spectral sequence of this fibration satisfies $E_2 = E_\infty$, leading to a proof that

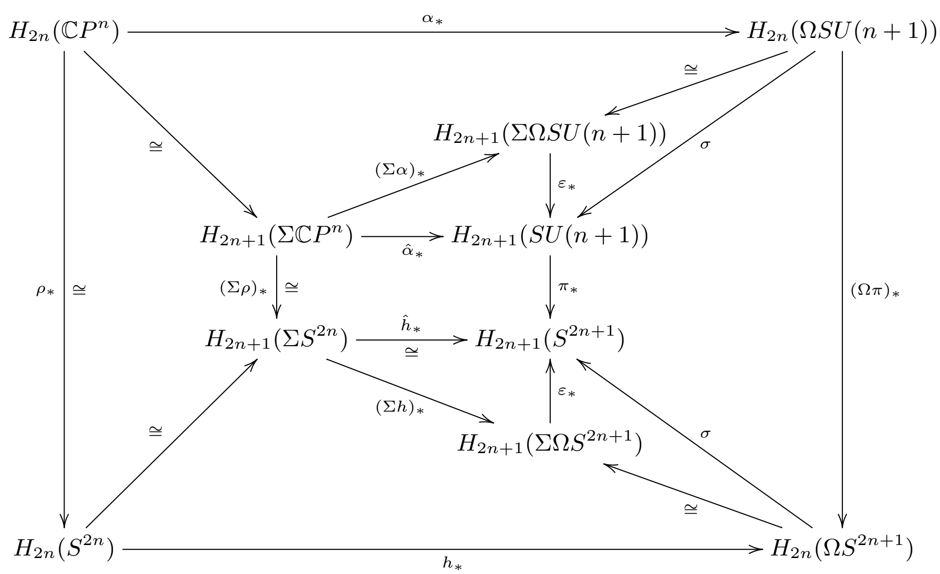
$$H_*(SU(n+1)) = E\{y_{2i+1} | 1 \leq i \leq n\}$$

as a Hopf algebra, where y_{2i+1} has degree $2i+1$ and $\pi_*(y_{2n+1})$ is a generator of $H_{2n+1}(S^{2n+1})$.

Using the homology Serre spectral sequence of the path space fibration over $SU(n+1)$, we conclude that

$$H_*(\Omega SU(n+1)) \cong P\{\delta_i | 1 \leq i \leq n\},$$

where δ_i has degree $2i$. The rest is a mopping up exercise detailed in More Concise. The key diagram is included in the notes of this talk, which are now on line.



Here ε denotes the evaluation map of the (Σ, Ω) adjunction, and the suspension σ is defined to be the composite of ε_* and the suspension isomorphism. The algebra generator δ_n maps to a fundamental class under $\pi_*\sigma$. By the diagram, so does the basis element $x_{2n} \in H_{2n}(\mathbb{C}P^n)$. Therefore, modulo decomposable elements which are annihilated by σ, $\alpha_*(x_{2i}) = \delta_i$ as claimed.