Problems for the UChicago ATSS-Wednesday

July 27, 2016

Problem 1. Show that, if we allow non-compact manifolds in the definition of cobordism, then every manifold is cobordant to the empty set.

Problem 2. Prove Pontryagin's theorem for framed 0-dimensional manifolds. (Yes the problem is long, but if you do it, I promise you will come out the other end understanding the proof of Pontryagin's theorem.)

(a) Show that the Thom space of the normal bundle of a 0-dimensional manifold $X \subset \mathbb{R}^N$ is equivalent to a wedge of copies of S^N. Better, show that a choice of framing on the normal bundle yields an explicit equivalence.

(b) So the framing induces an isomorphism $\nu^+ \xrightarrow{\cong} \bigvee_{x \in X} S^N$, show that the composite of the collapse map with this isomorphism

$$p_X : S^N \longrightarrow \nu^+ \longrightarrow \bigvee_{x \in X} S^N$$

induces on $H_N(-, \mathbb{Z})$ the map $\mathbb{Z} \longrightarrow \bigoplus \mathbb{Z}$ given by $1 \mapsto (\pm 1, ..., \pm 1)$ where the signs are determined by the choice of framing.

(c) Finally, show that the map

$$S^N \longrightarrow S^N$$

corresponding to X in the Pontryagin-Thom theorem is just the composite

$$S^N \xrightarrow{p_X} \bigvee_{x \in X} S^N \xrightarrow{\nabla} S^N$$

so that the degree is just the sum of the integers ± 1 from the previous part.

(d) Show that a framed cobordism (W, X, X') induces a homotopy between the two resulting maps $p_X, p_{X'} : S^N \longrightarrow S^N$. So we have a well defined map:

$$\Theta : \Omega_0^{fr} \longrightarrow \pi_N(S^N), \quad N \geq 1.$$

(e) Show that every map $S^N \to S^N$ is homotopic to one obtained in this way. (You may use the fact that every map is homotopic to a smooth map, and that every smooth map has a regular value.) So Θ is surjective.

(f) Finally, show that Θ is injective. (Hint: the argument is just like surjectivity, but this time you're not trying to realize a map as coming from a manifold, you're trying to realize a homotopy as coming from a manifold with boundary.)

(g) Conclude that

$$\Omega_0^{fr} \cong \pi_N(S^N), \quad N \geq 1.$$

(h) Compute the left hand side 'geometrically'. Deduce that $\pi_n(S^n) = \mathbb{Z}$ and the degree can be compute by examining the preimage of a regular value and counting signed points.

Problem 3. Let W be a compact manifold with boundary M with $i : M \hookrightarrow W$ the inclusion. Show that $i_*[M] = 0$.

Problem 4. Let E be a vector bundle over B and E' be a vector bundle over B'. Denote by $E \times E'$ the exterior product of vector bundles, over $B \times B'$. Verify: $T(E \times E') \cong T(E) \wedge T(E')$, and that for any pullback square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

where $E \to B$ is a vector bundle produces a map $T(E') \to T(E)$.

Problem 5. Fix a space X and consider the set of pairs $\{(M, f)\}$ where M is a compact n-manifold and $f : M \longrightarrow X$ is a map. Let $MO_n(X)$ denote the quotient of this set by the equivalence relation of *bordism*: $(M, f) \sim (N, g)$ if there exists a compact $(n+1)$-dimensional manifold with boundary, W, and a map $H : W \longrightarrow X$ such that $\partial W \cong M \amalg N$ and we have equalities $H|_M = f$, $H|_N = g$.

(a) Prove that composition turns $MO_n(-)$ into a functor **Spaces** $\longrightarrow$ **Set**.

(b) Show that $MO_n(X)$ is an abelian group under disjoint union.

We denote by $MO_*(X)$ the graded ring $\bigoplus_{n \geq 0} MO_n(X)$.

Problem 6. Prove that $MO_*(-)$ is a homotopy invariant functor.

Problem 7. Check the surjectivity of the Pontrjagin–Thom construction. In other words, show that for any map $f : S^{n+k} \to T(\gamma_{m,k})$, if we first construct a manifold M using the inverse image construction and then apply the Pontrjagin–Thom construction, the map produced is homotopic to f. (Hint: once the manifold M is constructed it comes with a chosen embedding into $\mathbb{R}^{n+k}$. Now consider how you want f to relate to the normal bundle of M.)

Problem 8. Check the injectivity of the Pontrjagin–Thom construction. In particular, show that if the Pontrjagin–Thom construction produces a null-homotopic map on M then M is the boundary of a manifold. (Hint: if the transversality construction is applied to a map $S^{n+k} \to T(\gamma_k)$ which does not hit the zero section, what is the result? Now interpret what this means for a map $S^{n+k} \times I \to T(\gamma_k)$.)

Problem 9. Show that the Klein bottle is a boundary.

Problem 10. Show that every 3-manifold is a boundary.

Problem 11. Construct a 4-manifold whose boundary is $\mathbb{R}P^3$.

Problem 12. Show that every 4-manifold is cobordant to one of the four:

$$\varnothing, \quad \mathbb{R}P^2 \times \mathbb{R}P^2, \quad \mathbb{R}P^4, \quad (\mathbb{R}P^2 \times \mathbb{R}P^2) \amalg \mathbb{R}P^4$$

Problem 13. Using the previous problem's calculation, which of these is cobordant to $\mathbb{C}P^2$? How about $\mathbb{C}P^2 \# \mathbb{C}P^2$ (the connect sum)? $S^2 \times S^2$?

Problem 14. Show by an explicit construction that $SO(n)$ and $SU(n)$ are boundaries.

Problem 15. Are all Lie groups boundaries?

Problem 16. Use the results from yesterday's problem session to show that $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$. More explicitly, show that if we think of S^2 as $\mathbb{C}P^1$ then $K^0(S^2)$ is generated by the tautological line bundle and the trivial bundle. (Actually, to appease algebraic geometers, we usually take the generator to be the *dual* of the tautological line bundle...)