Problems for the UChicago ATSS-Monday

July 25, 2016

1 Main Problems

Problem 1. Let (X, x_0) be a pointed space. Let $CX = \frac{X\times I}{X\times\{0\}}$ denote the cone on X. Give a (pointed) homeomorphism $S^1 \wedge X \cong \frac{CX}{X\cup(\{x_0\}\times I)}$.

Problem 2. Prove that if $\alpha \in H^q(X, R)$ has odd degree q, then $2\alpha^2 = 0$ in $H^{2q}(X, R)$.

Problem 3. Let S^n denote the one-point compactification of $\mathbb{R}^n$. Explicitly, this is the set $\mathbb{R}^n \amalg \{\infty\}$ topologized so that the complements of compact sets in $\mathbb{R}^n$ are open. (You can think of these complements as 'neighborhoods of ∞').

(a) This is not an abuse of notation, i.e. S^n is homeomorphic to the subspace of $\mathbb{R}^{n+1}$ consisting of unit vectors. You can try to show this using stereographic projection if you want, or just skip to the next part.

(b) Write down a map

$$S^n \wedge S^m \longrightarrow S^{n+m}$$

and show it's a homeomorphism. (This should be easier to do with the one-point compactification definition than with the unit vector definition.)

(c) More generally, if V and W are vector spaces, we can define the one-point compactifications S^V and S^W in the same way. Show that, in this language, we can rewrite (b) as a natural homeomorphism

$$S^V \wedge S^W \cong S^{V\oplus W}.$$

Problem 4. Let A be a set with two associative, unital binary operations: $\boxtimes, \star : A \times A \to A$. Suppose they distribute past each other in the following way:

$$(a \star a') \boxtimes (b \star b') = (a \boxtimes b) \star (a' \boxtimes b'),$$

and suppose their units coincide. Then show that $\boxtimes = \star$ and the operation is commutative. (This is called the *Eckmann-Hilton argument*.)

Problem 5. Use the previous problem to show that $\pi_2(X)$ is an abelian group. (Hint: Think of $\pi_2(X)$ as $[(I^2, \partial I^2), (X, x_0)]$ and build two ways of concatenating maps from the square: one via stacking vertically and the other horizontally. Also, draw pictures.)

Problem 6. (a) Describe $H^*(\mathbb{R}P^n, \mathbb{Z})$ as a ring.

(b) Use Poincaré duality to prove that $H^*(\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}[c]/c^{n+1}$ for $c \in H^2$. (Hint: To compute $\mathbb{C}P^n$ as a graded abelian group, recall that it has a cell structure with exactly one cell e_{2n} in each even dimension.)

(c) What about $H^*(\mathbb{C}P^n, \mathbb{Z}/2)$?

Problem 7. (a) Describe $H^*(S^1 \times X)$ in terms of $H^*(X)$.

(b) Prove that all cup products in $\widetilde{H}^*(\Sigma X)$ are zero.

(c) Show that, for pointed spaces X and Y, if $x \in H^*(X)$ and $y \in H^*(Y)$, then $xy = 0 \in H^*(X \vee Y)$. Hint: Use naturality.

Problem 8. Compute the cohomology ring of the n–torus $\mathbb{T}^n = (S^1)^{\times n}$ with coefficients in $\mathbb{Z}$.

Problem 9. Show that $\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$ even though these spaces have the same cohomology groups.

Problem 10. (a) Show that a *category* with one object is the same data as a set M with an associative, unital multiplication $M \times M \to M$ (i.e. a monoid.)

(b) Show that a *functor* between one-object categories is the same as a homomorphism of monoids.

(c) Let A and B be two one-object categories, $F, G : A \to B$ two functors between them. Let M, N, f, and g be the associated monoids and homomorphisms. Show that a *natural transformation* $\eta : F \to G$ is the same data as an element $n \in N$ such that, for all $m \in M$, $n \cdot f(m) = g(m) \cdot n$.

(d) If we think of a group G as a one-object category, then the set of natural isomorphisms from the identity functor to itself has a name that you know already from group theory. What is it?

Problem 11. Use the proof outlined in class to check that reduced homology is homotopy invariant.

Problem 12. We can define unreduced homology by setting $H_n(X)$ to be $H_n(X_+)$, X with a disjoint basepoint added. We define relative homology $H_n(X, A)$ for a CW pair (X, A) to be the homology of the chain complex $C_n(X)/C_n(A)$.

(a) Use the snake lemma to show that there is a long exact sequence in homology

$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}.$$

(b) Assume that for a CW pair (X, A) and any subset Z of A such that the closure of Z is contained in the interior of A we have $H_n(X, A) = H_n(X - Z, A - Z)$. Prove that for a CW pair (X, A), the reduced $H_n(X/A)$ is the same as $H_n(X, A)$.

Problem 13. Compute $H_m(X; \mathbb{Z}/p)$ for all primes p and X being the torus, S^n, $\mathbb{R}P^n$ and $\mathbb{C}P^n$. Do the same for the lens space $L(p', q)$. This is the quotient of S^3 by the action of $\mathbb{Z}/p'$, where we consider it as the unit sphere in $\mathbb{C}^2$ and 1 in $\mathbb{Z}/p'$ acts by sending (w, z) to $(e^{2\pi i/p'} w, e^{2\pi i q/p'} z)$.

For ease of reference, here are the Adem relations:

$$Sq^a Sq^b = \sum_{c \geq 0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c, \quad a < 2b$$

(Remember that the Steenrod algebra is an algebra over $\mathbb{F}_2$, so the binomial coefficients are taken mod 2).

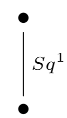
Problem 14. Use the Adem relations to verify these formulae:

(a) $Sq^1 Sq^1 = 0$,

(b) $Sq^1 Sq^{2n} = Sq^{2n+1}$,

(c) $Sq^1 Sq^2 Sq^1 = Sq^2 Sq^2$

Problem 15. Here is a picture of the subalgebra of the Steenrod algebra generated by Sq^1:



The bottom dot is $1 = Sq^0$, and the top dot is $Sq^1 \cdot 1 = Sq^1$. It is one step higher than the first dot because it is an operation of degree 1. The line indicates that we multiplied by Sq^1 to get from the bottom dot to the top dot. Draw a picture of the subalgebra generated by Sq^1 and Sq^2; it is 8 dimensional over $\mathbb{F}_2$. Bonus points if your picture is pretty.

Problem 16. For $x \in H^n(X)$, write

$$Sq(x) = Sq^0(x) + Sq^1(x) + \ldots + Sq^n(x).$$

Prove that the Cartan formula implies that

$$Sq(xy) = Sq(x)Sq(y).$$

Use this to give a simple formula for $Sq(c^i)$, where $c \in H^2(\mathbb{C}P^n, \mathbb{Z}/2)$ is a generator.

2 Extra problems

Problem 17. Show that the map $S^{p+q} = S^p \wedge S^q \to S^q \wedge S^p = S^{p+q}$ has degree $(-1)^{pq}$

Problem 18. Let X be a pointed space. There is a natural map $\epsilon : X \longrightarrow \Omega\Sigma X$ which sends a point $x \in X$ to the loop $I \times \{x\} \subset I \times X \to \Sigma X$. Show that the Freudenthal suspension theorem is equivalent to the statement that, if X has $\pi_j X = 0$ for $j \leq k-1$, then

$$\epsilon_* : \pi_n X \longrightarrow \pi_n \Omega\Sigma X$$

is an isomorphism for $n \leq 2k-2$ and a surjection for $n = 2k-1$.

Problem 19. Let X be a connected, smooth 1-manifold. Convince yourself (but try to be rigorous) that X is diffeomorphic to either $\mathbb{R}$ or S^1. So there are no exotic circles.

Problem 20. Given a space X let $\Gamma(X, \underline{\mathbb{Z}})$ denote the ring of locally constant functions from X to $\mathbb{Z}$. Show that this assignment gives a functor $\mathbf{hSpaces}^{\mathrm{op}} \longrightarrow \mathbf{CRing}$ where **hSpaces** denotes the homotopy category of spaces and **Cring** denotes the category of commutative rings. Show that, when restricted to spaces where path components are the same as ordinary components, there is a natural isomorphism $\Gamma(X, \underline{\mathbb{Z}}) \cong H^0(X, \mathbb{Z})$. Hint: The only thing that makes this problem difficult is words. You can do it.

Problem 21. If X is a pointed space, let $\widetilde{\Gamma}(X, \underline{\mathbb{Z}})$ denote the abelian group of locally constant functions that send the basepoint to 0. Then show that, for *any* subspace $A \subset B$, the following sequence is exact:

$$0 \to \widetilde{\Gamma}(B/A, \underline{\mathbb{Z}}) \to \Gamma(B, \underline{\mathbb{Z}}) \to \Gamma(A, \underline{\mathbb{Z}}).$$

Problem 22. Let's try to build a cohomology theory on the category of sets. We'll say that a *cohomology theory on the category of sets* is a functor $F : \mathbf{Set}^{op} \to \mathbf{Ab}$ satisfying the following properties: (i) $F(*) = \mathbb{Z}$, (ii) If $A \subset X$ then the sequence $0 \to \widetilde{F}(A/X) \to F(X) \to F(A)$ is exact, (iii) if $X = \coprod X_\alpha$ for some indexing set then the natural map $F(X) \to \prod F(X_\alpha)$ is an isomorphism.

Show that:

(a) Axiom (ii) is redundant and the last map in the exact sequence is automatically surjective.

(b) There is only one such functor F up to natural isomorphism.

(c) This functor is given by $X \mapsto \mathrm{Hom}_{\mathbf{Set}}(X, A)$ for some fixed object A. What is it?

Problem 23. Prove that Sq^{2^n} is indecomposable. (Compute its effect on $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2)$).