# Problems for the UChicago ATSS-Tuesday

July 26, 2016

**Problem 1.** Let Vect$(X)$ denote the monoid of vector bundles on $X$ under direct sum. Show that rank defines a natural transformation:

$$\mathrm{rk} : \mathrm{Vect}(-) \longrightarrow \Gamma(-, \underline{\mathbb{Z}}).$$

(Recall that $\Gamma(X, \underline{\mathbb{Z}})$ denotes the group of locally constant functions from $X$ to $\mathbb{Z}$.)

**Problem 2.** Show that there is exactly one nontrivial, real vector bundle of rank 1 on $S^1$, up to isomorphism.

**Problem 3.** Suppose given a map $f : S^1 \to GL_n$. Let $U_0$ and $U_1$ denote $S^2 - \{(0, 0, -1)\}$ and $S^2 - \{(0, 0, 1)\}$, respectively. Define $E_i := U_i \times \mathbb{R}^n$. Identify $U_0 \cap U_1$ with $S^1 \times I$ in your favorite way. Then define $\phi_f : E_0|_{S^1 \times I} \to E_1|_{S^1 \times I}$ by

$$\phi_f((z, t, v)) = (z, t, f(z) \cdot v).$$

(a) Show that $E_f := (E_0 \amalg E_1) / (e_0 \sim \phi_f(e_0))$ defines a vector bundle over $S^2$.

(b) Show that a homotopy between maps $f$ and $g$ yields an isomorphism of vector bundles $E_f \cong E_g$ so that we get a map $\pi_1 GL_n \longrightarrow \mathrm{Vect}_n(S^2)$.

(c) Show that this is an bijection. (You may assume the result that vector bundles over a contractible space are trivial.)

(d) Compute the group of real and complex rank 1 bundles on $S^2$.

(e) Bonus: Identify the tangent bundle (which has rank *one* as a complex line bundle) in terms of this classification. Hint: What is the derivative of $z^{-1}$?

**Problem 4.** Show that cobordism is an equivalence relation on compact $n$-manifolds.

**Problem 5.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Prove that the tangent bundle $TG$ is trivializable.

**Problem 6.** Are the Möbius bundle and tangent bundles on $S^1$ isomorphic? Prove your answer. Are the Möbius bundles and the tautological bundle on $S^1$ isomorphic? Prove your answer.

**Problem 7.** Show that the unit sphere $S^n$ admits a nonvanishing vector field if $n$ is odd. Show that the normal bundle for the usual embedding into $\mathbb{R}^{n+1}$ is trivial for all $n$.

**Problem 8.** Show that if $S^n$ admits a nonvanishing vector field, then the identity map $S^n \to S^n$ is homotopic to the antipodal map $x \mapsto -x$. For n even, show that the antipodal map induces the map $-1$ on top homology. Conclude that the tangent bundle of $S^n$ is not trivializable when $n$ is even.

**Problem 9.** Let $f \colon M \to N$ be a map of smooth manifolds. Assume that the Jacobian $Df \colon TM \to TN$ is surjective. Show that there exists a sub-bundle $\ker(f) \subset TM$ (with fiber at $x$ equal to $\ker(Df_x)$), and that $TM \cong \ker(f) \oplus f^*TN$.

**Problem 10.** Show that the set of real (or complex) vector bundles of rank 1 over a space forms a group with the product operation given by tensor product. Extra: Show that all elements of this group have order at most two (hint: show that every real line bundle admits a Euclidean metric).

**Problem 11.** Let $E \to X$ be a vector bundle of rank $n$ and let $\pi\colon Fl(E) \to X$ be the flag bundle, i.e. the bundle whose fiber at $x \in X$ consists of all sequences of linear subspaces $L_1 \subset \cdots \subset L_n = E_x$ such that $\dim(L_i) + 1 = \dim(L_{i+1})$. Show that $\pi^* E \to Fl(E)$ is isomorphic to a direct sum of line bundles.

**Problem 12.** Show that $\mathrm{Gr}_n(\mathbb{C}^{n+k}) \cong \mathrm{Gr}_k(\mathbb{C}^{n+k})$ (and similarly for real Grassmannians. In particular, conclude that $\mathbb{CP}^n \cong \mathrm{Gr}_n(\mathbb{C}^{n+1})$.

**Problem 13.** Show that the assignment $E \mapsto E \times \mathbb{C}^1$ for complex vector bundles $E$ determines a map

$$\mathrm{Gr}_n(\mathbb{C}^k) \to \mathrm{Gr}_{n+1}(\mathbb{C}^{k+1}).$$

In particular, show that the map $\mathbb{CP}^n \to \mathbb{CP}^{n+1}$ induces an isomorphism on cohomology up to degree $2n$.

**Problem 14.** Prove that $w(\mathbb{R}P^n) = 1$ if and only if $n + 1 = 2^k$ for some $k$. Conclude that, if $\mathbb{R}P^n$ has a trivial tangent bundle, then $n = 2^k - 1$.

**Problem 15.** (a) Compute $w(\mathbb{R}P^9)$.

(b) Find a polynomial $p(w_1) \in \mathbb{Z}/2[w_1]$ such that

$$w(\mathbb{R}P^9)p(w_1) = 1.$$

(c) Suppose that $\mathbb{R}P^9$ can be immersed in $\mathbb{R}^{9+k}$. Give a lower bound for $k$. (Hint: Use the Whitney Product Theorem).

(d) Do the same exercise, replacing $n$ by $2^r$. Compare what you find with the theorem of Whitney which states that for $n > 1$, then any $n$–dimensional manifold can be immersed in $\mathbb{R}^{2n-1}$.

**Problem 16.** Prove that $Sq^r$ is indecomposable if and only if $r = 2^k$. (Hint: Compute its effect on $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2)$ for one direction and use the Adem relations for the other).

**Problem 17.** (a) Compute $\widetilde{H}^*(\mathbb{R}P^2 \wedge \mathbb{R}P^2, \mathbb{Z}/2)$ and $\widetilde{H}^*(\Sigma\mathbb{R}P^2 \vee \Sigma^2\mathbb{R}P^2, \mathbb{Z}/2)$ together with all of the Steenrod operations. Conclude that $\mathbb{R}P^2 \wedge \mathbb{R}P^2 \not\simeq \Sigma\mathbb{R}P^2 \vee \Sigma^2\mathbb{R}P^2$.

(b) Show that there is a ring isomorphism $H^*(\Sigma(\mathbb{R}P^2 \wedge \mathbb{R}P^2), \mathbb{Z}/2) \cong H^*(\Sigma^2\mathbb{R}P^2 \vee \Sigma^3\mathbb{R}P^2, \mathbb{Z}/2)$, but that, nonetheless, $\Sigma(\mathbb{R}P^2 \wedge \mathbb{R}P^2) \not\simeq \Sigma^2\mathbb{R}P^2 \vee \Sigma^3\mathbb{R}P^2$.