VISCOSITY SOLUTIONS OF ELLIPTIC EQUATIONS

LUIS SILVESTRE

These are the notes from the summer course given in the Second Chicago Summer School In Analysis, in June 2015.

We introduce the notion of viscosity solutions for elliptic equations, focusing on second order equations with also a section on nonlocal equations at the end. We do not intend to present a comprehensive development of the subject nor to prove the main theorems in their most general form. Instead, the simplicity of exposition is a priority. References are given for further study of the subject. Regularity results are stated without proofs.

Contents

1. Introduction

Some partial differential equations do not have a solution in the classical sense. There is no function, which is smooth enough to compute its derivatives of the required order and plug them to verify the equation. In some cases, they do have a solution in a generalized sense. These generalized solutions are functions which do not possess the required regularity and their derivatives may not exist. The notion of viscosity solutions allows us to make sense of how a non smooth continuous function may solve an elliptic PDE.

The standard reference for the main results in the theory of viscosity solutions is the *User's guide* [4].

1.1. **Acknowledgments.** We thank Trevor Leslie, Joey Zou, Tobias Shin and Junichi Koganemaru for spotting some typos in these notes.

2. A review of the Laplace equation

Elliptic equations are, by definition, those that share some common properties with the Laplace equation. We start by a quick review about the properties of solutions to the Laplace equation: the harmonic functions.

In this section, we consider a functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded open set. The Dirichlet problem for the Laplace equation is

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

The boundary condition f should be a given function. It is well known that the Dirichlet problem is solvable if f is continuous and Ω satisfies the exterior ball condition.

2.1. **Interpretation of the Laplace equation: random walks.** Recall that the Laplacian is given by the following infinitesimal expression.

$$\Delta u(x) = \lim_{r \to 0} \frac{c}{r^{n+2}} \int_{B_r(x)} u(y) - u(x) \, \mathrm{d}y.$$

Let us consider the following random walk. Let r be an arbitrary small parameter. We start at a point $X_0 = x \in \Omega$. At each step, we move from X_k to any point $X_{k+1} \in B_r(X_k)$. We choose these points with a uniform distribution in the ball of radius r centered at X_k. Whenever the segment from X_k to X_{k+1} crosses $\partial\Omega$, we stop at that point in the boundary. Our purpose is to compute the expected value of the given function f at that point on the boundary $\partial\Omega$.

This expected value is a function of the initial point $x \in \Omega$ and $r > 0$. We call this function $u_r(x)$. Of course, if we started with a point $x \in \partial\Omega$, then the process would not move and we would get $u_r(x) = f(x)$. If we start at any point x in the interior of Ω, the value of $u_r(x) = u^k(X_0)$ equals the average of the values of u_r at the point we land after the first step $u_r(X_1)$. Thus, provided that r is smaller than the distance between x and $\partial\Omega$,

$$u_r(x) = \frac{1}{|B_r|} \int_{B_r(x)} u_r(y) \, \mathrm{d}y.$$

Therefore, we get

$$\int_{B_r(x)} u_r(y) - u_r(x) \, \mathrm{d}y = 0.$$

We observe that the function u_r, for this random walk, will converge to the solution to the Laplace equation (2.1).

As you can imagine, there are several variants of this problem depending on the rules for the random walk and the stopping condition. Consequently, there are many elliptic equations which arise from problems in probability.

2.2. **Some basic properties.** One of the most fundamental properties of harmonic functions is the mean value property.

Theorem 2.1 (Mean value property). *If $\Delta u = 0$ in Ω and $B_r(x) \subset \Omega$, then*

$$u(x) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S(y) = \frac{1}{|B_r|} \int_{B_r(x)} u(y) \, \mathrm{d}y.$$

The mean value property is exclusive of harmonic functions. It will not hold for other elliptic equations.

There are important properties of harmonic functions which follow from the mean value property. One of them is about their regularity.

Theorem 2.2. *Every harmonic function is C^∞. Moreover, for any positive integer k there is a constant C (depending only on k and dimension) such that if $\Delta u = 0$ in B_r, then*

$$\max_{B_{r/2}(x)} |D^k u| \leq \frac{C}{r^k} \max_{B_r} |u|.$$

If we replace the condition $\Delta u = 0$ for the inequality $\Delta u \geq 0$, the mean value property turns also into an inequality.

Proposition 2.3. *The function $u : \Omega \to \mathbb{R}$ satisfies $\Delta u \geq 0$ in Ω if every time $B_r(x) \subset \Omega$, then*

$$u(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y) \, \mathrm{d}y.$$

Moreover, the right hand side is monotone increasing in r, for as long as $B_r(x) \subset \Omega$.

Another consequence of the mean value property is the maximum principle.

Theorem 2.4 (Maximum principle). *Assume $\Omega \subset \mathbb{R}^n$ is bounded. If $\Delta u \geq 0$ in Ω and $u \in C(\overline{\Omega})$, then*

$$\max_{\partial\Omega} u = \max_{\overline{\Omega}} u.$$

Since the Laplace equation is linear, we deduce the comparison principle from the maximum principle.

Corollary 2.5 (Comparison principle)**.** *Assume $\Omega \subset \mathbb{R}^n$ is bounded. If $\Delta u \geq \Delta v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in the whole domain $\overline{\Omega}$.*

2.3. **Sub- and super-harmonic functions.** A function is sub-harmonic when $\Delta u \geq 0$ and super-harmonic when $\Delta u \leq 0$. We saw in Proposition 2.3 that sub-harmonic functions are characterized by the inequality in the mean value property. The reversed inequality characterizes super-harmonic functions.

We want to determine a way to make sense of the inequality $\Delta u \geq 0$ without requiring the function u to be C^2. The idea is to identify some property of functions which is equivalent to $\Delta u \geq 0$ when u is smooth, but can be checked even if u is very rough.

In this case, Proposition 2.3 gives us a possible answer. We could define that $\Delta u \geq 0$ in a *weak* sense if the quantity

$$\frac{1}{|B_r|} \int_{B_r(x)} u(y) \, \mathrm{d}y \tag{2.2}$$

is monotone increasing in r every time $B_r(x) \subset \Omega$.

This is a very permissive definition because we only need u to be locally integrable in order to check it.

There is a similar definition for the inequality $\Delta u \leq 0$ requiring the quantity above monotone decreasing. A harmonic function is a function for which (2.2) is constant in r.

It turns out that these definitions imply some regularity in the functions u. Any harmonic function will be C^∞. In the case of sub-/super-harmonic functions, we can only prove that they are semicontinuous.

Definition 2.6. *A function $u : \Omega \to \mathbb{R}$ is upper semicontinuous if for all $x \in \Omega$, we have*

$$u(x) = \lim_{r \to 0} \sup_{B_r(x)} u.$$

An equivalent definition is that for all sequences x_k in Ω converging to x, we have

$$u(x) \geq \limsup_{k \to \infty} u(x_k).$$

Also, we can define upper semicontinuity in term of preimages of semi-infinite intervals. That is, for all $a \in \mathbb{R}$,

$$u^{-1}((-\infty, a)) \text{ is an open set.}$$

One of the most classical results concerning semicontinuity is the following.

Proposition 2.7. *Let u_k be a decreasing sequence of continuous functions in Ω. Then the limit $u = \lim_{k \to \infty} u_k$ is upper semicontinuous.*

We will use the previous proposition to prove that subharmonic functions are upper semicontinuous.

Proposition 2.8. *Any function u which satisfies $\Delta u \geq 0$ in the weak sense described above is upper semi-continuous in Ω.*

Proof. For any $r > 0$, let

$$u_r = \frac{1}{|B_r|} u * \mathbb{1}_{B_r}.$$

The convolution above is explicitly given by the formula

$$\begin{aligned} u_r(x) &= \frac{1}{|B_r|} u * \mathbb{1}_{B_r}(x), \\ &= \frac{1}{|B_r|} \int u(y) \mathbb{1}_{B_r}(x - y) \, \mathrm{d}y, \\ &= \frac{1}{|B_r|} \int_{y \in B_r(x)} u(y) \, \mathrm{d}y. \end{aligned}$$

Our definition of $\Delta u \geq 0$ is exactly that this quantity is monotone increasing respect to r.

For every $r > 0$, it is possible to prove that the convolution u_r is a continuous function. Therefore, as $r \to 0$, we obtain a decreasing sequence of continuous functions. The limit $u = \lim_{r \to 0} u_r$ will be upper semicontinuous because of Proposition 2.7.

Note that for all $u \in L^1_{loc}$, the limit $u = \lim_{r\to 0} u_r$ holds almost everywhere because of Lebesgue differentiation theorem. Strictly speaking, we have proved that there is an upper semicontinuous function which equals u except at most for a set of measure zero. □

The previous proposition suggests that the right space to make a general definition of $\Delta u \geq 0$ is the upper semicontinuous functions. In this space, it makes sense to evaluate u at a point x. Therefore, we can write the slightly simpler definition.

Definition 2.9. *We say $u : \Omega \to \mathbb{R}$ is subharmonic if it is upper semicontinuous and every time $B_r(x) \subset \Omega$,*

$$u(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y) \,\mathrm{d}y.$$

A function u satisfying the opposite inequalities is called *superharmonic*. Equivalently, u is superharmonic when $-u$ is subharmonic.

The mean value property is what lets us prove most of the major properties of functions such that $\Delta u \geq 0$. In this case, we can reproduce those results. For example, the following comparison principle holds.

Proposition 2.10. *Let $u, v : \overline{\Omega} \to \mathbb{R}$. Assume $u \leq v$ on $\partial\Omega$. Let u be upper semicontinuous and v lower semicontinuous in $\overline{\Omega}$. Moreover, let us assume that u is subharmonic and v is superharmonic in Ω. Then, $u \leq v$ in Ω.*

2.4. **Exercises.**

Exercise 2.1. *Prove Proposition 2.7.*

Exercise 2.2. *Let $u \in L^1_{loc}(\mathbb{R}^n)$. Prove that the function $u * \mathbb{1}_{B_1}$ is continuous.*

Exercise 2.3. *Let $u : K \to \mathbb{R}$ be an upper semicontinuous function. Assume that K is compact. Prove that u achieves its maximum in K.*

Exercise 2.4. *Let u be an upper semicontinuous function. Prove that u is subharmonic if and only if every time $B_r(x) \subset \Omega$ and there is a function $v \in C^2(B_r(x)) \cap C(\overline{B_r(x)})$ such that $\Delta v \leq 0$ in $B_r(x)$ and $v \geq u$ on $\partial B_r(x)$, then $u \leq v$ in $B_r(x)$.*

Exercise 2.5. *Prove that the maximum of two subharmonic functions is subharmonic.*

Exercise 2.6. *Let $u : \overline{\Omega} \to \mathbb{R}$ be an arbitrary bounded function. The upper semicontinuous envelope, which we write u^*, is the smallest upper semicontinuous functions which is larger or equal to u. Prove that*

$$\begin{aligned} u^*(x) &= \inf\{v(x) : v \geq u \text{ in } \overline{\Omega} \text{ and } v \text{ is upper semicontinuous.}\}, \\ &= \sup\{\limsup_{k\to\infty} u(x_k) : x_k \to x\}. \end{aligned}$$

Exercise 2.7. *Let $\mathcal{U}$ be a bounded sequence of subharmonic functions in a domain Ω. Prove that*

$$\bar{u}(x) = \left(\sup_{u\in\mathcal{U}} u(x)\right)^*,$$

is subharmonic.

3. Fully nonlinear elliptic equations

A fully nonlinear elliptic equation is an expression of the form

$$F(D^2u, \nabla u, u, x) = 0.$$

Here D^2u stands for the Hessian matrix of the function u, and ∇u is its gradient. The function F is an arbitrary continuous function $F : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$. It must satisfy the following two conditions. [1]

F **is elliptic:** For any two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$ so that $A \geq B$, we have

$$F(A, p, u, x) \geq F(B, p, u, x),$$

for any values of $p \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $x \in \Omega$.

[1] In some places, the oposite monotonicity convention is used (so that $-\Delta u = 0$ is elliptic instead of $\Delta u = 0$). We follow the same ellipticity rule as in [3]. In [4], they use the opposite choice.

F **is proper:** For any two real values $u \geq v$, we have

$$F(A, p, u, x) \leq F(A, p, v, x),$$

for any values of $A \in \mathbb{R}^{n \times n}$, $p \in \mathbb{R}^n$, and $x \in \Omega$.

The Laplace equation corresponds to the case $F(A, p, u, x) = \mathrm{tr}A$.

Note that the definition above allows some degenerate cases to hold. For example, the function $F(A, p, u, x)$ is not required to depend on A. First order equations like

$$|\nabla u|^2 - u = 0,$$

correspond to $F(A, p, u, x) = |p|^2 - u$ and fit the theory. It is also possible to consider the heat equation as an example in which one of the coordinates is special and we call it "t". Even the case $F \equiv 0$ fits the definition above (in which case every function u solves $F(D^2u, \nabla u, u, x) = 0$).

3.1. **Variation of the random walk problem.** In section 2.1, we discussed how to derive the Laplace equation from a random walk problem. In this section we discuss some variations of that problem that lead to other elliptic equations.

Recall that we defined a random sequence of points $x := X_0, X_1, X_2, \ldots$. In section 2.1, the point X_{k+1} was chosen randomly with a uniform distribution in a ball of radius r around X_k.

Here, we want to use a different distribution of points for the choice of X_{k+1}. We can either choose a different distribution of points in $B_r(X_k)$, or a different shape around X_k where we choose X_{k+1} with a uniform distribution. Let us proceed with the latter variant. We choose X_{k+1} in an ellipsoid centered at X_k with a uniform distribution. We call this ellipsoid $E_r(X_k)$. The value of r is a scale parameter that makes $E_r(X_k)$ shrink to a point as $r \to 0$.

As before, we call $u_r(x)$ the expected value of the function f at the first point where the random walk hits the boundary $\partial\Omega$. In this case the point x refers to the starting point in the random walk. This function u_r satisfies

$$\int_{E_r(x)} u_r(y) - u_r(x) \, \mathrm{d}y = 0.$$

If we take the limit as $r \to 0$, we would obtain the function u which solves the equation

$$\sum_{ij} a_{ij} \partial_{ij} u = 0 \qquad \text{in } \Omega,$$

$$u = f \qquad \text{on } \partial\Omega.$$

The coefficients a_{ij} depend on the shape of $E_r(x)$. If we choose a different ellipsoid $E_r(x)$ at every different point x, that would lead to x-dependent coefficients.

$$\sum_{ij} a_{ij}(x) \partial_{ij} u = 0 \qquad \text{in } \Omega,$$

$$u = f \qquad \text{on } \partial\Omega.$$

The coefficients $\{a_{ij}(x)\}$ will always be positive matrices. This is a restriction intrinsic to the construction.

Let us now consider the following optimization problem. Let us say that we can choose, at every point $x \in \Omega$, any coefficient a_{ij} from within a given family $\{a_{ij}^\alpha : \alpha \in \mathcal{A}\}$. How large can we make the value of $u(x)$?

It turns out that there is one choice of coefficients that maximizes the value of $u(x)$ for all $x \in \Omega$. It corresponds to solving the problem

$$F(D^2u) := \sup_\alpha \sum_{ij} a_{ij}^\alpha \partial_{ij} u = 0 \qquad \text{in } \Omega,$$

$$u = f \qquad \text{on } \partial\Omega.$$

This is called the **Bellman equation**. The function F is elliptic and is given by a supremum of an arbitrary family of linear functions. Note that any convex function is the supremum of a family of linear functions, therefore, this model leads to any equation of the form $F(D^2u) = 0$ with F elliptic and convex (the simple equation above is also homogeneous of degree one, but that is not essential for the Bellman equation).

A more general model involves a two player game. Here, the coefficients are chosen from a two parameter family $a_{ij}^{\alpha\beta}$. The first player chooses α trying to maximize the value of $u(x)$, and then the second player chooses β trying to minimize the value of $u(x)$. The optimal choice for both players will correspond to the solution of the following equation

$$\begin{aligned} F(D^2u) := \inf_\beta \sup_\alpha \sum_{ij} a_{ij}^{\alpha\beta} \partial_{ij} u = 0 \qquad &\text{in } \Omega, \\ u = f \qquad &\text{on } \partial\Omega. \end{aligned}$$

This is called **Isaacs Equation**. In this case F is an elliptic function which is the infimum of arbitrary convex functions. Any Lipschitz function of D^2u can be written as an infimum of convex functions. Therefore, the model allows us to realize arbitrary equations of the form $F(D^2u) = 0$ with F elliptic.

3.2. **Definition of viscosity solution.** A classical solution of $F[u] = 0$ is a C^2 function u so that for every point $x \in \Omega$, we have the equality $F(D^2u(x), \nabla u(x), u(x), x) = 0$. A viscosity solution is going to be a generalized notion of solution where we only require a priori that our function u be continuous.

Definition 3.1. *We say that a function $u : \Omega \to \mathbb{R}$ satisfies the inequality $F(D^2u, \nabla u, u, x) \geq 0$ in Ω in the viscosity sense if u is upper semicontinuous in Ω and every time there exists a ball $B_r(x) \subset \Omega$ and a C^2 function $\varphi : B_r(x) \to \mathbb{R}$ such that $\varphi(x) = u(x)$ and $\varphi \geq u$ in $B_r(x)$, then $F(D^2\varphi(x), \nabla\varphi(x), \varphi(x), x) \geq 0$.*

A function u satisfying $F(D^2u, \nabla u, u, x) \geq 0$ in Ω in the viscosity sense is called a *subsolution* of the equation.

Reversing all the inequalities in the previous definition, we define $F(D^2u, \nabla u, u, x) \leq 0$, and we call those functions *supersolutions*. Note that u is a supersolution (i.e. $F[u] \leq 0$) if and only if $-u$ is a subsolution for the function $\tilde{F}(A, p, u, x) = -F(-A, -p, -u, x)$ (i.e. $\tilde{F}[u] \geq 0$). In particular, supersolutions are lower semicontinuous.

A viscosity solution to the equation $F(D^2u, \nabla u, u, x) = 0$ is a continuous function u which is at the same time a subsolution and a supersolution.

The idea of the definition is to check a condition that would be equivalent to the equation when u is smooth. If $u \in C^2$, that u is a subsolution in the viscosity sense is equivalent to $F(D^2u, \nabla u, u, x) \geq 0$ in the classical sense. Indeed, if u is a classical C^2 subsolution and there exist a test function φ as in Definition 3.1, we would have

$$\begin{aligned} \varphi(x) &= u(x), \\ \nabla\varphi(x) &= \nabla u(x), \\ D^2\varphi(x) &\geq D^2u(x). \end{aligned}$$

Based on the ellipticity assumption on the function F, this would imply that

$$F(D^2\varphi(x), \nabla\varphi(x), \varphi(x), x) \geq F(D^2u(x), \nabla u(x), u(x), x) \geq 0.$$

Conversely, if u is C^2 and is a viscosity subsolution, then using itself as a test function (i.e. $\varphi = u$), we observe that $F[u] \geq 0$ classically.

The point of the definition is to transfer the requirement of existence of second derivatives from the function u to the test functions φ. The definition makes sense even though u is merely upper semicontinuous.

The following is a semi-obvious variation of Definition 3.1 which is convenient to write down proofs (but not necessarily to think about them)

Definition 3.2. *We say that a function $u : \Omega \to \mathbb{R}$ satisfies the inequality $F(D^2u, \nabla u, u, x) \geq 0$ in Ω in the viscosity sense if u is upper semicontinuous in Ω and every time there exists a C^2 function $\varphi : \Omega \to \mathbb{R}$ such that $u - \varphi$ has a local maximum at a point $x \in \Omega$, then $F(D^2\varphi(x), \nabla\varphi(x), u(x), x) \geq 0$.*

Note that in this second definition we use the value of $u(x)$ instead of $\varphi(x)$. That is because it would be necessary to make a vertical translation of φ in order to transform it into a test function for Definition 3.1.

There is a corresponding notion of supersolution by reversing all inequalities and using a test function φ so that $u - \varphi$ has a local minimum at x.

3.3. **Half relaxed limits.** The definitions of viscosity sub- and supersolution is compatible with the natural limits of semicontinuous functions. We call them half relaxed limits and are defined below.

Definition 3.3. *Let u_k be a bounded sequence of functions. We write*

$$\limsup{}^* u_k(x) = \sup\{\limsup_{k\to\infty} u_k(x_k) : x_k \to x\}.$$

This half relaxed limit *is always an upper semicontinuous function.*

Likewise, when u_k is a bounded sequence of functions, we write

$$\liminf{}_* u_k(x) = \inf\{\liminf_{k\to\infty} u_k(x_k) : x_k \to x\}.$$

This half relaxed limit *is always a lower semicontinuous function.* [2]

We now state a rather simple property of half relaxed limits which will be used in a later proposition.

Lemma 3.4. *Let $K \subset \mathbb{R}^n$ be a compact set and $u_k : K \to \mathbb{R}$ be a sequence of functions such that $\limsup^* u_k(x) = u(x)$. Then, for all $\varepsilon > 0$, there exists a k_0 such that $u_k(x) \leq \max_K u + \varepsilon$ for all $k > k_0$.*

Note that the function u achieves its maximum on K due to exercise 2.3.

Proof. Assume the opposite. That is, there is a sequence of indexes k_j and points x_j so that $k_j \to \infty$ and $u_{k_j}(x_j) > \max_K u + \varepsilon$.

Since K is compact, we can assume that, after taking a further subsequence, the points x_j converge to some point x_∞. Thus, using the definition of $\limsup^*$,

$$u(x_\infty) \geq \limsup u_{k_j}(x_j) \geq \max_K u + \varepsilon.$$

We arrived to a contradiction and finished the proof. □

Lemma 3.5. *Let $u = \limsup^* u_k$, for a sequence of upper semicontinuous functions u_k in a domain Ω. Assume that u has a strict local maximum at the point x_0. Then there is a sequence of indexes k_j and points x_j such that*

1. *u_{k_j} has a local maximum at x_j.*
2. *$u_{k_j}(x_j)$ converges to $u(x_0)$.*
3. *x_j converges to x_0.*

Proof. We assumed that u has a strict local maximum at x_0. That means that for some $r > 0$, $u(y) < u(x_0)$ for all $y \in \overline{B_r(x_0)} \setminus \{x_0\}$.

Let $\rho > 0$ be an arbitrarily small radius. We have that

$$\max_{y\in\overline{B_r(x_0)\setminus B_\rho}} u(y) = u(x_0) - \delta,$$

with $\delta > 0$.

Using Lemma 3.4, we obtain that for sufficiently large k, $u_k \leq u(x_0) - \delta/2$ in $\overline{B_r(x_0) \setminus B_\rho}$.

From the definition of $\limsup^*$, there is a sequence k_j, and points $y_j \to x$ such that $u_{k_j}(y_j) \to u(x_0)$.

Let x_j be the point where the maximum of u_{k_j} is achieved in $\overline{B_r(x_0)}$. This maximum cannot be smaller than $u_{k_j}(x_j)$, which converges to $u(x_0)$. Therefore, for large enough k_j, we will have $x_j \in B_\rho$. Since $\rho > 0$ is arbitrary, we conclude that $x_j \to x_0$ as $j \to \infty$.

From the construction of the x_j, necessarily $u_{k_j}(x_j) \geq u_{k_j}(y_j) \to u(x_0)$. From the definition of $\limsup^*$, we conclude that necessarily $u_{k_j}(x_j) = u(x_0)$. □

The purpose of the last two lemmas was to prove the following proposition.

Proposition 3.6. *Let $F : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a continuous, elliptic, proper, function (equation). Let u_k be a bounded sequence of subsolutions $F[u_k] \geq 0$ in Ω in the viscosity sense. Then, the half relaxed limit*

$$u(x) = \limsup{}^* u_k(x)$$

is also a viscosity subsolution of $F[u] \geq 0$ in Ω.

[2] In the classical language of the calculus of variations, $\liminf_* u_k = \Gamma\text{-}\liminf u_k$.

Likewise, if u_k be a bounded sequence of supersolutions $F[u_k] \leq 0$ in Ω in the viscosity sense. Then, the half relaxed limit

$$u(x) = \liminf{}_* u_k(x)$$

is also a viscosity supersolution of $F[u] \leq 0$ in Ω.

Proof. We prove the first statement, which concerns subsolutions. The second statement is proved analogously (reversing all inequalities).

We use Definition 3.2. Let φ be any C^2 function for which $u - \varphi$ has a local maximum at the point $x_0 \in \Omega$.

The function $u(x) - \varphi(x) - |x - x_0|^4$ has a strict local maximum at the point x_0. Let us call $\tilde{\varphi}(x) = \varphi(x) + |x - x_0|^4$, which is also a valid C^2 test function.

Since $\tilde{\varphi}$ is continuous, we have $\limsup^* u_k - \tilde{\varphi} = u - \tilde{\varphi}$.

Applying Lemma 3.5, there is a sequence of indexes $k_j \to \infty$ and points $x_j \to x$ so that $u_{k_j} - \tilde{\varphi}$ achieves its local maximum at x_j and $u_{k_j}(x_j) \to u(x_0)$.

From the definition of viscosity solution (Definition 3.2), we have

$$F(D^2\tilde{\varphi}(x_j), \nabla\tilde{\varphi}(x_j), u_{k_j}(x_j), x_j) \geq 0.$$

Passing to the limit as $j \to \infty$, using the continuity of F, we obtain

$$F(D^2\tilde{\varphi}(x_0), \nabla\tilde{\varphi}(x_0), u(x_0), x_0) \geq 0.$$

But note that $D^2\tilde{\varphi}(x_0) = D^2\varphi(x_0)$ and $\nabla\tilde{\varphi}(x_0) = \nabla\varphi(x_0)$. Therefore

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \geq 0.$$

Thus, we have verified that Definition 3.2 holds for u, and then u is a viscosity subsolution. □

3.4. **Exercises.**

Exercise 3.1. *Consider the boundary value problem*

$$\begin{aligned} |u_x|^2 - 1 = 0 \text{ in } [-1,1], \\ u(x) = 0 \text{ for } x = -1, 1. \end{aligned}$$

Which one is the right viscosity solution, $u(x) = |x| - 1$ or $u(x) = 1 - |x|$?

Exercise 3.2. *Prove that a function u is subharmonic in the sense of the previous section if and only if it satisfies $\Delta u \geq 0$ in the viscosity sense.*

Exercise 3.3. *Let $e \in \mathbb{R}^n$ be an arbitrary unit vector and $u : \Omega \to \mathbb{R}$ be a viscosity solution of $F[u] = 0$ in $\Omega \setminus \{x \cdot e = 0\}$. That is, the function u solves the equation on both sides of the hyperplane $\{x \cdot e = 0\}$. Prove that if u is differentiable at any point on $\{x \cdot e = 0\}$, then u solves the equation $F[u] = 0$ in the full domain Ω.*

Hint. If $u - \varphi$ has a local maximum at a point on $\{x \cdot e = 0\}$, what about the function $u - \varphi + \varepsilon|x \cdot e|$?

Exercise 3.4. *Prove that for any sequence of functions u_k that are uniformly bounded, the half relaxed limit $\limsup^* u_k$ is upper semicontinuous.*

Exercise 3.5. *Let $K \subset \mathbb{R}^n$ be a compact set and $u_k : K \to \mathbb{R}$ be a sequence of upper semicontinuous functions such that $\limsup^* u_k(x) = u(x)$. Assume that u is a continuous function. Prove that, for all $\varepsilon > 0$, there exists a k_0 such that $u_k(x) \leq u + \varepsilon$ for all $k > k_0$.*

Make an example to show that the conclusion may not be valid if u is not continuous.

(i.e. we could not prove it in class because it was not true)

Exercise 3.6. *Let u_k be a sequence of functions in a compact set K. Assume that*

$$u = \limsup{}^* u_k = \liminf{}_* u_k.$$

Prove that u_k converges to u uniformly in K.

Exercise 3.7. *Let $F_k : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a sequence of elliptic, proper, functions (equations) so that $\limsup^* F_k = F$. Let u_k be a bounded sequence of subsolutions $F_k[u_k] \geq 0$ in the viscosity sense. Prove that the half relaxed limit*

$$u(x) = \limsup{}^* u_k(x)$$

is also a viscosity subsolution of $F[u] \geq 0$ in Ω.

Exercise 3.8. *Let $u_k : \overline{\Omega} \to \mathbb{R}$ be a sequence of upper semicontinuous functions. Prove that*

$$\int_\Omega \liminf{}_* u_k(x)\,\mathrm{d}x \leq \liminf \int_\Omega u_k(x)\,\mathrm{d}x.$$

Show an example in which the inequality is strict even though $u_k \to u$ in $L^1(\Omega)$.

4. The comparison principle

The comparison principle is the basic mechanism that guarantees the uniqueness of solutions of the Dirichlet problem for elliptic equations. The general form of the comparison principle is the following.

The comparison principle. Assume that F is a function satisfying the ellipticity and properness conditions of section 3 plus some extra nondegeneracy condition (to be determined later).

Let $u, v : \overline{\Omega} \to \mathbb{R}$ be two functions. The function u is assumed to be upper semicontinuous on $\overline{\Omega}$ and v is lower semicontinuous in $\overline{\Omega}$. Assume that $F[u] \geq 0$ and $F[v] \leq 0$ in the viscosity sense and also $u \leq v$ on $\partial\Omega$. Then, also, $u \leq v$ in Ω.

There is necessarily a required nondegeneracy condition for F so that we rule out cases like $F \equiv 0$. We need, for example, F to be uniformly elliptic or strictly proper. These are conditions that would make the inequality strict in the definition of ellipticity in section 3 when $A \neq B$ (that's uniform ellipticity) or the inequality strict in the definition of properness when $u > v$. Parabolic equations are also nondegenerate and the comparison principle always holds for them provided that the equation depends smoothly on x and t.

In this note we will prove a simple, but representative, case of the comparison principle. We will consider F independent of ∇u and x, like $F(D^2u, u)$, and strictly proper. That is, we assume that for any symmetric matrix A and $u > v$,

$$F(A, u) < F(A, v).$$

4.1. Sup- and inf-convolutions.

The comparison principle is relatively simple to prove when u and v are C^2 functions. In fact, when only one of them is C^2, the comparison principle can be deduced quickly from the definition of viscosity solution.

In order to prove the comparison principle when both functions satisfy the equation in the viscosity sense, we will use a regularization procedure. Unfortunately, this regularization does not give us a C^2 function right away, but we will prove that we get a function which is second differentiable almost everywhere.

Definition 4.1. *If u is upper semicontinuous in $\overline{\Omega}$, we define*

$$u^\varepsilon(x) = \max_{y \in \overline{\Omega}} u(y) - \frac{1}{\varepsilon}|x - y|^2.$$

If u is lower semicontinuous in $\overline{\Omega}$, we define

$$u_\varepsilon(x) = \min_{y \in \overline{\Omega}} u(y) + \frac{1}{\varepsilon}|x - y|^2.$$

The useful fact about inf- and sup-convolutions is that they preserve the condition that a function is a super- or sub-solution.

Lemma 4.2. *If u is a subsolution of $F(D^2u, \nabla u, u) \geq 0$ in Ω in the viscosity sense, then u^ε is also a subsolution of $F(D^2u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon) \geq 0$ in Ω^ε in the viscosity sense.*

Here Ω^ε is the subset of those point $x \in \Omega$ so that the maximum in the definition of $u^\varepsilon(x)$ is achieved for some y in the interior of Ω. In particular $x \in \Omega^\varepsilon$ if $\operatorname{dist}(x, \partial\Omega) > (\varepsilon \operatorname{osc} u)^{1/2}$. [3]

Proof. Let φ be a test function that touches u^ε from above at the point $x_0 \in \Omega^\varepsilon$.

Let $y_0 \in \Omega$ be the point where the maximum is achieved in the definition of $u^\varepsilon(x_0)$. We have that

$$\varphi(x_0) = u(y_0) - \frac{1}{\varepsilon}|x_0 - y_0|,$$
$$\varphi(x) \geq u^\varepsilon(x) \geq u(x - x_0 + y_0) - \frac{1}{\varepsilon}|x_0 - y_0|^2.$$

[3] We use the notation $\operatorname{osc}_A u = \sup_A u - \inf_A u$.

Therefore, the function $\tilde{\varphi}(x) = \varphi(x+x_0-y_0) + \frac{1}{\varepsilon}|x_0-y_0|^2$ is a valid test function which touches the function u from above at the point y_0. Applying the definition of viscosity subsolution for the function u, we get

$$F(D^2\tilde{\varphi}(y_0), \nabla\tilde{\varphi}(y_0), \tilde{\varphi}(y_0)) \geq 0.$$

This is the same as

$$F\left(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0) + \frac{1}{\varepsilon}|x_0-y_0|^2\right) \geq 0.$$

Using that F is proper, we deduce

$$F\left(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0)\right) \geq 0.$$

This finished the verification that u^ε is a subsolution. □

Naturally, there is a straight forward modification of Lemma 4.2 which says that if u is a supersolution, then u_ε is also a supersolution.

Note that Lemma 4.2 involves an equation of the form $F(D^2u, \nabla u, u)$ which is independent of x. This is important for the result in this lemma. Adding x dependence in the equation that u satisfies would force us to modify the equation that u^ε satisfies.

As we can see from its formula, the *sup-convolution* u^ε is the envelope of a family of paraboloids of opening $-1/\varepsilon$. The following proposition is a consequence of that geometric understanding.

Proposition 4.3. *The function $u^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2$ is convex.*

Proof. For any $x_0 \in \overline{\Omega}$, let y_0 be the point where the maximum is achieved in the definition of $u^\varepsilon(x_0)$.

For any other value of $x \in \overline{\Omega}$, we have

$$u^\varepsilon(x) \geq u(y_0) - \frac{1}{\varepsilon}|x-y_0|^2,$$

since y_0 is an eligible value of y in the maximum. Therefore

$$\begin{aligned} u^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2 &\geq u(y_0) - \frac{1}{\varepsilon}\left(|x-y_0|^2 - |x|^2\right), \\ &= u(y_0) + \frac{1}{\varepsilon}\left(2y_0 \cdot x - |y_0|^2\right). \end{aligned}$$

The equality holds when $x = x_0$. Therefore, we deduce that for any $x_0 \in \Omega$, the function $u^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2$ is above a tangent plane at the point x_0. Thus, $u^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2$ is convex. □

Convexity has some mild regularity implications. First of all, all convex functions are locally Lipschitz. We will use a more delicate property about the point of second differentiability that we now state.

Theorem 4.4 (Alexandrov theorem)**.** *Any convex function $v : \Omega \to \mathbb{R}$ is second differentiable almost everywhere.*

Here, second differentiability at a point x means that there is a vector p $(= \nabla v(x))$ and a matrix A $(= D^2u(x))$ such that

$$\lim_{y \to x} \frac{|u(y) - u(x) - p \cdot (y-x) - \frac{1}{2}\langle A(y-x), (y-x)\rangle|}{|x-y|^2} = 0.$$

This is a classical result from geometric measure theory (it may be explained in Marianna Csornyei's lectures).

Corollary 4.5. *For any $\varepsilon > 0$, the functions u^ε or u_ε are second differentiable almost everywhere.*

Naturally, the sup- and inf-convolutions are approximations of the original function u as $\varepsilon \to 0$. We leave this as an exercise.

Exercise 4.1. *If u is upper semicontinuous in $\overline{\Omega}$, then*

$$\limsup_{\varepsilon \to 0}{}^* u^\varepsilon = u.$$

If u is lower semicontinuous in $\overline{\Omega}$, then

$$\liminf_{\varepsilon \to 0}{}_* u_\varepsilon = u.$$

We now state and prove a version of the comparison principle.

Theorem 4.6. *Let F be strictly proper. That is, for any symmetric matrix A and $u > v$, we have $F(A, u) < F(A, v)$.*

Let $u, v : \overline{\Omega} \to \mathbb{R}$. Assume that u is bounded and upper semicontinuous in $\overline{\Omega}$ and satisfies $F(D^2u, u) \geq 0$ in Ω in the viscosity sense. Assume that v is bounded and lower semicontinuous in $\overline{\Omega}$ and satisfies $F(D^2v, v) \leq 0$ in Ω in the viscosity sense. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ also in Ω.

Proof. Assume the result is not true. That is, assume that $\max u - v = u(x_0) - v(x_0) > 0$ for some $x_0 \in \Omega$.

Let us start by considering u^ε and v_ε. Using Lemma 3.5 and Exercise 4.1, we observe that for ε sufficiently small, there will be a point x_1, arbitrarily close to x_0, so that

$$\max_{\overline{\Omega}} u^\varepsilon - v_\varepsilon = u^\varepsilon(x_1) - v_\varepsilon(x_1) > 0.$$

Moreover, $u^\varepsilon(x_1) - v_\varepsilon(x_1)$ is strictly larger than any value of $u^\varepsilon(x) - v_\varepsilon(x)$ when $x \in \Omega \setminus \Omega^\varepsilon$.

If both u^ε and v_ε were second differentiable at the point x_1, we would evaluate the equation at that point and finish the proof. There is no way we can guarantee this fact. We only know u^ε and v_ε are differentiable almost everywhere. So, we must look for more points where the equation would give a simple contradiction.

Let $w := u^\varepsilon - v_\varepsilon$ and Γ be the positive concave envelope of w in Ω^ε. That is, Γ is the lower envelope of all planes that are nonnegative and above w in Ω^ε.

$$\Gamma(x) = \min\{a \cdot x + b : a \in \mathbb{R}^n, b \in \mathbb{R}, a \cdot x + b \geq \max(w(x), 0) \text{ for all } x \in \Omega^\varepsilon\}. \tag{4.1}$$

Let $A = \{x : \Gamma(x) = w(x)\}$. Our objective is to show that there is some point $x \in A$ such that u^ε and v_ε are second differentiable at x. Indeed, we can see that for any such $x \in A$, $w(x) > 0$ and $D^2w(x) \leq 0$, therefore

$$u^\varepsilon(x) > v^\varepsilon(x) \text{ and } D^2u^\varepsilon(x) \leq D^2v_\varepsilon(x).$$

Therefore, $F(D^2u^\varepsilon(x), u^\varepsilon(x)) < F(D^2v^\varepsilon(x), v^\varepsilon(x))$, contradicting the equation.

We are left to show that such point $x \in A$ exists. Since u^ε and v_ε are second differentiable almost everywhere, it suffices to show that A has positive measure (i.e. $|A| > 0$).

Note that for any point $x \in A$, there is a quadratic polynomial of opening $-2/\varepsilon$ that touches w, and therefore Γ at x. Since Γ is concave, then Γ must be differentiable at x and $\nabla\Gamma(x)$ is well defined.

In order to prove that $|A| > 0$, we will study the set

$$\nabla\Gamma(A) := \{\nabla\Gamma(x) : x \in A\}.$$

This is the set of slopes a of all planes in (4.1) that touch w at some point.

In order to compare the measure $|A|$ with the image measure $|\nabla\Gamma(A)|$, we use the following lemma.

Lemma 4.7. *If $x, y \in A$, then*

$$|\nabla\Gamma(x) - \nabla\Gamma(y)| \leq \frac{C}{\varepsilon}|x - y|.$$

In other words, the map $\nabla\Gamma$ is Lipschitz on A with Lipschitz constant C/ε.

The idea of this lemma is that in some sense $0 \geq D^2\Gamma \geq 2/\varepsilon$ in A. The proof is more involved because we cannot assume that Γ is second differentiable and we do not know the shape of the set A.

Proof. At every point in A, Γ has a tangent paraboloid from below, with opening $2/\varepsilon$. Moreover, since Γ is concave, all its tangent planes stay above Γ everywhere. Therefore, in particular, for any $z \in \Omega^\varepsilon$,

$$\Gamma(x) + (z - x) \cdot \nabla\Gamma(x) \geq \Gamma(z) \geq \Gamma(x) + (z - x) \cdot \nabla\Gamma(x) - \frac{2}{\varepsilon}|x - z|^2,$$

$$\Gamma(y) + (z - y) \cdot \nabla\Gamma(y) \geq \Gamma(z) \geq \Gamma(y) + (z - y) \cdot \nabla\Gamma(y) - \frac{2}{\varepsilon}|z - y|^2$$

Subtracting the second and third inequalities, we get

$$0 \geq \Gamma(x) + (z - x) \cdot \nabla\Gamma(x) - \frac{2}{\varepsilon}|x - z|^2 - \Gamma(y) + (y - z) \cdot \nabla\Gamma(y).$$

Moreover, using $z = y$ in the first inequality

$$\Gamma(y) \leq \Gamma(x) + (y - x) \cdot \nabla\Gamma(x).$$

Therefore,

$$0 \geq (z-y)\cdot(\nabla\Gamma(x)-\nabla\Gamma(y)) - \frac{2}{\varepsilon}|x-z|^2.$$

We conclude the proof choosing

$$z = y + \gamma(\nabla\Gamma(x)-\nabla\Gamma(y)),$$

where γ is a positive real number chosen so that $\gamma|\nabla\Gamma(x)-\nabla\Gamma(y)| \approx |x-y|$ and $z \in \Omega^\varepsilon$. □

It is a fact that Lipschitz functions map sets of measure zero into sets of measure zero. Therefore, by proving that $\nabla\Gamma(A)$ has positive measure, we prove that A has positive measure. More precisely, the following identity holds

$$|\nabla\Gamma(A)| = \int_A |\det D^2\Gamma(x)| \,\mathrm{d}x \leq \left(\frac{C}{\varepsilon}\right)^n |A|.$$

We are left to prove that $|\nabla\Gamma(A)| > 0$. The key of this step is that $\max\Gamma = \Gamma(x_1)$ is strictly larger than the value of Γ on the boundary $\partial\Omega^\varepsilon$.

Let

$$m := \Gamma(x_1) - \max_{\partial\Omega^\varepsilon}\Gamma = w(x_1) - \max_{\partial\Omega^\varepsilon} w > 0,$$
$$d := \operatorname{diam}(\Omega^\varepsilon).$$

We claim that any $a \in \mathbb{R}^n$ such that $|a| < m/d$ belongs to $\nabla\Gamma(A)$. Indeed, the oscillation of the linear function $a\cdot x$ would be smaller than m in Ω^ε. Therefore, the maximum of $\Gamma(x) - a\cdot x$ must be achieved at an interior point $x_2 \in \Omega^\varepsilon$. Thus, $\nabla\Gamma(x_2) = a$. Recall that the concave function Γ is differentiable at any point in A because there is a tangent paraboloid from below.

This means that $\nabla\Gamma(A)$ contains a small ball. Therefore, it has positive measure and we conclude the proof of Theorem 4.6. □

4.2. **Exercises.**

Exercise 4.2. *Assume that $u : \overline{\Omega} \to \mathbb{R}$ is a α-Holder continuous function. That means that*

$$|u(x)-u(y)| \leq C_0|x-y|^\alpha,$$

for some constant C_0, and $\alpha \in [0,1]$. Prove that

$$u^\varepsilon(x) - u(x) \leq C\varepsilon^{\frac{\alpha}{2-\alpha}},$$

where C depends on α and the constant C_0 above.

Moreover, prove also that the point y where the maximum is achieved in Definition 4.1 for u^ε satisfies

$$|y-x| \leq C\varepsilon^{\frac{1}{2-\alpha}}.$$

Exercise 4.3. *Prove that if $u : B_r \to \mathbb{R}$ is a convex function, then for all $x, y \in B_{r/2}$,*

$$u(x) - u(y) \leq 2\frac{|x-y|}{r}\operatorname*{osc}_{B_r} u.$$

Exercise 4.4. *Let u be a solution to an equation which is independent of x.*

$$F(D^2u, \nabla u, u) = 0 \text{ in } \Omega.$$

Assume that the comparison principle holds for this equation and that u is Lipschitz on the boundary. That means that for all $x \in \partial\Omega$ and $y \in \overline{\Omega}$,

$$|u(x)-u(y)| \leq C|x-y|.$$

Prove that u is Lipschitz in the whole domain $\overline{\Omega}$.

Hint. For any fixed value of $h \in \mathbb{R}^n$, compare the functions $u(x)$ with $u(x+h) + C|h|$.

Exercise 4.5. *Let u be a subsolution to the x-dependent equation*

$$F(D^2u, \nabla u, u, x) \geq 0 \qquad \text{in } \Omega.$$

Prove that u^ε is a subsolution to the modified equation

$$F^\varepsilon(A, p, u, x) = \sup\{F(A, p, u, y) : |y-x| \leq r\},$$

in the domain Ω^ε, where $r = (\varepsilon \operatorname{osc} u)^{1/2}$.

Exercise 4.6. *Let $\Gamma : \overline{\Omega} \to \mathbb{R}$ be a concave function so that $\Gamma \leq 0$ on $\partial\Omega$. Prove the inequality*

$$\max_{\Omega} \Gamma \leq C \operatorname{diam} \Omega \left(\int_{\Omega} |\det D^2\Gamma| \, \mathrm{d}x \right)^{1/n},$$

for some constant C which depends on the dimension n only.

5. Existence of solutions

We now turn to the issue of existence of solutions. The discuss Perron's method which is a very general approach for constructing solutions of equations that satisfy the comparison principle.

Our objective is to find the unique viscosity solution of the equation.

$$F(D^2u, \nabla u, u, x) = 0 \text{ in } \Omega, \tag{5.1}$$

$$u = f \text{ on } \partial\Omega \tag{5.2}$$

Perron's method is based on the following three assumptions.

(1) The comparison principle holds.
(2) There exists a supersolution u_0 of (5.1) so that $u_0 \geq f$ on $\partial\Omega$.
(3) There exists a subsolution u_1 of (5.1) so that $u_1 \leq f$ on $\partial\Omega$.

5.1. Generalized boundary conditions.

It turns out that the boundary condition (5.2) is not always achievable. This will happen when the equation is degenerate elliptic. The most extreme case is if we want to solve

$$u_x = 0 \text{ in } (0,1),$$
$$u(0) = 0 \text{ and } u(1) = 1.$$

The equation tells us that the function must be constant. But the boundary condition at 0 and 1 does not coincide. Obviously, there cannot be a solution u that achieves the boundary condition at both points.

The heat equation is a more subtle example. When we study a parabolic equation in the cylinder that $x \in B_1$ and $t \in [0,1]$, we cannot prescribe the boundary condition on the final time $t = 1$. The only part of the boundary where the value of the function can be given is the parabolic boundary. The equation will determine the value of the solution on the final time.

Depending on the equation, there will be some parts of the boundary where we can prescribe the values of the function, and other parts where the equation holds and the boundary condition does not count. We now define a generalized boundary condition that reflects this.

Definition 5.1. *We say that u is a sub-solution of* (5.1) *and* (5.2) *in the viscosity sense if it is upper semicontinuous in $\overline{\Omega}$, $F(D^2u, \nabla u, u, x) \geq 0$ in Ω in the viscosity sense, and whenever there is a C^2 function $\varphi \geq u$ and a point $x_0 \in \partial\Omega$ so that $\varphi(x_0) = u(x_0)$, then either*

$$\varphi(x_0) \leq f(x_0) \qquad \textbf{or} \qquad F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) \geq 0.$$

A lower semicontinuous function is a supersolution of (5.1) and (5.2) in the viscosity sense if it satisfies the corresponding condition where all inequalities are reversed. As before, a solution is a continuous function which is a sub- and supersolution at the same time.

With this generalized definition of the boundary condition (5.2) we let every equation decide where the boundary condition will be honored, and where the equation will determine the values of the solution u. In some cases (when barrier functions exist), it can be shown that the Dirichlet boundary conditions necessarily holds. For some degenerate cases like the ones we describe above, some parts of the boundary will be invisible to the solution.

This generalized notion of boundary condition is also stable with respect to half-relaxed limits. For example, the following proposition is proved by an argument similar to Proposition 3.6.

Exercise 5.1. *Assume u_k is a sequence of subsolutions of* (5.1) *and* (5.2) *in the viscosity sense. Then also $u = \limsup^* u_k$ is a subsolution in the viscosity sense (as in Definition 5.1).*

For carrying out Perron's method below, we will assume that the comparison principle holds for this type of viscosity sub- and supersolutions of (5.1) and (5.2). That is our assumption number (1).

5.2. **Perron's method.** The comparison principle tells us that the solution to the equation (5.1)-(5.2) is larger or equal than all its subsolutions and smaller or equal to all supersolutions. Thus, it is reasonable to construct the solution u as a supremum of all subsolutions of the problem. We define

$$\mathcal{A} := \{v : \overline{\Omega} \to \mathbb{R} : v \text{ a viscosity subsolution of (5.1) and (5.2)}\}.$$

Note that by our assumption on the existence of one subsolution u_1 and one supersolution u_0, we know that the set $\mathcal{A}$ is non-empty and uniformly bounded above.

The solution u will be the maximum of all functions in $\mathcal{A}$. At this point in the proof we cannot say such maximum exists. We define u in the following way

$$u(x) := \left(\sup_{v \in \mathcal{A}} v(x)\right)^* .$$

The superscript $*$ stands for the upper semicontinuous envelope. In other words, the function u is given by the expression

$$u(x) = \sup\{\limsup v_j(x_j) : v_j \in \mathcal{A} \text{ and } x_j \to x\}.$$

The first step is to show that u is a subsolution of (5.1) and (5.2).

Lemma 5.2. *The upper semicontinuous envelope of a family of subsolutions of the equation* (5.1) *and* (5.2) *is also a subsolution to* (5.1) *and* (5.2) *in the viscosity sense.*

Proof. Let φ be a C^2 function so that $\varphi \geq u$ and $\varphi(x_0) = u(x_0)$.

Reviewing the definition of u, we observe that there exists a sequence of points $x_j \to x_0$ and a sequence of functions $v_j \in \mathcal{A}$ such that

$$\lim_{j \to \infty} v_j(x_j) = u(x_0).$$

In particular $\limsup^* v_j(x_0) \geq u(x_0)$. By construction $u \geq \limsup^* v_j$ for any sequence of functions $v_j \in \mathcal{A}$. Therefore, in this case, $\limsup^* v_j(x_0) = u(x_0) = \varphi(x_0)$. For all other points, we have $\varphi \geq u \geq \limsup^* v_j$.

Using Exercise 5.1 (which is the version up to the boundary of Proposition 3.6), we know that $\limsup^* v_j$ is a subsolution. Therefore, by definition,

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) \geq 0,$$

if $x_0 \in \Omega$, or

$$\varphi(x_0) \leq f(x_0) \qquad \textbf{or} \qquad F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) \geq 0.$$

if $x_0 \in \partial\Omega$. In any case, we verify that u is a viscosity subsolution of (5.1) and (5.2). [4] □

The previous lemma tells us that the function u belongs to the set $\mathcal{A}$. From our construction, $u \geq v$ for any $v \in \mathcal{A}$. Therefore, we are now in a position to say that indeed u is the maximum element in this set.

$$u = \max_{v \in \mathcal{A}} v.$$

We are left to prove that u is a supersolution. For that, we will temporarily consider its lower semicontinuous envelope u_*.

Lemma 5.3. *The lower semicontinuous envelope u_* is a supersolution of* (5.1) *and* (5.2).

Proof. Assume the contrary. That is, there must be some point $x_0 \in \overline{\Omega}$, and some function $\varphi \in C^2$, so that $\varphi \leq u_*$ and $\varphi(x_0) = u_*(x_0)$ but

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) > 0,$$

if $x_0 \in \Omega$, or

$$\varphi(x_0) < f(x_0) \qquad \textbf{and} \qquad F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) > 0.$$

if $x_0 \in \partial\Omega$.

Like we did in previous proofs, we can assume that $\varphi(x) < u_*(x)$ for all $x \neq x_0$, otherwise we would replace $\varphi(x)$ with $\varphi(x) - |x - x_0|^4$.

In any case, the strict inequality must hold in a neighborhood of x_0, since $\varphi \in C^2$ and F is continuous.

Moreover, for a small enough $\delta > 0$, the strict inequality also holds if we replace φ for $\tilde{\varphi}(x) := \varphi(x) + \delta$.

[4] We acknowledge Jackson Hance for an idea simplifying the proof of Lemma 5.2

For any $\delta > 0$, $\tilde{\varphi}(x_0) > u_*(x_0)$. Since u_* is the lower semicontinuous envelope, this does not necessarily imply that $\tilde{\varphi}(x_0) > u(x_0)$ but it does imply that $\tilde{\varphi}(x) > u(x)$ for some values of x near x_0.

Moreover, since $\varphi(x) < u_*(x) \leq u(x)$ for all $x \neq x_0$, then, for small enought δ, $\tilde{\varphi} < u$ outside of a small neighborhood around x_0.

Let $w(x) = \max(u(x), \tilde{\varphi}(x))$. We claim that w is a subsolution of the equation. Indeed, any test function that touches w from above, touches either u from above, or $\tilde{\varphi}$ at some point in a neighborhood of x_0. Since $\tilde{\varphi}$ is a strict subsolution in a neighborhood of x_0, any of the two cases implies the right inequality for the test function.

But this is a contradiction because u is the maximum of all subsolutions, and we have just constructed another one, w, which is larger than u at some points. □

At this point we have shown in Lemma 5.2 that u is a subsolution and in Lemma 5.3 that u_* is a supersolution of the same equation. The comparison principle implies that $u_* \geq u$. But the lower semicontinuous envelope of any function is less or equal than the original function. Therefore, $u_* = u$, and it is a continuous solution to the equation (5.1)-(5.2).

This finishes the discussion of Perron's method. At the end we obtain a continuous solution to our equation assuming only that a comparison principle holds that is compatible with the notion of boundary condition that we consider, and the existence of one sub- and one supersolution.

5.3. **Exercises.**

Exercise 5.2. *We say that the equation* (5.1)-(5.2) *has barriers at the point* $x_0 \in \partial\Omega$ *if there exists two continuous functions* U *and* L *so that*

- U *is a supersolution of* (5.1)-(5.2).
- L *is a subsolution of* (5.1)-(5.2).
- $U(x_0) = L(x_0) = f(x_0)$.

Prove that if barriers exists at the point x_0*, then the solution* u *satisfies* $u(x_0) = f(x_0)$.

Exercise 5.3. *Let* F *be a function satisfying the following hypothesis.*

- F *is elliptic and proper.*
- F *is Lipschitz continuous with respect to all variables.*
- F *is uniformly elliptic with respect to* D^2u *(see definition in the next section)*
- *For any point* $x \in \overline{\Omega}$, $F(0,0,0,x) = 0$.

Assume also that the set Ω *has a tangent ball from outside at any point on its boundary.*

Prove that barriers exists at any point $x_0 \in \partial\Omega$.

Note. The construction solving the previous exercise involves a somewhat tedious computation. The formulas for U and L are closely related to the ones used to prove Hopf's lemma for uniformly elliptic equations (See section 6.4.2 in Evans book [5]).

Exercise 5.4. *Compute the unique viscosity solution to the equation*

$$u_x = 0 \text{ in } (0,1),$$
$$u(0) = 0 \text{ and } u(1) = 1.$$

6. Regularity issues

In the previous sections we discussed general results concerning the existence and uniqueness of viscosity solutions to boundary value problems. This is a very well established framework that provides us with continuous functions that solve elliptic PDEs in the viscosity sense. The natural question that follows is the regularity of these solutions. Are they merely continuous functions or are they differentiable? Would these solutions be C^2 and actually solve the equations classically? The answer depends on extra assumptions for the equations.

A standard reference for regularity results about fully nonlinear elliptic equations is the book of Caffarelli and Cabre [3].

6.1. **Uniform ellipticity.** The key concept in developing general regularity results for elliptic PDE is uniform ellipticity.

Definition 6.1. *We say that a function $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ is uniformly elliptic if there exist constants $\Lambda \geq \lambda > 0$ such that for any positive definite matrix $A \geq 0$ and any $X \in \mathbb{R}^{n\times n}$, we have the inequalities*

$$\lambda \mathrm{tr} A \leq F(X+A) - F(X) \leq \Lambda \mathrm{tr} A.$$

When the function F is differentiable, the definition above is equivalent to the inequalities

$$\lambda I \leq \frac{\partial F}{\partial X_{ij}} \leq \Lambda I.$$

6.2. **Survey of results.** In this section we survey some regularity results for uniformly elliptic equations. We concentrate in equations of the form $F(D^2u) = 0$, which are independent of any lower order values $(\nabla u, u)$ and x. There are, of course, regularity results involving lower order and x dependency. In order to keep the presentation relatively simple, we restrict to the most basic results concerning equations depending on D^2u only.

The following result says that solutions to uniformly elliptic equations in 2D are always classical.

Theorem 6.2 (Nirenberg, 1953)**.** *Assume that F is uniformly elliptic. The solutions to the equation*

$$F(D^2u) = 0 \text{ in } B_1 \subset \mathbb{R}^2$$

are always $C^{2,\alpha}$ for some $\alpha > 0$. They are classical solutions with Hölder continuous second derivatives. Moreover, an estimate holds

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Here C and α depend only on the ellipticity constants λ, Λ.

This theorem was obtained long before the development of viscosity solutions. The original result was in the form of an a priori estimate, from which one can prove that the Dirichlet problem is solvable classically. From the uniqueness of viscosity solutions, this classical $C^{2,\alpha}$ solution must coincide with the viscosity solution.

In higher dimension, we can only prove the C^2 regularity of solutions when we make further assumptions on the equation. The following result was obtained independently by Krylov and Evans in 1983.

Theorem 6.3. *Assume that F is uniformly elliptic and* **convex**. *The solutions to the equation*

$$F(D^2u) = 0 \text{ in } B_1$$

are always $C^{2,\alpha}$ for some $\alpha > 0$. Moreover, an estimate holds

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Here C and α depend only on dimension and the ellipticity constants λ, Λ.

Without the convexity assumption, we can only prove that viscosity solutions to uniformly elliptic equations are $C^{1,\alpha}$ for some $\alpha > 0$. The following result was proved originally by Krylov and Safonov in 1980. Their original proof was an a priori estimate for strong solutions. The proof was later rewritten (and slightly refined) by Caffarelli in a way that it could be applied to all viscosity solutions.

Theorem 6.4. *Assume that F is uniformly elliptic. The solutions to the equation*

$$F(D^2u) = 0 \text{ in } B_1$$

are always $C^{1,\alpha}$ for some $\alpha > 0$. Moreover, an estimate holds

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty}.$$

Here C and α depend only on dimension and the ellipticity constants λ, Λ.

There are examples of non C^2 solutions of uniformly elliptic equations in high dimensions. The optimal example to date was obtained by Nadirashvili, Tkachev and Vladuts [6]. It tells us that there exists a non C^2 function which solves a uniformly elliptic equation in the viscosity sense in 5D. Whether there exist non classical viscosity solutions of a uniformly elliptic equation in 3D or 4D is a remarkable open problem.

Another condition that can lead to further regularity is when Λ and λ are sufficiently close. For example, the following result holds.

Theorem 6.5. *Assume that F is uniformly elliptic and $\Lambda/\lambda \leq 1+\varepsilon_0$ for a small constant ε_0 (depending on dimension only). The solutions to the equation*

$$F(D^2u) = 0 \text{ in } B_1$$

are always $C^{2,\alpha}$ for some $\alpha > 0$. Moreover, an estimate holds

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Here C and α depend only on dimension and the ellipticity constants λ, Λ.

The previous result is a consequence of a classical estimate by Cordes and Nirenberg. It is unfortunately difficult to find a reference for an explicit proof of the result as stated.

In the example by Nadirashvili, Tkachev and Vladuts [6], the ratio Λ/λ is approximately 10000.

6.3. **Some ideas used for regularity results.** Let us start by discussing a few ideas related to Theorem 6.4.

Assume we have a solution of $F(D^2u) = 0$ and that both u and F are smooth. If we want to obtain an a priori estimate for the derivatives of u, a natural idea is to differentiate the equation and look for an equation satisfied by the derivatives of u. Let $v = \partial_e u$, we get [5]

$$\frac{\partial F}{\partial X_{ij}}(D^2u)\partial_{ij}v = 0$$

We want to use this equation to get an estimate for v in C^α. It is convenient to call $a_{ij}(x) = \frac{\partial F}{\partial X_{ij}}(D^2u(x))$, so that we write

$$a_{ij}(x)\partial_{ij}v(x) = 0.$$

Because of the uniform ellipticity assumption, we know that for all x,

$$\lambda I \leq \{a_{ij}(x)\} \leq \Lambda I. \tag{6.1}$$

Since the coefficients $a_{ij}(x)$ depend on the value of $D^2u(x)$, we cannot assume any a priori regularity for them. Indeed, we intend to use this approach to prove a modulus of continuity for ∇u. We cannot use anything about D^2u. The convenience of our assumptions is that (6.1) holds regardless of the value of $D^2u(x)$.

The key to prove Theorem 6.4, is to prove the following theorem first about equations with *rough* coefficients.

Theorem 6.6. *Assume that $v \in C^2$ solves*

$$a_{ij}(x)\partial_{ij}v(x) = 0 \qquad \text{in } B_1. \tag{6.2}$$

No regularity is assumed for the coefficients a_{ij}, only the uniform ellipticity condition

$$\lambda I \leq \{a_{ij}(x)\} \leq \Lambda I. \tag{6.3}$$

Then v satisfies

$$\|v\|_{C^\alpha(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}.$$

Theorem 6.6 is very difficult to prove. It is a version of DeGiorgi-Nash-Moser theorem, in non-divergence form.

We do not assume any regularity for the coefficients a_{ij}. The result holds even if the coefficients are discontinuous.People commonly stress this fact by saying that the coefficients are only assumed to be *measurable*. Amusingly, the measurability of the coefficients is not at all used in the proof and is therefore not necessary either.

We want to reformulate Theorem 6.6 is a form suitable for viscosity solution methods. Our definition of viscosity solutions does not allow us to consider equations like (6.2) if the coefficients are discontinuous. In order to overcome that problem, we define the *Pucci* operators.

[5] We use the convention that repeated indexes denote summation

Definition 6.7. *Let $P^+, P^- : \mathbb{R}^{n\times n} \to \mathbb{R}$ be the following nonlinear functions*

$$P^+(M) = \sup \left\{ \sum_{ij} a_{ij} M_{ij} : \lambda I \leq \{a_{ij}\} \leq \Lambda I \right\},$$

$$P^-(M) = \inf \left\{ \sum_{ij} a_{ij} M_{ij} : \lambda I \leq \{a_{ij}\} \leq \Lambda I \right\}.$$

Now, assuming that (6.2) holds for some uniformly elliptic coefficients $a_{ij}(x)$ is exactly the same as assuming the two inequalities $P^+(D^2v) \geq 0$ and $P^-(D^2v) \leq 0$. Indeed, for all values of x we have that $P^+(D^2v) \geq a_{ij}\partial_{ij}v \geq P^-(D^2v)$, so (6.2) implies the two inequalities for the Pucci operators. Conversely, if $P^+(D^2v) \geq 0$ and $P^-(D^2v) \leq 0$, then at every point x there are values of a_{ij} that would make $a_{ij}\partial_{ij}v(x)$ either positive or negative. We can therefore choose some intermediate value of $a_{ij}(x)$ so that $a_{ij}(x)\partial_{ij}v(x) = 0$.

The advantage of the two inequalities $P^+(D^2v) \geq 0$ and $P^-(D^2v) \leq 0$ over the equation (6.2) is that they are well defined in the viscosity sense. Thus, we reformulate Theorem 6.6 in the following way.

Theorem 6.8. *Assume v is a continuous function in $\overline{B_1}$ which satisfies the following two inequalities in the viscosity sense*

$$P^+(D^2v) \geq 0 \qquad \text{and} \qquad P^-(D^2v) \leq 0 \qquad \text{in } B_1.$$

Then v is actually Hölder continuous C^α in $B_{1/2}$ for some $\alpha > 0$ (depending on λ, Λ and dimension). Moreover, an estimate holds

$$\|v\|_{C^\alpha(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}.$$

Here, the constant C depends on λ, Λ and dimension only.

Theorem 6.8 is the same as Theorem 6.6 in the case that $v \in C^2$. The only difference is that Theorem 6.8 can also be applied when we are working with a viscosity solution v that is not any more regular than continuous, a priori.

In general, it is fair to expect that a priori estimates for classical solutions that do not depend on the size of high derivatives of the solution (like Theorem 6.6) should also hold for viscosity solutions. The proofs are usually based on the same essential facts, but with a few more technical difficulties in the viscosity solution setting.

6.4. **Exercises.**

Exercise 6.1. *Let M be a symmetric matrix. Prove the following expression for the Pucci operators.*

$$P^+(M) = \Lambda\langle \text{sum of positive eigenvalues of } M\rangle + \lambda\langle \text{sum of negative eigenvalues of } M\rangle,$$

$$P^-(M) = \lambda\langle \text{sum of positive eigenvalues of } M\rangle + \Lambda\langle \text{sum of negative eigenvalues of } M\rangle$$

Exercise 6.2. *In terms of λ, Λ and n, find a number q so that the functions $u(x) = |x|^q$ solves $P^+(D^2u) = 0$ in $\mathbb{R}^n \setminus \{0\}$. Repeat the same question with P^-.*

Exercise 6.3. *Prove that a function $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ is uniformly elliptic as in definition 6.1 if for any two symmetric matrices X and Y we have*

$$P^-(Y) \leq F(X+Y) - F(X) \leq P^+(Y).$$

Exercise 6.4. *Let u_k be a bounded sequence of functions satisfying the inequalities*

$$P^+(D^2u_k) \geq 0 \qquad \text{and} \qquad P^-(D^2u_k) \leq 0 \qquad \text{in } B_1,$$

with ellipticity constants $\lambda = 1 - 1/k$ and $\Lambda = 1 + 1/k$.

Prove that the sequence u_k converges to a harmonic function uniformly over compact subsets of B_1.

Exercise 6.5. *Let $u, f : \overline{B_1} \to \mathbb{R}$ be a continuous functions satisfying the following inequalities in the viscosity sense*

$$P^+(D^2u) \geq f \qquad \text{and} \qquad P^-(D^2u) \leq f \qquad \text{in } B_1.$$

Prove that $u = u_1 + u_2$ where u_1 and u_2 satisfy

$$\begin{aligned} &P^+(D^2u_1) \geq 0 \qquad \text{and} \qquad P^-(D^2u_1) \leq 0 \qquad \text{in } B_1, \\ &P^+(D^2u_2) \geq f \qquad \text{and} \qquad P^-(D^2u_2) \leq f \qquad \text{in } B_1, \\ &u_2 = 0 \qquad \text{on } \partial B_1. \end{aligned}$$

Note. I don't know how to solve the previous exercise. Let me know if you find a valid proof.

Exercise 6.6. *Prove that Theorem 6.8 can be deduced from the following lemma.*

Lemma 6.9 (Improvement of oscillation)**.** *Assume v is a continuous function in $\overline{B_1}$ which satisfies the following two inequalities in the viscosity sense*

$$P^+(D^2v) \geq 0 \qquad \text{and} \qquad P^-(D^2v) \leq 0 \qquad \text{in } B_1.$$

Then

$$\operatorname*{osc}_{B_{1/2}} v \leq (1-\delta) \operatorname*{osc}_{B_1} v,$$

where $\delta > 0$ depends only on λ, Λ and the dimension n.

Hint. *Apply the lemma to rescaled versions of the function v and obtain a decay of $\operatorname{osc}_{B_r} v$ as $r \to 0$.*

Exercise 6.7. *Prove that the lemma of the previous exercise can be deduced from this other lemma.*

Lemma 6.10 (Weak Harnack inequality)**.** *Assume $u : B_1 \to \mathbb{R}$ is a nonnegative and upper semicontinuous supersolution of $P^-(D^2u) \leq 0$ in B_1. Assume also that*

$$|\{x \in B_1 : u(x) \geq 1\}| \geq \mu > 0.$$

Then $u \geq \delta > 0$ in $B_{1/2}$, where $\delta > 0$ depends only on μ, λ, Λ and the dimension n.

Note. The core of the proof of Theorem 6.8 is in the proof of Lemma 6.10.

7. Nonlocal equations

This section focuses on nonlocal equations. A good source for a variety of results about nonlocal equations is the wiki page [2], in particular the lecture notes from another summer course [7].

In sections 2.1 and 3.1 we discussed how linear and nonlinear elliptic equations can be derived from a random walk problem as a scale parameter r converges to zero. The equations that we consider in this section include also those integro-differential equations that we have before we take the limit $r \to 0$. Second order elliptic equations are extremal cases of nonlocal equations.

In the linear case, we would want to study equations of the general form

$$a_{ij}\partial_{ij}u + b \cdot \nabla u + \int_{\mathbb{R}^n} (u(x+h) - u(x) - h \cdot \nabla u(x) \mathbb{1}_{B_1}(h)) \, K(h) \, \mathrm{d}h = 0 \qquad \text{for all } x \in \Omega.$$

The operators on the left hand side appear in probability theory as the generators of *Levy processes*. These are generalizations of diffusions, of Brownian motion, that are also allowed to have discontinuities. These random processes *jump* from a point to another point far away. The kernel K indicates the frequency of these jumps.

Naturally, we could also consider the equations where the coefficients a_{ij}, b or the kernel K depend on x. We can also build nonlinear problems as in the Bellman or Isaacs equations that correspond to stochastic control or stochastic game problems.

From a purely analytic point of view, these are the only equations that satisfy the global maximum principle. In order to explain this properly, we will give some definitions first.

Definition 7.1. *Let F be an arbitrary map which takes bounded functions on $\mathbb{R}^n$, $C^2(\Omega)$, and gives us a continuous function in $C(\Omega)$. We say this map is elliptic if any time two functions u and v satisfy*

$$\max u - v = u(x_0) - v(x_0) \geq 0 \text{ for some point } x_0 \in \Omega,$$

then $F[u](x_0) \leq F[v](x_0)$.

A particular case are the operators we were dealing with before

$$F[u](x) = F(D^2u, \nabla u, u, x),$$

with F elliptic and proper as before. The difference is that nonlocal operators F are allowed to depend on all the values of u and not only the ones in a neighborhood of x.

The Dirichlet problem for a nonlocal equation would have the following form

$$\begin{aligned} F[u] &= 0 \text{ in } \Omega, \\ u &= f \text{ in } \mathbb{R}^n \setminus \Omega. \end{aligned}$$

Note that the boundary condition is given in the whole complement of the domain of the equation Ω. It would make no sense to give boundary values on $\partial\Omega$ only. This is because the values of u outside Ω will influence the values of $F[u]$ inside Ω. In terms of the models in probability, it is because Levy processes may exit the domain Ω jumping to any point outside.

There is an old result by Courrege (1965) which says that the only linear operators which are elliptic in the sense of Definition 7.1 have the form

$$F[u](x) = a_{ij}(x)\partial_{ij}u + b(x)\cdot\nabla u - c(x)u + \int_{\mathbb{R}^n} (u(x+h) - u(x) - h\cdot\nabla u(x)\mathbb{1}_{B_1}(h))\, K(x,h)\, \mathrm{d}h. \tag{7.1}$$

Here, for all $x \in \Omega$, $a_{ij}(x)$ is a positive matrix, $b(x)$ is an arbitrary vector, $c(x) \leq 0$, and the kernel $K(x,h) \geq 0$ must satisfy

$$\int_{\mathbb{R}^n} K(x,h)\min(1,|h|^2)\, \mathrm{d}h < +\infty. \tag{7.2}$$

(Actually the kernel $K(x,h)$ should be understood as a nonnegative measure in h which depends on x and could be singular)

The condition (7.2) is what guarantees that the integral in the definition of $F[u]$ converges if $u \in C^2$ and bounded. Indeed, for h large

$$(u(x+h) - u(x) - h\cdot\nabla u(x)\mathbb{1}_{B_1}(h)) = u(x+h) - u(x) \text{ is bounded.}$$

For h small,

$$(u(x+h) - u(x) - h\cdot\nabla u(x)\mathbb{1}_{B_1}(h)) \approx \langle D^2u(x)h, h\rangle = O(|h|^2).$$

The term $h\cdot\nabla u(x)$ is there to provide the necessary cancellation around the origin $h = 0$.

The nonlinear version of Courrege theorem was recently obtained by Guillen and Schwab under the extra hypothesis that $F[u]$ is *Frechet differentiable*. It says that all nonlinear elliptic operators F have the form

$$F[u](x) = \min_a \max_b L_{ab}u(x),$$

where L_{ab} is a family of linear operators of the form (7.1). Interestingly, these are the operators which correspond to the general Isaacs equation driven by Levy processes.

The natural definition of viscosity solution follows.

Definition 7.2. *We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies the inequality $F[u] \geq 0$ in Ω (subsolution) in the viscosity sense if u is upper semicontinuous in Ω and every time there exists a function φ which is C^2 in a neighborhood of x, $\varphi \geq u$ in $\mathbb{R}^n$, and $\varphi(x) = u(x)$, then $F[\varphi](x) \geq 0$.*

The opposite inequality (supersolution) is defined reversing all inequalities. A solution is a function which is a sub- and supersolution at the same time.

The definition is essentially the same as the one given before. The only detail to take into account is that the test function φ must be defined and satisfy the inequality $\varphi \geq 0$ in the full space $\mathbb{R}^n$, but it is only required to be C^2 in a neighborhood of the touching point x. This is convenient because, for all practical purposes, we only need to consider test functions φ which are equal to u outside of some neighborhood of x.

The theory of viscosity solutions, and several regularity results for classical elliptic PDE, have natural extensions to nonlocal equations.

The Laplace operator is the first example of a classical elliptic operator. In the same fashion, the first example of a nonlocal operator is the fractional Laplacian. For $s \in (0,1)$, we define

$$\Delta^s u(x) = c_{n,s}\int_{\mathbb{R}^n} (u(x+h) - u(x))\frac{1}{|h|^{n+2s}}\, \mathrm{d}h.$$

In this case we are omitting the correction term $h \cdot \nabla u(x) \mathbb{1}_{B_1}(h)$ because it is odd and would integrate to zero. Nonetheless, if $s \geq 1/2$, this term should be there implicitly to ensure the convergence of the integral around the origin.

The name is motivated from the following identity after taking the Fourier transform

$$\widehat{\Delta^s u}(\xi) = -|\xi|^{2s} \hat{u}(\xi). \tag{7.3}$$

The constant $c_{n,s}$ must be chosen so that the identity above holds.

When $s \to 1$, we recover the usual Laplacian. Here, the number s should be understood simply as a parameter. The operator Δ^s, is a nonlocal elliptic operator of order $2s$.

7.1. **Regularity issues.** A key concept in the development of regularity results for classical elliptic equations was *uniform ellipticity*, given in Definition 6.1. For linear equations, the uniform ellipticity tells us that the coefficients $\{a_{ij}\}$ are comparable at every point with the identity matrix (as in (6.1)). For nonlocal equations, we may define uniform ellipticity of order $\sigma \in (0, 2)$ by requiring our operators to be comparable to the operators of the fractional Laplacian $\Delta^{\sigma/2}$.

Consider a linear nonlocal equation of the form

$$\int_{\mathbb{R}^n} \left(u(x+h) - u(x) - h \cdot \nabla u(x) \mathbb{1}_{B_1}(h) \right) K(x,h) \, \mathrm{d}h = 0.$$

In order to simplify our notation, let us make the symmetry assumption $K(x,h) = K(x,-h)$. Thus, the term $h \cdot \nabla u(x) \mathbb{1}_{B_1}(h) K(x,h)$ is odd in h and should integrate to zero. We rewrite the equation as

$$PV \int_{\mathbb{R}^n} \left(u(x+h) - u(x) \right) K(x,h) \, \mathrm{d}h = 0.$$

The integral may be singular at the origin now. We must understand it in the principal value sense.

At this point, it makes sense to say that the linear integro-differential equation will be uniformly elliptic of order $\sigma \in (0,2)$ if the kernel $K(x,h)$ is comparable to the kernel of Δ^s. That means that there are constants $\Lambda \geq \lambda > 0$ such that

$$\lambda \frac{c_{n,s}}{|h|^{n+\sigma}} \leq K(x,h) \leq \Lambda \frac{c_{n,s}}{|h|^{n+\sigma}}. \tag{7.4}$$

Although this choice seems natural, it is somewhat arbitrary. The class of nonlocal operators is very rich. While a second order equation consists of a matrix $\{a_{ij}\} \in \mathbb{R}^{n \times n}$ for every point x, a nonlocal equations may have any kernel function $K(x, \cdot)$ at every point x. There are many variations.

We write a version of Theorem 6.6 for nonlocal equations.

Theorem 7.3. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution to the equation*

$$PV \int_{\mathbb{R}^n} \left(u(x+h) - u(x) \right) K(x,h) \, \mathrm{d}h = 0 \qquad \text{for all } x \in B_1.$$

We assume that $K(x,h) = K(x,-h)$ and that K satisfies (7.4). There is no regularity assumption of K with respect to x or h. Then, the following estimate holds

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}.$$

The constants $\alpha > 0$ and C depend on λ, Λ, the dimension n and σ. These constants degenerate as $\sigma \to 0$ but not as $\sigma \to 2$.

We define $\mathcal{L}_0$ to be a class of nonlocal operators. We say $L \in \mathcal{L}_0$ if the operator L has the form

$$L[u](x) = PV \int_{\mathbb{R}^n} \left(u(x+h) - u(x) \right) K(h) \, \mathrm{d}h,$$

where $K(h) = K(-h)$ and

$$\lambda \frac{c_{n,\sigma/2}}{|h|^{n+\sigma}} \leq K(h) \leq \Lambda \frac{c_{n,\sigma/2}}{|h|^{n+\sigma}}.$$

Using this class $\mathcal{L}_0$, we define a nonlocal version of the Pucci operators.

$$M^+_{\mathcal{L}_0}[u](x) = \sup\{L[u](x) : L \in \mathcal{L}_0\},$$
$$M^-_{\mathcal{L}_0}[u](x) = \inf\{L[u](x) : L \in \mathcal{L}_0\}.$$

For nonlocal equations, the extremal operators $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$ play the same role as the Pucci operators do for classical elliptic equations. In particular, we can rewrite the theorem above as

Theorem 7.4. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function that satisfies the following two inequalities in the viscosity sense*

$$\begin{aligned} M^+_{\mathcal{L}_0}[u](x) &\geq 0 \qquad \text{in } B_1, \\ M^-_{\mathcal{L}_0}[u](x) &\leq 0 \qquad \text{in } B_1. \end{aligned}$$

Then $u \in C^\alpha(B_{1/2})$ for some $\alpha > 0$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

The constants $\alpha > 0$ and C depend on λ, Λ, the dimension n and σ. These constants degenerate as $\sigma \to 0$ but not as $\sigma \to 2$.

Theorem 7.4 is a nonlocal version of Theorem 6.8, whereas Theorem 7.3 is a nonlocal version of Theorem 6.6. The results are very similar. The most remarkable difference is that the right hand side of the estimate depends on the supremum of $|u|$ in the full space $\mathbb{R}^n$ instead of just the domain of the equation. This is a natural consequence of the nonlocality of the equation. All points in $\mathbb{R}^n$ are visible by the equation from the inside of B_1.

See [1] for a list of variations of the theorems above.

We also use the extremal operators $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$ to define the uniform ellipticity of an arbitrary nonlocal operator.

Definition 7.5. *We say that a nonlocal operator F is elliptic with respect to the class $\mathcal{L}_0$ if for any two functions u and v, bounded in $\mathbb{R}^n$ and C^2 around the point x, we have*

$$M^-_{\mathcal{L}_0}[v](x) \leq F[u+v](x) - F[u](x) \leq M^+_{\mathcal{L}_0}[v](x).$$

Using this definition, we can state a nonlocal version of Theorem 6.4.

Theorem 7.6. *Let F be a nonlocal operator, elliptic with respect to $\mathcal{L}_0$ and translation invariant (i.e. $F[u(\cdot + h)](x) = F[u](x+h)$). If $u : \mathbb{R}^n \to \mathbb{R}$ is a bounded function that satisfies*

$$F[u] = 0 \text{ in } B_1 \text{ in the viscosity sense.},$$

then $u \in C^{1,\alpha}(B_1)$ for some $\alpha > 0$. Moreover, an estimate holds

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

Here C and α depend only on dimension and the ellipticity constants λ, Λ and σ.

7.2. **Exercises.**

Exercise 7.1. *Prove that the statement of Proposition 3.6 also holds when F is an arbitrary nonlocal elliptic operator as in Definition 7.1 and $\limsup^* u_k = u$ holds in the full space $\mathbb{R}^n$.*

Exercise 7.2. *We say that a sequence of nonlocal operators F_k converges weakly to F if for any bounded function $\varphi : \mathbb{R}^n \to \mathbb{R}$ which is C^2 in a ball $B_r(x) \subset \Omega$, we have $F_k[\varphi] \to F[\varphi]$ uniformly in $B_{r/2}(x)$.*

Assume that $u_k : \mathbb{R}^n \to \mathbb{R}$ is a uniformly bounded sequence of upper semicontinuous functions so that

- $\limsup^* u_k \to u$ *in* $\mathbb{R}^n$.
- $F_k[u] \geq 0$ *in the viscosity sense in* Ω.
- $F_k \to F$ *weakly.*

Prove that $F[u] \geq 0$ in Ω in the viscosity sense.

Exercise 7.3. *Let $k : \partial B_1 \to \mathbb{R}$ be a continuous function. Prove that there exists a positive matrix $\{a_{ij}\} \in \mathbb{R}^{n\times n}$ so that, for any bounded, C^2 function u,*

$$a_{ij}\partial_{ij}u(x) = \lim_{\sigma\to 2}(2-\sigma)\int_{\mathbb{R}^n} (u(x+h) - u(x) - h\cdot\nabla u(x)\mathbb{1}_{B_1}(h))\,\frac{k(h/|h|)}{|h|^{n+\sigma}}\,\mathrm{d}h.$$

Exercise 7.4. *Prove that the formula (7.3) holds.*
Note. *This computation is not as easy as it seems. It requires good Fourier analysis skills.*

Exercise 7.5. *Prove the following expression for* $M^+_{\mathcal{L}_0}$

$$M^+_{\mathcal{L}_0}[u](x) = \lambda \Delta^s u(x) + \frac{(\Lambda - \lambda)}{2} \int_{\mathbb{R}^n} (u(x+h) + u(x-h) - 2u(x))^+ \frac{c_{n,\sigma/2}}{|h|^{n+\sigma}} \,\mathrm{d}h.$$

The superscript + denotes the positive part (i.e. $a^+ = (a + |a|)/2$*).*

Exercise 7.6. *Let* F *be a nonlocal elliptic operator as in Definition 7.1. Assume that for any two bounded functions* u, v *which are* C^2 *in a neighborhood of* x*, we have*

$$P^+(D^2 v(x)) \geq F[u+v](x) - F[u](x) \geq P^-(D^2 v(x)).$$

Prove that then F *is a classical uniformly elliptic operator of the form* $F[u](x) = F(D^2 u, x)$*.*

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