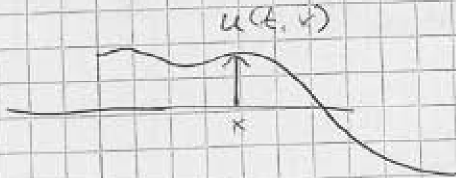


d'Alement

$F = ma$

$u_{tt} = u_{xx}$

$$\boxed{u_{tt} - u_{xx} = 0}$$

$$\boxed{\begin{array}{l} u(0, x) = f(x) \\ u_t(0, x) = g(x) \end{array}}$$

Cauchy problem

<u>Ask for:</u>

Existence

Uniqueness

continuous

depence ondata

$f, g \in C^{\infty}(\mathbb{R})$

$$\Box = \partial_{tt} - \partial_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x) \underset{\substack{\uparrow \\ \text{commute}}}{=} (\partial_t + \partial_x)(\partial_t - \partial_x)$$

$\Box u = 0 \qquad (\partial_t - \partial_x) v(x, t) = 0$

$$v(0, x) = \phi(x)$$

$$\Rightarrow \quad v(t, x) = \phi(t + x)$$

$$u(t, x) = \frac{1}{2}\left(f(x+t) + f(x-t)\right)$$

$u_t(t, x) = 0$

$u(0, x) = f$

$\Box u = 0 \qquad$ OK if $g \equiv 0$

If $g \neq 0$, we set $w = u_t$
$\quad t=0$

note: $\Box w = 0$

$w(0,x) = g(x)$

$w_t(0,x) = u_{tt}(0,x) = u_{xx}(0,x) = f''(x) = 0$

Super Position principle:
sum of two solutions is another solution

Exercise: $w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy$

$$u(t,x) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy$$

Uniqueness $\quad E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(u_x^2 + u_t^2\right)(t,x)\,dx_{\text{energy}}$

$$\Rightarrow E'(t) = \int_{-\infty}^{\infty} \left(u_x u_{xt} + u_{tt} u_t\right)(t,x)\,dx =$$

$$= \int_{-\infty}^{\infty} \left(-u_{xx} + u_{tt}\right) u_t(t,x)\,dx = 0$$

imagine we have two solutions u & $\tilde{u}$ to CP

$\Rightarrow \Box(u - \tilde{u}) = 0$

$(u - \tilde{u})\big|_{t=0} = 0$

$\partial_t (u - \tilde{u})\big|_{t=0} = 0$

But then energy of $u-\tilde{u} \equiv 0$

To fix this (knowing that the energy $\underline{\underline{\text{is}}}$ finite)

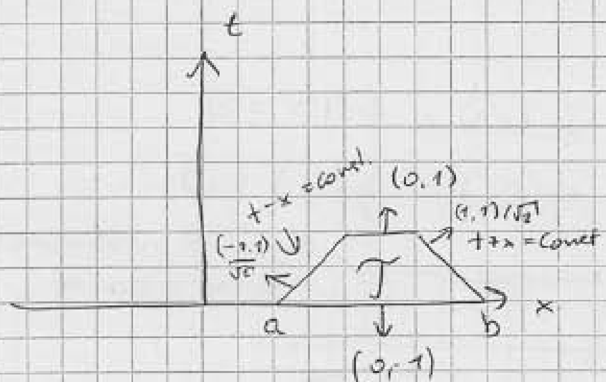
$$\left(\int_{-\infty}^{\infty} (f')^2 + g^2 \, dx < \infty \right)$$

Idea is: Localise

$e(t,x) = \frac{1}{2}\left(u_t^2 + u_x^2\right)$ energy density

$$\partial_t e = u_t\left(u_{tt} - u_{xx}\right) + \underbrace{u_t u_{xx} + u_x u_{xt}}_{(u_t u_x)_x}$$

$$\operatorname{div}_{t,x}\left(e, -u_t u_x\right)$$



$$\cancel{\iint_T \operatorname{div}_{t,x}\left(e, -u_t u_x\right) = 0}$$

$$0 = \iint_T \operatorname{div}_{x,t}\left(-u_t u_x, e\right) dx\,dt =$$

$$= \int_{a+T}^{b-T} e(x,T)\,dx - \int_a^b e(x,0)\,dx + \frac{1}{\sqrt{2}} \int_{\text{rhs}} (e - u_t u_x)\,ds$$

$$+ \frac{1}{\sqrt{2}} \int_{\text{Lhs}} \left(e + u_t u_x\right) ds$$

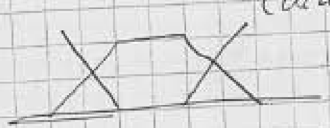
$$e \pm u_t u_x = \frac{1}{2}\left(u_t^2 + u_x^2 \pm 2 u_t u_x\right) = \frac{1}{2}\left(u_t \pm u_x\right)^2 \geq 0$$

FLUX ≥ 0

$$\Rightarrow \int_{a+T}^{b-t} e(x,T)\,dx \leq \int_a^b e(x,0)\,dx$$

$\partial_t(u - \tilde{u}) = 0$

$u - \tilde{u} = 0$



$\Rightarrow$ uniqueness

$\Rightarrow$ cont. dep on data

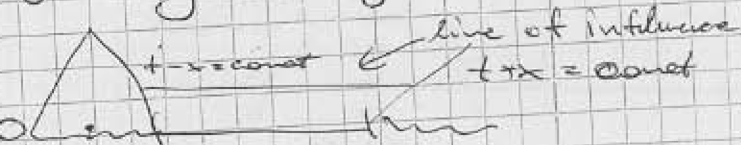
$u(0,t) = 0 \quad u(L,t)$

$\forall t > 0$

Boundary Conditions

$u_{tt} - u_{xx} = 0 \qquad f(0) = f(L) = 0$

$u|_{t=0} = f, \ \partial u|_{t=0} = g \qquad g(0) = g(L) = 0$



we make $f = g = 0$

$$E(t) = \int_{-\infty}^{\infty} e(x,t)\,dx$$

$u \equiv 0$

duolingo

Fourier Transform

$$f(x) = \sum_{n=1}^{\infty} \hat{f}(n) \sin\left(n \frac{\pi}{L} x\right)$$

$$g(x) = \sum_{n=1}^{\infty} \hat{g}(n) \sin\left(n \frac{\pi}{L} x\right) \qquad L = 1$$

$$\hat{f}(n) = \int_0^1 f(x) \sin(n\pi x)\, dx$$

$$(n\pi)^2 \hat{f}(n) = \int_0^1 f(x) \left(-\frac{d}{dx}\right)^2 \sin(n\pi x)\, dx = \int_0^1 f''(x) \sin(n\pi x)\, dx$$

$$\Rightarrow \pi^2 n^2 |\hat{f}(n)| \leq \int_0^1 |f''(x)|\, dx$$

$$u(x,t) = \sum_{n=1}^{\infty} e_n(t) \sin(n\pi x)$$

$\left\{\sin(n\pi x)\right\}_{n=1}^{\infty}$ ON-base in Hilbert space $L^2([0,1])$

$-\frac{d^2}{dx^2} = A \qquad \langle Af, g\rangle_{L^2} = (f, Ag)$

$f, g \in C^2([0,1]) \quad f = g = 0$

at $x = 0, 1$

$$A e_n = n^2 \pi^2 e_n$$

$$\underset{\partial_t^2 + A}{\underset{\|}{\Box}} u = \sum_{n=1}^{\infty} \underbrace{\left(\ddot{a}_n(t) + n^2\pi^2 a_n(t) \right)}_{= 0} e_n(x)$$

$\ddot{a}_n(t) + n^2\pi^2 a_n(t) = 0$

$a_n(0) = \hat{f}(n)$

$\dot{a}_n(0) = \hat{g}(n)$

Fundamental system
$\cos(n\pi t)$, $\sin(n\pi t)$

$$a_n(t) = \hat{f}(n)\cos(n\pi t) + \frac{\hat{g}(n)}{n\pi}\sin(n\pi t)$$

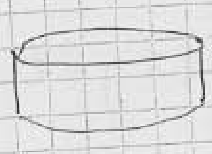
$$g = 0 \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \hat{f}(n) \underbrace{\cos(n\pi t)\sin(n\pi x)}_{\frac{1}{2}\left(\sin(n\pi(x+t)) + \cos(\pi n(x-t))\right)}$$

$$= \frac{1}{2}\left(f(x+t) + f(x-t) \right)$$

$f(t) = f(-t)$
(odd extension)

$f(1+t) + f(1-t) = 0$

Two-dim wave eq.



$u_{tt} - \Delta u = 0$

$$\text{CP}\begin{cases}(t,x)\in\mathbb{R}^{1+d} & u_{tt}-\Delta u=0\\ u|_{t=0}=f\\ \partial_t u|_{t=0}=g\end{cases}$$

We use $d=3$ ~~for simplicity~~

Kirchhoff formula in $d=3$

$$u(x,t)=\partial_t\left(\frac{1}{4\pi t}\int_{x+t\,\mathbb{S}^2} f(y)\,\sigma(dy)\right)+\frac{1}{4\pi t}\int_{x+t\,\mathbb{S}^2} g(y)\,\sigma(dy)$$

surface area on the sphere

$\boxed{f=0}$ $\quad u(x,t)=t\displaystyle\fint_{x+t\,\mathbb{S}^2} g\,d\sigma$

Problem # 1 $\quad f=0,\ g\in C_0^\infty(\mathbb{R}^3)$

Show that $\hat{u}(t,\xi)=\displaystyle\int_{\mathbb{R}^3} e^{-ix\cdot\xi}\,u(t,x)\,dx$

Satisfies $\partial_t^2\hat{u}(t,\xi)+|\xi|^2\hat{u}(t,\xi)=0$

$\hat{u}(0,\xi)=0$

$\partial_t\hat{u}(0,\xi)=\hat{g}(\xi)$

$$\Rightarrow \hat{u}(t,\xi)=\frac{\sin(t|\xi|)}{|\xi|}\,\hat{g}(\xi)$$

Show $\frac{1}{4\pi}\hat{\sigma}_{S^2}(\xi) = \frac{\sin(|\xi|)}{|\xi|}$

then conclude that

$$F^{-1}\left(\frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi)\right) = \frac{1}{4\pi t}\,\sigma_{tS^2} * g\,(x) \quad \forall t>0$$

$$= \frac{1}{4\pi t}\int_{tS^2} g(x-y)\,\sigma_{tS^2}(dy)$$

Problem #2. Let u be a radial solution of $\Box u = 0$ in $\mathbb{R}^{1+3}$

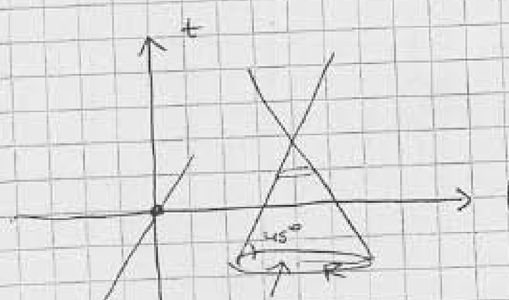
Show that $ru(t,\overset{|x|}{\overset{\shortparallel}{r}})$ solves the 1-dim $\Box = 0$

$$e(t,x) = \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)(t,x)$$

$$\partial_t e = u_t u_{tt} + \nabla u \cdot \nabla u_t$$

$$= u_t(u_{tt} - \Delta u) + \underbrace{u_t \operatorname{div}(\nabla u) + \nabla u \cdot \nabla u_t}_{\operatorname{div}(u_t \nabla u)}$$

$$\operatorname{div}_{x,t}\left(-u_t \nabla u, e\right) = u_t \Box u = 0$$



$$\int_{\mathcal{R}} \operatorname{div}_{x,t}(-u_t \nabla u, e)\,dx\,dt =$$

$\overset{=}{\text{div thm}} \quad \int_{\text{top}} e - \int_{\text{bottom}} e \quad + \text{flux}$

$$\begin{cases} \Box u = 0 \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = g \end{cases} \qquad f, g \in C_0^\infty \ (\text{or } \mathcal{S})$$

$$\widehat{u(t, \xi)} = \int_{\mathbb{R}^d} u(t, x) e^{-ix\xi} \, dx$$

$$\partial_t^2 \widehat{u(t, \xi)} + |\xi|^2 \widehat{u(t, \xi)} = 0$$

$$\widehat{u(0, \xi)} = \hat{f}(\xi)$$

$$\partial_t \widehat{u(0, \xi)} = \hat{g}(\xi)$$

fundamental system

$$e^{\pm it|\xi|}$$

$$\widehat{u(t, \xi)} = c_+ e^{it|\xi|} + c_- e^{-it|\xi|}$$

$$= \hat{f}(\xi) \cos(t|\xi|) + \hat{g}(\xi) \sin(t|\xi|)$$

$$u(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} \cos(t|\xi|) \hat{f}(\xi) \, d\xi$$

$$(*) \qquad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi) \, d\xi$$

Problem #3 Show that for $f, g \in \mathcal{S}(\mathbb{R}^d)$ $(*)$ defines a solution $u \in C^\infty(\mathbb{R}^{1+d}_{t,x})$

and $x \mapsto u(t, x) \in \mathcal{S}(\mathbb{R}^d)$

and $u(0) = f$, $\partial_t u(0) = g$ (In 1d get D back)

Problem #4 $\quad d=2$

compute the Fourier free version of $(*)$

$$u(t,x) = \frac{1}{\pi} \int_{|x-y|\le t} \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\, dy = (K_t * g)(x)$$

K_t: fundamental solution

$$0 = \int_{|x-x_0|\le R-T} e(x,T)\,dx - \int_{|x-x_0|\le R} e(x,0)\,dx$$ ← see cone picture

$$+\frac{1}{\sqrt{2}} \int_{\text{mantle}} e - u_t \nabla u \cdot \vec{v}$$

$$e = \frac{1}{2}(u_t^2 + |\nabla u|^2) = \frac{1}{2}\left| u_t \vec{v} - \nabla u\right|^2 + u_t \nabla \vec{u} \cdot \vec{v}$$

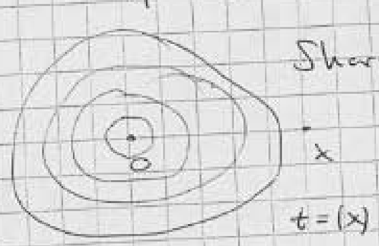
$$\int_{|x-x_0|\le R-T} e(x,T)\,dx = \int_{|x-x_0|\le R} e(x,0)\,dx - \underbrace{\text{flux}}_{}$$

$$\text{flux} = \frac{1}{2\sqrt{2}} \int_{\text{mantle}} |u_t \vec{v} - \vec{\nabla} u|^2\, d\sigma \ge 0$$

$$E(T) \le E(0)$$

$$f=0 \qquad u(t,x) = \frac{1}{4\pi t}\, \sigma_{tS^2} * g\,(x)$$

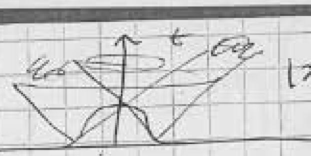
Sharp Huygens Principle



~~dispersion~~ dispersion:

(heuristic)

$d \geq 2$



$|x| = t \pm R$

$\mathbb{R}^d$

$\text{area}(t\,S^2)\cdot 2R \approx t^2 \quad (t^{d-1})$

$$\|u(t,\cdot)\|_\infty^2 \; t^2 \approx \text{const}$$

$$\|u(t,\cdot)\|_\infty \lesssim \frac{1}{t} \qquad \left(\frac{1}{t^{\frac{d-1}{2}}}\right)$$

Proposition: Let u be a C^∞ solution of $\Box u = 0$ in $\mathbb{R}^{1+3}_{t,x}$ data $(u|_{t=0}, u_t|_{t=0}) = (f,g) \in \mathcal{J}$

$$\Rightarrow \|u(t,\cdot)\|_\infty \leq \frac{C}{t}\left(\|D^2 f\|_{L^1(\mathbb{R}^3)} + \|\nabla g\|_{L^1}\right) \qquad C \text{ - absolute constant}$$

$\uparrow \; \sum \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^1}$

Proof: let $f = 0$

$$u(t,x) = \frac{1}{4\pi t}\int_{x+tS^2} g(y)\,\sigma(dy) =$$

$$= \frac{1}{4\pi t}\int_{x+tS^2} \underbrace{g(y)\frac{y-x}{t}}_{v(y)} \cdot \vec{n}(y)\,\sigma(dy) =$$

$\operatorname{div} \vec{v} = \nabla g(y)\cdot \frac{y-x}{t} + \frac{3}{t} g$

$$= \frac{1}{4\pi t}\int_{|y-x|\leq t} \operatorname{div}(v(y))\,dy =$$

$$= \overbrace{\frac{1}{4\pi t}\int_{|x-y|\leq} \nabla g(y)\cdot \frac{y-x}{t}\,dy}^{A} + \overbrace{\frac{3}{4\pi t^2}\int_{|x-y|\leq t} g(y)\,dy}^{B}$$

$$|A| \leq \frac{1}{4\pi t}\int |\nabla g(y)| \, \Big|\overbrace{\frac{y-x}{t}}^{\leq 1}\Big|\, dy \leq \frac{1}{4\pi t}\|\nabla g\|_{L^1(\mathbb{R}^3)}$$

$$|B| \le \frac{3}{4\pi t^2}\left(\int_{\mathbb{R}^3} |g(y)|^{3/2}\,dy\right)^{3} \left(\int_{|x-y|\le t} 1^3\right)^{1/3} \le$$

Hölder $p = \frac{3}{2}$ $p' = 3$

$\frac{4}{3}\pi t^3$

$$\le \frac{C}{t}\|g\|_{L^{3/2}(\mathbb{R}^3)}$$

By Sobolev embedding:

$$g \in \mathcal{S} \Rightarrow \|g\|_{L^{3/2}(\mathbb{R}^3)} \le C\|\nabla g\|_{L^1}$$

$$\|f\|_{L^{3/2}} \le C\|Df\|_{L^1}$$

$$\|g\|_{L^3} \le C\|Dg\|_{L^{3/2}} \le C\|D^2 g\|_{L^1}$$

Under the scaling symmetry

$f \rightsquigarrow f_\lambda$ $\qquad \lambda > 0$

$g \rightsquigarrow g_\lambda$ $\qquad x \mapsto x\lambda \quad \lambda = e^\sigma$

$$u(t,x) \rightsquigarrow u(\lambda t, \lambda x) := u_\lambda$$

$$\Box u = 0 \iff \Box u_\lambda = 0 \iff \lambda^2(\Box u)_\lambda = 0$$

$$\Rightarrow u_\lambda(0,x) = u(0,\lambda x) = f_\lambda(x) = f(\lambda x)$$

$$\partial_t(u_\lambda)(0,x) = \lambda g_\lambda(x) \qquad (Dg_\lambda)(x) = \lambda(Dg)_\lambda(x)$$

$$\underbrace{\|(u_\lambda)(t,\cdot)\|_\infty}_{\|u(\lambda t,\cdot)\|_\infty} \le \frac{C}{t}\|D^2 f_\lambda\|_{L^1} + \|\underbrace{D(\lambda g_\lambda)}\|_{L^1}$$

$$\lambda^2\|(Dg)_\lambda\|_{L^1} =$$

$$= \lambda^2\lambda^{-3}\|Dg\|_{L^1} = \frac{1}{\lambda}\|Dg\|_{L^1}$$

$$\|\varphi_\lambda\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\varphi(\lambda x)|\,dx = \lambda^{-3}\|\varphi\|_{L^1}$$

Start on $\mathbb{R}$

Claim $\begin{cases} \|f\|_{L^\infty} \le C\|f'\|_{L^1} \\ f \in \mathcal{S}(\mathbb{R}) \end{cases}$

$$f(x) = \int_{-\infty}^{x} f'(x)\,dx$$

$$|f(x)| \le \int_{-\infty}^{\infty} |f'(x)|\,dx$$

$$\overline{X = \{ f \in \mathcal{S}(\mathbb{R}) \mid \|f'\|_{L^1} < \infty \}}$$ & $\|f'\|_{L^1}$ $\in$ completion

Question: What exactly is this complete X

$f_n \in \mathcal{S}(\mathbb{R})$ for $f_n \in C^1$ with cpt support

Cauchy relative to $\|f_n' - f_m'\|_{L^1(\mathbb{R})} \to 0$ as $m,n \to \infty$

$f_n' \to h \in C^1$

$\|f_n - f_m\|_\infty \le \|f_n' - f_m'\|_{L^1} \to 0$ as $n,m \to \infty$

$f_n \to f \in L^\infty \cap C^0$

AC - absolutely continuous

$$f_n(x) = \int_{-\infty}^{x} f_n'(u)\,du$$

$$\downarrow \qquad\qquad \downarrow$$

$$f(x) = \int_{-\infty}^{x} h(u)\,du \quad \forall x$$

$\Rightarrow f \in AC(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$

AC: $\forall \varepsilon > 0 \ \exists \delta > 0$ if $\sum_i |x_i - y_i| < \delta$

$$\Rightarrow \sum_i \left| f(x_i) - f(y_i) \right| < \varepsilon$$

Theorem: $X = \left\{ f \in AC(\mathbb{R}) \,\middle|\, f' \in L^1(\mathbb{R}) \right\} = W^{1,1}(\mathbb{R})$

Sobolev space

cut off fcn

Example: $f(x) = \sqrt{|x|}\, \chi(x)$

$$f'(x) = \sqrt{|x|}\, \chi'(x) + \frac{1}{2} |x|^{-1/2} \operatorname{sgn}(x)\, \chi(x) \in L^1(\mathbb{R})$$

$f' \notin L^2$

<u>Lemma</u>: $f \in C^1(\mathbb{R}^2)$ compact support

$\Rightarrow \|f\|_{L^2(\mathbb{R}^2)} \le C \|Df\|_{L^1}$

note: $f \rightsquigarrow f_\lambda(\cdot) = f(\lambda\,\cdot)$ $\quad (\lambda^{-2})^{1/2} = \lambda^{-1}$

$\lambda \lambda^{-2}$

<u>proof</u>: $$f^2(x_1, x_2) \le \int_{-\infty}^{\infty} |\partial_1 f(u_1, x_2)|\, du_1 \cdot \int_{-\infty}^{\infty} |\partial_2 f(x_1, u_2)|\, du_2$$

$$\int_{\mathbb{R}^2} f^2(x_1, x_2)\, dx_1 dx_2 \le \iint |\partial_1 f(u_1, x_2)|\, du_1 dx_2 \cdot \iint |\partial_2 f(x_1, u_2)|\, dx_1\, du_2 \le \|Df\|^2_{L^1(\mathbb{R}^2)}$$

Lemma: $f \in C_0^1(\mathbb{R}^d)$

$$\|f\|_{L^{d/(d-1)}(\mathbb{R}^d)} \leq C\|Df\|_{L^1(\mathbb{R}^d)}.$$

Exercise: check $f \rightsquigarrow f_\lambda$ scales correctly

Proof: $d=3$

$$\left|f^3(x_1,x_2,x_3)\right|^{1/2} \leq \left(\int_{-\infty}^{\infty}|\partial_1 f(u_1,x_2,x_3)|\,du_1\right.$$

$$\left.\int_{-\infty}^{\infty}|\partial_2 f(x_1,u_2,x_3)|\,du_2 \cdot \int_{-\infty}^{\infty}|\partial_3 f(x_1,x_2,u_3)|\,du_3\right)^{1/2}$$

$$\int_{-\infty}^{\infty}|f(x_1,\underbrace{x_2,x_3}_{\text{frozen}})|\,dx_1 \leq \left(\int_{-\infty}^{\infty}\partial_1 f(u_1,x_2,x_3)\,du_1\right)^{1/2}.$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\partial_2 f(x_1,u_2,x_3)|\,du_2 \cdot \int_{-\infty}^{\infty}|\partial_3 f(x_1,x_2,u_3)|\,du_3\right)dx_1$$

$$\left(\int_{-\infty}^{\infty}|(\varphi\cdot\psi)(x)|^{1/2}\,dx \leq \left(\int|\varphi|\right)^{1/2}\left(\int|\psi|\right)^{1/2}\right)$$

$$\leq \left(\int_{-\infty}^{\infty}|\partial_1 f(u_1,x_2,x_3)|\,du_1\right)^{1/2}\left(\iint|\partial_2 f(x_1,u_2,x_3)|\,du_2\,dx_1\right)^{1/2}$$

$$\left(\iint|\partial_3 f(x_1,x_2,u_3)|\,du_3\,dx_1\right)^{1/2}$$

$\Rightarrow$ etc $\displaystyle\int_{\mathbb{R}^3} |f(x)|^{3/2} dx \le \|Df\|_{L^1}^{3/2}$

Other examples of Sobolev estimates:

$f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \hat{f} \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |f|^2 dx \quad \text{Plancherel}$$

$f \in L^2(\mathbb{T})$ $\displaystyle\int |f(x)|^2 dx = \sum_{\mathbb{Z}} |\hat{f}(n)|^2$

Def: $\dot{H}^s(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{\dot{H}^s} < \infty \right\}$

Proposition: Let $s > \frac{d}{2}$ $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$

ie $\|f\|_{L^\infty(\mathbb{R}^d)} \le C \|f\|_{\dot{H}^s(\mathbb{R}^d)}$

Exercise: prove false if $s = \frac{d}{2}$ by scaling

OBS! ~~Ha~~ $\|f\|_{H^s} = \| \langle \xi \rangle \hat{f}(\xi) \|_{L^2}$

<u>Lemma</u>: $\|f\|_{\dot H^1(\mathbb{R}^d)} = \|\nabla f\|_{L^2(\mathbb{R}^d)}$

Proof: $\xi_1 \hat f(\xi) = \int e^{-ix\cdot\xi}\, i\frac{\partial}{\partial x_1} f(x)\,dx$

$\forall f \in C_0^1$

$\Rightarrow \; -i\,\widehat{\nabla f}(\xi) = \vec{\xi}\hat f(\xi) =$

$\|\nabla f\|_{L^2} = \|\widehat{\nabla f}\|_{L^2} = \|\vec{\xi}\hat f\|_{L^2} = \|\,|\xi|\,\hat f(\xi)\|_2 = \|f\|_{\dot H^1}$

<u>Lemma</u>: $C^{-1}\left(\|f\|_{L^2} + \|\nabla f\|_2\right) \le \|f\|_{H^1(\mathbb{R}^d)} \le C\left(\|f\|_{L^2} + \|\nabla f\|_{L^2}\right)$

<u>Proof</u>: $f \in C_0^1$, $\|f\|_2^2 + \|\nabla f\|_2^2 =$

$$\int |\hat f(\xi)|^2\,d\xi + \int |\xi|^2 |\hat f(\xi)|\,d\xi = \int \langle\xi\rangle^2 |\hat f(\xi)|^2\,d\xi$$

$= \|f\|_{H^1}^2$

$$\|f\|_\infty = \Big\|\int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat f(\xi)\,d\xi\Big\|_{L^\infty} \le \|\hat f\|_{L^1} =$$

$$= \int |\hat f(\xi)|\,\langle\xi\rangle^{-s}\langle\xi\rangle^{s}\,d\xi \overset{\text{Cauchy Schwarz}}{\le}$$

$$\le \underbrace{\left(\int |\hat f(\xi)|^2 \langle\xi\rangle^{2s}\,d\xi\right)^{1/2}}_{\|f\|_{H^s(\mathbb{R}^d)}} \underbrace{\left(\int \langle\xi\rangle^{-2s}\,d\xi\right)^{1/2}}_{<\infty \text{ if } s>\frac{d}{2}}$$

Let us assume that $\|f\|_\infty \le C\|f\|_{H^{3/2}}$ in $\mathbb{R}^3$ $\frac{3}{2}$ derivatives in L^2
(false)

Interpolate with ~~$\|f\|_{L^{3/2}} \le C\|Df\|_{L^1}$~~

$\|f\|_2 \le \|f\|_2$
0 derivatives in L^2

1 der in L^2 $\quad \frac{2}{3}\cdot\frac{3}{2} + \frac{1}{3}\cdot 0 = 1$

$$\frac{2}{3}\cdot\frac{1}{\infty} + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$\|f\|_{L^6} \le C\|\nabla f\|_{L^2}$

Theorem: $\forall f \in C^1_0$ we have

$$\underbrace{\|f\|_{L^6} \le C\|Df\|_{L^2}}_{\text{check: scaling invariant}} = C\|f\|_{\dot H^1} \le \left(C\|f\|_{H^1}\right)$$

(26)

$$\begin{cases} \|f\|_{L^{3/2}(\mathbb{R}^3)} \le C\|Df\|_{L^1(\mathbb{R}^3)} \\ \|f\|_{L^6(\mathbb{R}^3)} \le C\|Df\|_{L^2(\mathbb{R}^3)} \end{cases}$$

$$\longrightarrow \frac{1}{3/2} = \frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{2} + \frac{\theta}{2} \qquad \theta = \frac{2}{6} = \frac{1}{3}$$

$$= \frac{2}{3}\theta + \frac{1-\theta}{6} = \frac{2}{3}\frac{1}{3} + \frac{1}{6}\cdot\frac{2}{3} = \frac{2}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

$$\|f\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}$$

$$\Updownarrow$$

$$\|\nabla^{-1} f\|_{L^6(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)}$$

$$\| \quad \int_{\mathbb{R}^3} \frac{\hat{f}(\xi)}{|\xi|} e^{ix\cdot\xi}\, d\xi, \qquad a(\xi) = \frac{1}{|\xi|}$$

$$I_1(f) = \int_{\mathbb{R}^3} \frac{f(x-y)}{|y|^2}\, dy = \int \frac{f(x-y)}{|y|^{3-1}}\, dy$$

<u>HLS Hardy-Littlewood-Sobolev inequality</u>

$s>0$

$$\|I_s f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \qquad \frac{1}{p} - \frac{1}{q} = \frac{s}{n}$$

$$I_s(f) = \int \frac{f(x-y)}{|y|^{n-s}}\, dy$$

$$I_1: \ \frac{1}{2} - \frac{1}{6} \overset{?}{=} \frac{1}{3} \quad \checkmark$$

$A : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ $\quad$ $(A \neq 0)$

translationinvariant $\overset{\text{linear}}{\vee}$ operator $\quad 1 \le p, q \le \infty$

$(T_\tau f)(x) = f(x+\tau)$, so $\quad A T_\tau = T_\tau A$

Problem: Show $q \ge p$

(note $Id : L^2(\mathbb{T}) \to L^1(\mathbb{T})$ $\quad \|f\|_{L^1} \le \|f\|_{L^2}$)

Question: What is A in the above

$\|Af\|_q \le C\|f\|_p$

Answer: $A = D^{-1}$

define $\quad A\varphi := \left(|\xi|^{-1} \hat{\varphi}(\xi)\right)^\vee \qquad \varphi \in \mathcal{S}(\mathbb{R}^d)$

Here $\quad \hat{\varphi}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} \varphi(x) dx$

$$\check{\Phi}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} \Phi(\xi) d\xi \qquad (\hat{\varphi})^\vee = \varphi$$

~~CLAIM~~ Lemma: (26) $\Leftrightarrow$ $A : L^2(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$

proof: Assume (26) holds $\forall f \in \mathcal{S}(\mathbb{R}^3)$

Claim $\|(|\xi|^{-1} \hat{f}(\xi))^\vee\|_6 \le C\|f\|_2$

$\hat{f}(\xi) = |\xi| \hat{g}(\xi)$

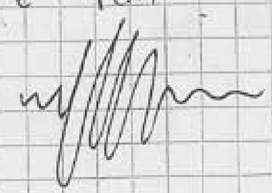
$\|(\hat{g})^\vee\|_6 = \|g\|_6 \le C\| |\xi| \hat{g}(\xi)\|_2 = C\|g\|_{\dot{H}^1(\mathbb{R}^3)}$
$\le C\|Dg\|_2$

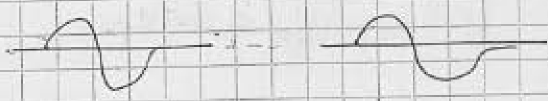
Converse: Assume $A: L^2 \to L^6$ is bounded

$$\|(|\xi|^{-1}\hat f(\xi))^\vee\| \le C\|\hat f\|_2 \qquad \hat f(\xi) = |\xi|\,\hat g(\xi)$$

$$\|g\| \le C\||\xi|\hat g\| = C\|Dg\|_2$$

$e^{-ix\xi_0} f(x)$





Lemma: Let $f \in \mathcal{S}(\mathbb{R}^d)$ with supp $\hat f \subset B(0,1)$

$$\Rightarrow \exists \text{ const}: \|f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$$

proof: $\hat f = \hat f(\xi)\chi \leftarrow$ cut off

$$f = (\hat f\chi)^\vee = \underbrace{f * \check\chi}_{Tf} \in \mathcal{S}$$

Young's inequality $\|f\|_\infty \le \|f * \check\chi\|_\infty \le \|f\|_1\|\check\chi\|_\infty$

$$|f * \check\chi| \le \int |f(x-y)\check\chi(y)|\,dy \le \|\check\chi\|_\infty \int |f(x-y)|\,dy = \|\check\chi\|_\infty\|f\|_1$$

In fact $\|Tf\|_p = \|f * \check\chi\|_p \le \|f\|_p\|\check\chi\|_1$

$\mu = \sum_{j=1}^{N} a_j \delta_{x_j}$ $\quad x_j \in \mathbb{R}^d$ distinct points in $\mathbb{R}^d$

$f \in \mathcal{S}$ $\quad (f * \mu)(x) = \int f(x-y)\,\mu(dy) =$

$= \sum_{j=1}^{N} a_j\, f(x - x_j)$

$\| f * \mu \|_{L^p} \le \sum_{j=1}^{N} \| a_j\, f(\cdot - x_j) \|_{L^p} = \underbrace{\sum |a_j|}\, \|f\|_p$

$\|\mu\|$ = total variation of μ

$\|\mu\| = 1 \quad \| f * \mu \|_p \le \|f\|_p$

$\| f * g \|_p \le \|f\|_p \|g\|_1$

general Young ~~ineq~~ ineq.

$\| f * g \|_{L^r} \le \|f\|_p \|g\|_q \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

(2.6) is easy if $\operatorname{supp} \hat{f} \subset B(0,1)$

$\|f\|_6 \le C \|f\|_2 \le C \|f\|_{\dot{H}^1}$

Q: Suppose $\hat{f}(\xi)$ is supported on $2^{j-1} \le |\xi| \le 2^{j+1}$

$\|f\|_6^6 = \int |f|^6\, dx \le \|f\|_\infty^4 \int |f|^2 dx \le \|\hat{f}\|_{L^1}^4 \|\hat{f}\|_2^2$

$$\int_{2^{j-1} \le |\xi| \le 2^{j+1}} |\hat{f}(\xi)|\, d\xi \le \left(\int |\hat{f}(\xi)|^2 \right) \left(\int_{|\xi| \simeq 2^j} 1\, d\xi \right)^{1/2} \le C \|\hat{f}\|_2\, 2^{\frac{3j}{2}}$$

$\|f\|_6 \le C\, 2^j \|\hat{f}\|_2 \le C \|Df\|_2$

$$(*)\begin{cases} u_{tt} - \Delta u + u + f(u) = 0 & \mathbb{R}^{1+d}_{t,x} \\ f(0)=0, \quad f'(0)=0 \end{cases}$$

for example $f(u) = \pm u^2, \pm u^3$ etc.

Open problem: Does $u_{tt} - \Delta u + u^7 = 0$ have smooth solutions for smooth data for <u>all</u> times? $\mathbb{R}^{1+3}_{t,x}$

Cauchy problem

[CP] $\quad u\big|_{t=0} = u_0 \quad \partial_t u\big|_{t=0} = u$

Q: energy conservation?

Assume $u(t,x)$ is a smooth solution of (CP) for $0 \le t \le T$ $\qquad F'(u) = f(u)$

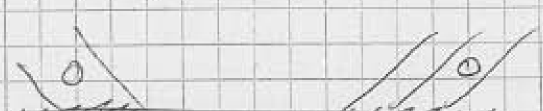
$$\operatorname{div}\Big(-u_t \nabla u, \underbrace{\tfrac{1}{2}\left(u_t^2 + |\nabla u|^2 + u^2\right) + F(u)}_{e}\Big)$$

$$= -u_t \Delta u - \cancel{\nabla u_t \cdot \nabla u} + u_t u_{tt} + \cancel{\nabla u \cdot \nabla u_t} + u u_t + f(u) u_t$$

$$= u_t\left(u_{tt} - \Delta u + u + f(u)\right) = 0$$

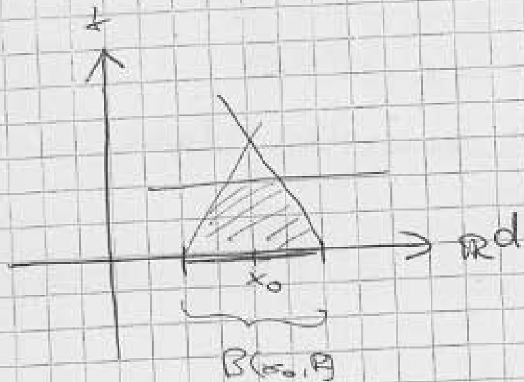
$$E = \int_{\mathbb{R}^d} \frac{1}{2}\left(u_t^2 + |\nabla u|^2 + u^2\right) + F(u)\, du = \text{const}$$

In fact, we still have the finite speed of propagation:



$$e_0(u) := \frac{1}{2}\left(u_t^2 + |\nabla u|^2 + u^2\right)$$

<u>Lemma:</u> Let $u, \tilde{u} \in C^2$ be solution $0 \le t < T$ of $(*)$ with equal data on $B(x_0, R)$



Then $u \equiv \tilde{u}(x,t)$ on $|x - x_0| < R - T$ $\forall t \in (0,T)$

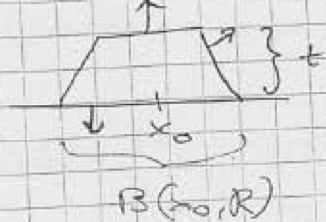
<u>Proof:</u> $w := u - \tilde{u}$

$$w_{tt} - \Delta w + w = -f(u) + f(\tilde{u})$$

$$w|_{t=0} = 0 \quad \partial_t w|_{t=0} = 0 \quad \text{on } B(x_0,R)$$

$$\dagger \begin{cases} \operatorname{div}_{x,t}\left(-w_t \nabla w, \frac{1}{2}\left(w_t^2 + |\nabla w|^2 + w^2\right)\right) = \\ = -\cancel{\nabla w_t \cdot \nabla w} - w_t \Delta w + w_t w_{tt} + \cancel{\nabla w_t \cdot \nabla w} + w_t w = \\ = w_t\left(w_{tt} - \Delta w + w\right) = w_t G \end{cases}$$

Integrate $\dagger$ over



$$\underbrace{\int_{|x-x_0| \le R-t} e_0(w)(x,t)\,dx}_{E_R(t)} + \underbrace{\text{Flux}}_{\ge 0} = \int_0^t \int_{|x-x_0| \le R-s} w_t\, G(x,s)\,dx\,ds$$

$|G| \le M|w|$

$$f(u) - f(\tilde{u}) = \int_0^1 \frac{d}{d\lambda} f\left(\tilde{u} + \lambda(u-\tilde{u})\right) d\lambda =$$

$$= \int_0^1 f'\left(\tilde{u} + \lambda(u-\tilde{u})\right) d\lambda (u-\tilde{u})$$

$$\left| \int_0^1 f'(\tilde{u} + \lambda(u-\tilde{u})(x,t) d\lambda \right| \le M \quad \text{uniform}$$

$(x,t) \in K$ compact subset of $\mathbb{R}^d \times [0,T)$

$$E_R(t) \le M \int_0^t \int_{|x-x_0| \le R-s} |w_t - w|(x,s)\,dx\,ds \qquad 0 \le t < T-\delta$$

$$\overbrace{\int_{|x-x_0| \le R-t} \frac{1}{2}\left(w_t^2 + |\nabla w|^2 + w^2\right)(x,t)\,dx}^{\varphi(t)} \le M_\delta \int_0^t \int_{|x-x_0| \le R-s} |w_t\, w|(x,s)\,dx\,ds$$

$$\le M \int_0^t \int_{|x-x_0| \le R-s} \frac{1}{2}\left(w_t^2 + w^2 + |\nabla w|^2\right)(x,s)\,dx\,ds \qquad |ab| \le \tfrac{1}{2} a^2 + b^2$$

$\varphi(t)$ continuous on $0 \le t < T-\delta$

$\varphi \ge 0$, and $\varphi(t) \le M \int_0^t \varphi(s)\,ds$

Claim: $\varphi \equiv 0$

Idea is that $\varphi \le \psi$ where $\psi(t) = M \int_0^t \psi(s)\,ds$

$$\left.\begin{array}{l} \psi' = M\psi(t) \\ \psi(0) = 0 \end{array}\right\} \Rightarrow \psi(t) = \psi(0) e^{Mt} = 0$$

Grönwall's lemma: ↑ see upstairs

φ as above $\quad \varphi(t) \le k + M\int_0^t \varphi(s)\,ds$
$k \ge 0$

$\Rightarrow \varphi(t) \le k e^{Mt}$.

Prove by iteration $\quad \varphi(t) \le k + M\int_0^t \left(k + M\int_0^s \varphi(u)\,du\right) ds$

$$\le \dots \le k e^{Mt} \frac{M^l}{l!} \max_{[0,T-\varepsilon]} \varphi$$

2) $\Phi(t) := \int_0^t \varphi(s)\,ds$

$\Phi'(t) \le k + M\,\Phi(t)$

$\left(\Phi(t) e^{-Mt}\right)' \le k e^{-Mt}$

$\Phi(t) e^{-Mt} - \Phi(0) \le k \int_0^t e^{-Ms}\,ds = \frac{k}{M}\left(1 - e^{-Mt}\right)$

$\Phi(t) \le e^{Mt} \frac{k}{M}\left(1 - e^{-Mt}\right)$

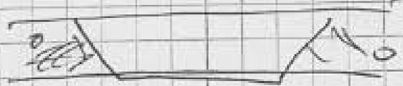
Remark: $\quad G = f(u) - f(\tilde{u}) = \underbrace{\int_0^1 f'\left(\tilde{u} + \lambda(u - \tilde{u})\right) d\lambda}_{L^\infty(K)}(u - \tilde{u})$

$\|u\|_{L^\infty(K)} \quad \|\tilde{u}\|_{L^\infty(K)} \le M$

Corollary: Suppose (CP) above with u_0, u_1 smooth of compact support has a small solution $u(t,x)$ for $0 \le t < T$. Then $x \mapsto u(x,t)$ is of compact support for $\forall t \in [0,T)$ and energy

$$E = \int_{\mathbb{R}^d} \left(\frac{1}{2}\left(u_t^2 + |\nabla u|^2 + u^2\right) + F(u) \right)(x,t)\, dx$$

is constant (i.e. a conserved quantity)



Theorem: The equation $u_{tt} - \Delta u + u - u^3 = 0$ does not have global (in time) smooth solution for all smooth compactly supported data

Proof: Step 1 $u(x,t) = v(t)$

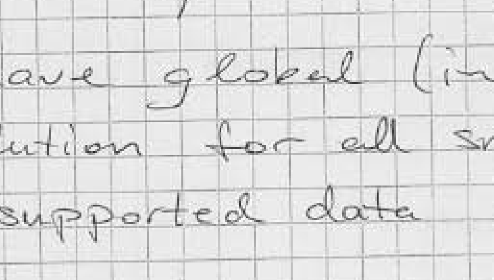
Step 2: Modify v so that it has compact support in x

$$\begin{cases} v'' + v - v^3 = 0 \\ v(0) = a \quad v'(0) = b \end{cases}$$

$$\underbrace{\begin{pmatrix} v \\ v' \end{pmatrix}}_{V}' = \begin{pmatrix} v' \\ v^3 - v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} v \\ v' \end{pmatrix} + \begin{pmatrix} 0 \\ v^3 \end{pmatrix}$$

$$\dot{V} = AV + \begin{pmatrix} 0 \\ V_1^3 \end{pmatrix} = \vec{F}(V) = 0$$

Phase portrait



as $0 = v'' v' + v v' - v^3 v' = \frac{d}{dt}\left(\frac{1}{2} v'^2 + \frac{1}{2} v^2 - \frac{1}{4} v^4 \right) \Rightarrow v'^2 + v^2 - \frac{1}{2} v^4 = \text{const}$

$$F(V_1, V_2) = (V_2, V_1^3 - V_1) = (0,0) \Leftrightarrow V_2 = 0, \quad V_1(V_1^2 - 1) = 0$$

$$V_1 = 0$$
$$V_1 = \pm 1$$

linear about $(0,0)$: $\pm i$

— " — $(1,0)$: $\left.\begin{matrix} \\ \\ \end{matrix}\right\} \pm\sqrt{2}$

— " — $(1,0)$:

$$\tilde{v}'' = \tilde{v}^3 \qquad \tilde{v}(t) = \mu t^{-\alpha}$$

$$\tilde{v}' = -\alpha\mu t^{-\alpha-1} \qquad \tilde{v}'' = \mu\alpha(\alpha+1)t^{-\alpha-2}$$

~~Fro~~ ... $\mu = \pm\sqrt{2}$ $\quad \alpha = 1$

$$\tilde{v}(t) = \sqrt{2}\,(T-t)^{-1} \qquad T > 0$$

$v = \tilde{v} + \text{correction}$

Claim: Those solutions $v(t)$ of (ODE) which satisfy $v(t) \to \infty$ as $t \to T$ have the property that $T < \infty$

$v, v' > 0$ for $t \geq t_0$

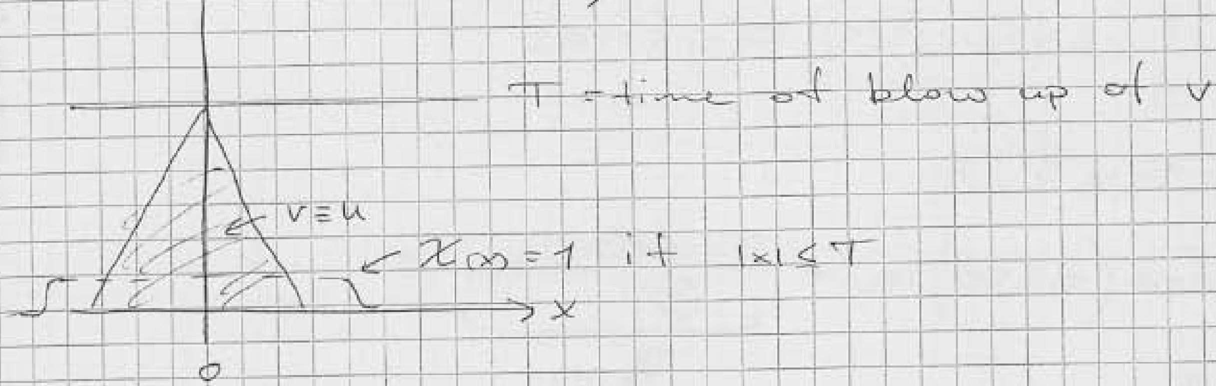
$$v' = \sqrt{\tfrac{1}{2}v^4 - v^2 + c} = \frac{v^2}{2}\sqrt{1 - \frac{2}{v^2} + \frac{2c}{v^4}} \geq \frac{v^2}{2} \qquad T > t > t_0$$

$$\int_{t_0}^{t_1} \frac{v'(t)}{v^2(t)}\,dt \geq \frac{1}{2}(t_1 - t_0)$$

$$-\frac{1}{v(t)}\Big|_{t_0}^{t_1} = \frac{1}{v(t_0)} - \frac{1}{v(t_1)} \geq \frac{1}{2}(t_1 - t_0)$$

$$\frac{1}{v(t_0)} \geq \frac{1}{2}(t - t_0) \qquad \forall t \in (t_0, T)$$

$$\left(\chi(x)\, v(0),\ \chi(x)\, v(0)\right) = (u_0, u_1)$$



Solve (CP) with data (u_1, u_2) $\Rightarrow$ gives you (Maximum) a smooth solution $u(t,x)$ for times $0 < t < T$

$$u_{tt} - \Delta u + u = F$$
$$u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1$$

1) $\underline{F=0}$ $\quad u(t) = \underbrace{\cos(t\langle\nabla\rangle)\, u_0}_{(2\pi)^{-d}\int e^{ix\xi}\cos(t\langle\xi\rangle)\,\hat{u}_0(\xi)\, d\xi} + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}\, u_1$

$$\partial_t^2 u(t, \hat{\xi}) + \underbrace{|\xi|^2 u(t,\hat{\xi}) + u(t,\hat{\xi})}_{\langle\xi\rangle^2 u(t,\hat{\xi})} = 0$$

$e^{\pm it\langle\xi\rangle}$ or $\cos(t\langle\xi\rangle), \frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}$ $\qquad u(t,\hat{\xi}) = \cos(t\langle\xi\rangle)\,\hat{u}_0(\xi) + \frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}\,\hat{u}_1(\xi)$

2) $F \neq 0$ $\quad u_0, u_1 = 0$

$$\partial_t^2 u(t,\hat{\xi}) + \langle\xi\rangle^2 u(t,\hat{\xi}) = F(t,\hat{\xi})$$

$$\Rightarrow u(t,\hat{\xi}) = \int_0^t \frac{\sin((t-s)\langle\xi\rangle)}{\langle\xi\rangle} F(s,\hat{\xi})\,ds$$

$$\partial_t u = \int_0^t \cos((t-s)\langle\xi\rangle) F(s,\hat{\xi})\,ds$$

$$\partial_t^2 u = F(t,\hat{\xi}) + \ldots$$

$$u(t) = \cos(t\langle\nabla\rangle) u_0 + \underbrace{\frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} u_1}_{B_t u_1} + \int_0^1 \frac{\sin((t-s)\langle\nabla\rangle)}{\langle\nabla\rangle} F(s)\,ds$$

$$m_t(\xi) := \frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}$$

$$B_t : L^2 \to H^1(\mathbb{R}^d)$$

$$\|B_t f\| = \|\underbrace{\langle\xi\rangle m_t(\xi)}_{\leq 1} \hat{f}\|_2 \leq \|f\|_2$$

$$B_t f = K_t * f$$

Problem: i) Compute $K_t(x) = \left(\frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}\right)^{\vee}(x)$

ii) note from your formula for K_t that you <u>lose</u> the sharp Huygens principle.

(for $u_{tt} - \Delta u = 0$ rather than $u_{tt} - \Delta u + u = 0$ we had $K_t^{old}(x) = \frac{1}{4\pi t}\sigma_{tS^2}$ $\quad K_t(x) = 0$ if $|x| > t$

Q: How to make sense of a solution of $u_{tt} - \Delta u + u + f(u) = 0$, how to construct any such solution? $f(0) = f'(0) = 0$

To start, we should solve for

$U = \begin{pmatrix} u \\ u_t \end{pmatrix}$

$$\partial_t U = \begin{pmatrix} u_t \\ \Delta u - u \end{pmatrix} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} U + N(U)$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + 1 & 0 \\ 0 & 1 \end{pmatrix} U + N(U) = J \mathcal{H} U + N(U)$$

Remark: if $N \equiv 0$, compute

$$\frac{d}{dt} \langle \mathcal{H} U, U \rangle_{L^2} = \langle \mathcal{H} J \mathcal{H} U, U \rangle + \langle \mathcal{H} U, J \mathcal{H} U \rangle =$$

$$= \langle J \mathcal{H} U, \mathcal{H} U \rangle - \langle J \mathcal{H} U, \mathcal{H} U \rangle = 0 \qquad \left\langle \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} \right\rangle := \int U_1 \tilde{U} + U_2 \tilde{U}_2 \, dx$$

$$\Rightarrow \langle \mathcal{H} U, U \rangle = \text{const}$$

$$= \int |\nabla u|^2 + u^2 + u_t^2 \, dx = \| \langle \nabla \rangle u \|_2^2 + \| u_t \|_2^2$$

$\partial_t U = J \mathcal{H} U$ flow is unitary — to norm $\| \langle \nabla \rangle U_1 \|_2^2 + \| U_2 \|_2^2$

$$\partial_t U = J\mathcal{H} U + G \qquad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

$$U(0) = U_0$$

$$U(t) = \underbrace{e^{tJ\mathcal{H}}}\, U_0 + \int_0^t e^{(t-s)J\mathcal{H}} G(s)\, ds$$

$$e^{t\begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos(t\langle\nabla\rangle) & \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} \\ -\langle\nabla\rangle \sin(t\langle\nabla\rangle) & \cos(t\langle\nabla\rangle) \end{pmatrix}$$

$$U(t_0) = e^{t_0 J\mathcal{H}} U_0 + \int_0^{t_0} e^{(t_0 - s)J\mathcal{H}} G(s)\, ds$$

$$e^{(t-t_0)J\mathcal{H}} U(t_0) + \int_{t_0}^t e^{(t-s)J\mathcal{H}} G(s)\, ds =$$

$$= e^{tJ\mathcal{H}} U_0 + \int_0^{t_0} e^{(t-s)J\mathcal{H}} G(s)\, ds + \int_{t_0}^t e^{(t-s)J\mathcal{H}} G(s)\, ds$$

<u>Definition</u>: We say that $u \in C([0,T), H^1) \cap C^1([0,T), L^2)$

is a strong solution of

$$\begin{cases} u_{tt} - \Delta u + u \pm u^3 = 0 \\ u|_{t=0} = u_0 \in H^1, \ \partial_t u|_{t=0} = u_1 \in L^2 \end{cases}$$

if $U = \begin{pmatrix} u \\ u_t \end{pmatrix}$ satisfies $U(t) = \underbrace{e^{tJ\mathcal{H}} U_0 \mp \int_0^t e^{(t-s)J\mathcal{H}} \begin{pmatrix} 0 \\ u^3(s) \end{pmatrix} ds}_{A(U)}$ $\quad U_1^3$

$U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \forall t \in [0,T)$

$$\|\cdot\|_X = \sup_{0<t<T} \|\cdot\|_Z$$

$U \in C([0,T), H^1 \times L^2) = X \qquad A : X \to X$

Lemma: Given $U_0 \in X$ $\exists T > 0$ s.t.

$A : B_R \to B_R$ where $B_R = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in X \;\middle|\; \sqrt{\|v_1\|^2_{H^1} + \|v_2\|^2_{L^2}} \le R \right\}$ for some $R = R(\|U_0\|_X)$

$$Z = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in X ;\ \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_Z < \infty \right\}$$

and A is a contraction

Proof: $(AU)(t) = e^{tJ\mathcal{H}} U_0 + \int_0^t e^{(t-s)J\mathcal{H}} \begin{pmatrix} 0 \\ \mp U_1^3 \end{pmatrix}(s)\, ds$

$$\|AU(t)\|_Z \le \|U_0\|_Z + \int_0^t \|U_1(s)\|_6^3\, ds \le \|U_0\|_Z + C\int_0^t \|U(s)\|_Z^3\, ds$$

$$\le \|U_0\|_Z + CT \sup_{0 \le s \le T} \|U(s)\|_Z^3$$

Take the sup in t!

$$\|AU\|_X \le \|U_0\|_Z + CT\|U\|_X^3$$

$\|U\|_X \le R$ we need that

$\underbrace{\|U_0\|_Z}_{R/2} + CTR^3 \le R$ let $R = 2\|U_0\|_Z$

$CTR^2 \le \frac{1}{2}$ $T = \frac{1}{2C\|U_0\|_Z^2}$

$u_{tt} - \Delta u + u \pm u^3 = 0$ nonlinear (cubic) Klein-Gordon where u is a smooth solution for times $0 \le t < T$

$U = \begin{pmatrix} u \\ u_t \end{pmatrix}$

$$U(t) = e^{tJ\mathcal{H}} U_0 + \int_0^t e^{(t-s)J\mathcal{H}} \begin{pmatrix} 0 \\ \pm u^3(s) \end{pmatrix} ds \quad \begin{pmatrix} \text{Duhamel} \\ \text{formula} \end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} -\Delta + Id & 0 \\ 0 & Id \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\underbrace{e^{tJ\mathcal{H}}}_{S_0(t)} = \begin{pmatrix} \cos(t\omega) & \frac{\sin(t\omega)}{\omega} \\ -\omega \sin(t\omega) & \cos(t\omega) \end{pmatrix} \qquad \omega = \langle \nabla \rangle \leadsto \langle \xi \rangle$$

$$X_T = C\left([0,T], Z\right) \qquad Z = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

$$(AU)(t) = S_0(t) U_0 + \int_0^t S_0(t) \begin{pmatrix} 0 \\ \mp U_1^3(s) \end{pmatrix} ds$$

$U_0 \in Z$ fixed

We showed using Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$

$$\|AU(t)\|_Z \le \|U_0\|_Z + \int_0^t \|U_1(s)\|_6^3 \, ds \le \|U_0\|_Z + CT\|U\|_X^3$$

if $\|U_0\|_Z \le \frac{R}{2}$ and $T \le \frac{1}{CR^2}$

$\Rightarrow \|AU\|_X \le R$ if $\|U\|_X \le R$

Step 2

$$\|AU - A\tilde{U}\|_X = \left\| \int_0^t S_0(t-s) \left[\begin{pmatrix} 0 \\ \mp U_1^3(s) \end{pmatrix} - \begin{pmatrix} 0 \\ \mp \tilde{U}_1^3(s) \end{pmatrix} \right] ds \right\|_X$$

$$\le \int_0^T \| \underbrace{U_1^3(s) - \tilde{U}_1^3(s)}_{(U_1 - \tilde{U}_1)(U_1^2 + U_1\tilde{U}_1 + \tilde{U}_1^2)} \|_2 \, ds$$

Side calculation:

$$\| (U_1 - \tilde{U}_1)(U_1^2 + U_1 \tilde{U}_1 + \tilde{U}_1^2) \|_{L^2(\mathbb{R}^3)}$$

Hölder

$$\leq \| U_{1}(s) - \tilde{U}_1(s) \|_{L^6(\mathbb{R}^3)} \| U_1^2 + U_1 \tilde{U}_1 + \tilde{U}_1^2 \|_{L^3(\mathbb{R}^3)} \leq$$

$\frac{1}{2} = \frac{1}{6} + \frac{1}{3}$

$$\leq C \| U - \tilde{U} \|_Z \left(\| U(s) \|_Z^2 + \| \tilde{U}(s) \|_Z \right)$$

$$\| AU - A\tilde{U} \|_X \leq C \int_0^T \| U(s) - \tilde{U}(s) \|_Z \left(\| U(s) \|_Z^2 + \| \tilde{U}(s) \|_Z \right) ds$$

$$\leq CT \| U - \tilde{U} \|_X R^2 \leq \frac{1}{2} \| U - \tilde{U} \| \quad \text{if } CTR^2 \leq \frac{1}{2}$$

$\Rightarrow \exists$ unique fixed point $U \in X$

Logic

Fix U_0, I pick $R \geq 2 \| U_0 \|_Z$, then i pick $T \leq \frac{1}{CR^2}$

$AU = U$

Why is $U_2 = \partial_t U_1$?

Write $AU = U$ in the form

$$U_1(t) = \cos(t\omega) u_0 + \frac{\sin(t\omega)}{\omega} u_1 + \int_0^t \frac{\sin((t-s)\omega)}{\omega} \left(\mp U_1^3(s) \right) ds$$

$$U_2(t) = -\omega \sin(t\omega) + \cos(t\omega) u_1 + \int_0^t \cos((t-s)\omega) \left(\mp U_1^3(s) \right) ds$$

$$\cancel{U_2} \Rightarrow \int_0^t U_2(s) \, ds \overset{\text{check}}{=} U_1(t) - \underbrace{U_1(0)}_{u_0}$$

Problem: $F: X \times Y \to X$

X: complete metric space with metric d

Y: metric d_Y

i) $d\left(F(x,y), F(\tilde{x},\tilde{y})\right) \leq \gamma\, d(x,\tilde{x})$

ii) $d\left(F(x,y), F(x,\tilde{y})\right) \leq M\, d_Y(y,\tilde{y})$

Conclusion fixed point $x(y)$ is Lipschitz

$F(x(y),y)$ $\quad d\left(x(y), x(\tilde{y})\right) \leq K\, d_Y(y,\tilde{y})$

<u>Careful statement:</u> locally in $Z = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$

(i.e. on any ball $\|U_0\|_Z \leq R$)

$\exists T = \frac{1}{CR^2}$ such that the fixed point

$U(t) = S_0(t)U_0 + \int_0^t S_0(t-s)\, N(U)(s)\, ds$

~~$\tilde{U}(t) = S_0(t)\tilde{U}_0$~~

$\|U - \tilde{U}\|_{X_T} \leq M \|U_0 - \tilde{U}_0\|_Z$

$\left(\frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}\right)^{\vee}(x) = C\, \frac{J_1\left(\sqrt{t^2-|x|^2}\right)}{\sqrt{t^2-x^2}}\, \mathbb{1}(|x|<t)$

Q: if $U_0 \in H^2 \times H^1(\mathbb{R}^3) \Rightarrow U(t) \in C\left([0, T_{max}), H^2 \times H^1\right)$

Def: T_{max} is the largest possible T with the property that $U \in C\big([0, T_{max}), H^1 \times L^2\big)$ and

$$AU = U, \quad U(0) = U_0$$

Prop: ~~Remarks~~: if $AU = U$, $U \in C\big([0,T), H^1 \times L^2\big)$ $T<\infty$

with $\|U\|_{L^3([0,T], L^6)} < \infty$

Then $T_{max} > T$

Proof: $$U(t) = S_0(t)\,U_0 + \int_0^t S_0(t-s)\begin{pmatrix} 0 \\ \pm U_1^3(s)\end{pmatrix} ds$$

Claim: $\lim_{t\to T} U(t) =: \tilde{U}_0 \in H^1 \times L^2$ exists

$$\|U(t) - U(t')\|_Z \le \|S_0(t)\,U_0 - S_0(t')\,U_0\|_Z + \int_t^{t'} \|U(s)\|_Z^3\, ds$$

$t<t'<T$

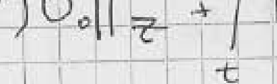
as $t, t' \to T$ $\qquad S_0(+t)^{-1} = S_0(-t)$

$$\|S_0(t)U_0 - S_0(t')U_0\|_Z = \|U_0 - S_0(t'-t)U_0\|_Z \to 0$$

Now $\int_0^T \|U(s)\|_Z^3\, ds < \infty \Rightarrow \int_t^{t'} \|U(s)\|_Z^3\, ds \to 0$ as $t,t' \to T$

Therefore $U(t) \to \hat{U}_0 \in Z$ as $t \to T$

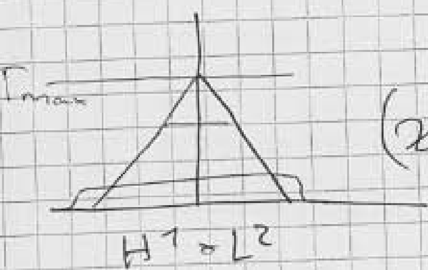
Now solve with data $\hat{U}_0$ starting at time T up to some $T^* > T$



$\boxtimes$

Yesterday we saw $\exists$ solutions $v(t)$ of
$v'' + v - v^3 = 0$ s.t. $v(t) \to \infty$ as $t \to T < \infty$

In fact $v(t) = \frac{1}{\sqrt{2}}(T-t)^{-1}(1+o(1))$ as $t \to T$
(see lemma 2.4 in [NS])



$(\chi(x)v(0), \chi(x)v'(0))$

Assume $T_{max} > T$ $\quad \|u(t)\|^6_{L^6(\mathbb{R}^3)} \geq \int_{|x|<T-t} |v(t)|^6 dx = C(T-t)^{-3}$

$$\int_0^T \|u(t)\|_6^3 \, dt = C \int_0^T (T-t)^{-3/2} \, dt = \infty$$

Prop: If $U_0 \in H^2 \times H^1(\mathbb{R}^3) \Rightarrow U \in C([0, T_{max}), H^2 \times H^1)$

Remark: How good is $H^2(\mathbb{R}^3)$?

Recall $H^2(\mathbb{R}^3) \hookrightarrow L^\infty \cap C(\mathbb{R}^3)$

$\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{H^s}$ if $s > \frac{3}{2}$ $\forall f \in \mathcal{S}(\mathbb{R}^3)$

$g \in H^s$ $\exists f_n \in \mathcal{S}$ $\quad f_n \xrightarrow{H^s} g \Rightarrow \|f_n - f_m\|_{L^\infty} \to 0$
as $n, m \to \infty$

$f_n \to f \in C \cap L^\infty$ $\quad f = g$

Proof: $U(t) = S_0(t) U_0 + \int_0^t S_0(t-s) \begin{pmatrix} 0 \\ \mp U_1^3(s) \end{pmatrix} ds$

$0 \le t < T_{max}$

$\partial_j U(t) = S_0(t) \partial U_0 + \int_0^t S_0(t-s) \begin{pmatrix} 0 \\ \mp 3 U_1^2 \partial U_1 \end{pmatrix} ds$

linear

$\Rightarrow \| \partial U(t) \|_2 \le \| \partial U_0 \|_2 + \int_0^t \| U_1^2 \partial U_1(s) \|_2 \, ds$

$\le \| \partial U_0 \|_2 + \int_0^t \underbrace{\| U(s) \|_2^2}_{M_\delta} \| \partial U_1(s) \|_2 \, ds$

$0 \le t \le T_{max} - \delta$

$\le \| U_0 \|_{H^2 \times H^1} + M \int_0^t \| \partial U(s) \|_2 \, ds$

By Grönwall $\Rightarrow \| \partial U(t) \|_2 \le \| U_0 \|_{H^2 \times H^1} e^{M_\delta t}$

In particular $U(t) \in H^2 \times H^1$ for $\forall t \in [0, T_{max})$

To fix $\dfrac{U(t, x + h e_j) - U(t, x)}{h} = U^{(h)}(t, x)$

By our Grönwall argument:

$\sup_{t \in [0, T_{max} - \delta]} \| U^{(h)}(t) \|_{H^1 \times L^2} \le K_\delta$

$\sup_{0 < h < 1}$ $\quad U^{(h_k)}(t) \xrightarrow{H^1 \times L^2} G_j$

$f \in L^2 \quad \vec{v} \in L^2$

$\langle \vec{v}, \vec{\varphi} \rangle = - \langle f, \operatorname{div} \vec{\varphi} \rangle$

$\vec{v} = \nabla w$ weak gradient

Prop: if $(u_0, u_1) \in H^1 \times L^2$ then the strong solution of

$0 \le t < T_{max}$

$$\begin{cases} u_{tt} - \Delta u + u \pm u^3 = 0 \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{cases}$$

Satisfies $E = \int \frac{1}{2}\left(|\nabla u|^2 + u^2 + u_t^2\right) + \frac{u^4}{4}\, dx = \text{const}$

$$\left(\int u^4\right)^{1/4} \le \left(\int u^6\right)^{\frac{\theta}{6}} \left(\int u^2\right)^{\frac{1-\theta}{2}} \qquad \frac{1}{4} = \frac{\theta}{6} + \frac{1-\theta}{2}$$

Claim: Choose $\underset{\text{Defocusing}}{+}$ above $\Rightarrow T_{max} = \infty$

For $\underset{\text{Focusing}}{-}$, you may have $T_{max} < \infty$

Def: consider an equation of the form

$(*)$ $u_{tt} - \Delta u + f(u) = 0 \qquad f(0) = 0 \qquad F'(u) = f(u)$

$$E = \int_{\mathbb{R}^d} \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right) + F(u)\, dx$$

We say that $(*)$ is energy subcritical if $\int |F(u)|\, dx \le C(\|u\|_{H^1})$ $\quad (+)$

Energy critical if $(+)$ holds but no estimate of the form

$\int |F(u)|\, dx \le C\|u\|_{H^s(\mathbb{R}^d)}$ with $s < 1$ is valid

Energy super critical if no estimate of one form $(+)$ exists

$H^s(\mathbb{R}^3) \to L^4(\mathbb{R}^3)$ $\quad \frac{1}{2} - \frac{1}{4} = \frac{s}{3} \quad s = \frac{3}{4}$

$\|f\|_{L^p} \leq C \|f\|_{\dot{H}^s} \quad \frac{1}{2} - \frac{1}{p} = \frac{s}{d}$

$H^s \hookrightarrow L^8(\mathbb{R}^3) \quad \frac{1}{2} - \frac{1}{8} = \frac{s}{3} \quad s = \frac{9}{8}$

$u_{tt} - \Delta u + u^5 = 0$

$u(t,x) \rightsquigarrow \sqrt{\lambda}\, u(\lambda t, \lambda x)$

$\lambda^2 \sqrt{\lambda} = \sqrt{\lambda}^5$

$\|\sqrt{\lambda}\, u(0, \lambda x)\|_{\dot{H}^1} = \|u(0,x)\|_{\dot{H}^1}$

$\lambda^{3/2} (\lambda^{-3})^{1/2}$

Corollary: If $(u_0, u_1) \in \mathcal{S} \times \mathcal{S}$

$\Rightarrow$ then your strong solution

$u \in C([0, T_{max}), H^1) \cap C^1([0, T_{max}), L^2)$

is in fact $C^\infty([0, T_{max}) \times \mathbb{R}^d)$

Proof: $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^3) \quad \omega = \langle \nabla \rangle$

$\Rightarrow \partial_t u(x,t) = -\omega \sin(\omega t) u_0 + \cos(\omega t) u_1$

$\mp \int_0^t \cos((t-s)\omega)\, u^5(s)\, ds \in L^2$, continuous in t

and in fact, cont. as a function $t \mapsto H^1$.

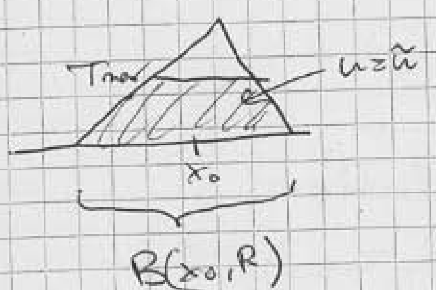
$\nabla \partial_t u$: $\int_0^t \cos((t-s)\omega) \underbrace{u^2 \nabla u(s)}_{\text{cont } s \mapsto L^2} ds$

$$\partial_t^3 u(x,t) = -\omega^2 \cos(\omega t) u_0 - \omega \sin(\omega t) u_1$$
$$\mp u^3(t) \pm \omega^2 \int_0^t \frac{\sin(\omega(t-s))}{\omega} u^3(s)\,ds =$$
$$= -(1-\Delta) u(t) \mp u^3(t) \in L^2$$

$$\partial_{tt} u - \Delta u + u \pm u^3 \underset{L^2}{=} 0$$

$$\bigcap_{s \geq 0} H^s \subset C^\infty \cap L^\infty$$

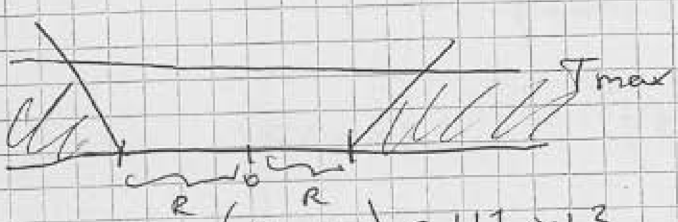
<u>Corollary:</u> Let (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2(\mathbb{R}^3)$ assume that they agree ae on $B(x_0, R)$. Let $u, \tilde{u}$ be the strong solution with respective data.



$f \in L^2$

$f * \varphi_\varepsilon \xrightarrow[\varepsilon \to 0]{L^2} f$

$f_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right) \quad \varphi \in C^\infty$



$(u_0, u_1) \in H^1 \times L^2$

$\underset{ae}{=} 0 \quad$ on $\quad \mathbb{R}^3 \setminus B(0, R)$

Q: $\|u\|_{L^3([0,T_{max}), L^6(\mathbb{R}^3))} < \infty$ if $T<\infty$

$\Rightarrow T_{max} > T$

Now, what can we say about the solution if i) $T_{max} = \infty$ ii) $\|u\|_{L^3([0,\infty), L^6)}$

$$U(t) = S_0(t) U_0 + \int_0^t S_0(t-s) \begin{pmatrix} 0 \\ \mp U_1^3(s) \end{pmatrix} ds$$

$$S_0(-t) U(t) = U_0 + \int_0^t S_0(-t) \begin{pmatrix} 0 \\ \mp u^3(s) \end{pmatrix} ds$$

$\forall t \in [0, \infty)$

$$\| S_0(-t) U(t) \|_{H^1 \times L^2} \leq \int_0^t \|u(s)\|^3_{L^6(\mathbb{R}^3)} ds$$

$$\| S_0(-t) U(t) - S_0(-t') \|_{H^1 \times L^2} \leq \int_t^{t'} \|u(s)\|^3_{L^6} ds$$

So ~~$S_0(-t)$~~ $S_0(-t) U(t) \xrightarrow[H^1 \times L^2]{} \tilde{U}_0$ as $t \to \infty$

and $\tilde{U}_0 = U_0 + \int_0^\infty S_0(-s) \begin{pmatrix} 0 \\ \mp u^3(s) \end{pmatrix} ds$

$$\| U(t) - S_0(t) \tilde{U}_0 \|_{H^1 \times L^2}$$

$$\| \underbrace{S_0(t) U_0 + \int_0^t S_0(t-s) N(s) ds} - S_0(t) U_0 - \int_0^\infty S_0(t-s) N(s) ds \|_{\mathcal{Z}}$$

$$= \| \int_t^\infty S_0(t-s) N(s) ds \|_{\mathcal{Z}} \leq \int_t^\infty \|u^3(s)\|_{L^2} ds =$$

$$= \|u\|^3_{L^3([t,\infty), L^6)}$$ is called scattering

Theorem: $\begin{cases} u_{tt} - \Delta u + u \pm u^3 = 0 & \text{in } \mathbb{R}^{1+3} \\ u(0,\cdot) = u_0 \in H^1(\mathbb{R}^3) \\ \partial_t u(0,\cdot) = u_1 \in L^2(\mathbb{R}^3) \end{cases}$

1) has a unique strong (Duhamel) solution $u \in C([0,T), H^1) \cap C^1([0,T), L^2)$

where $T \geq T_0 \, \|(u_0, u_1)\|_{H^1 \times L^2}$

The map $(u_0, u_1) \mapsto u$ is locally Lipschitz

2) If $(u_0, u_1) \in H^2 \times H^1$, then $u \in C([0,T), H^2) \cap$
$\cap \, C^1([0,T), H^1) \cap$
$\cap \, C^2([0,T), L^2)$

3) If $(u_0, u_1) \in \mathcal{S} \times \mathcal{S} \Rightarrow u \in C^\infty_{x,t}$

4) $E = \int_{\mathbb{R}^3} \left(\frac{1}{2}\left(|\nabla u|^2 + u^2 + u_t^2 \right) \pm \frac{1}{4} u^4 \right)(x,t)\, dx = \text{constant}$

5) Let T_{max} be the sup of $\forall T > 0$ as in 1) $\quad T_{max} < \infty \Leftrightarrow \|u\|_{L^8([0,T_{max}), L^6(\mathbb{R}^3))}$

6) If $T_{max} = \infty$ and $\|u\|_{L^8([0,\infty), L^6(\mathbb{R}^3))} < \infty$

then u scatters:

$\exists\, (v_0, v_1) \in H^1 \times L^2(\mathbb{R}^3)$ so that

$\|(u, u_t)(t) - \underbrace{(v, v_t)(t)}_{S_0(t)(v_0, v_1)}\| \to 0$ as $t \to \infty$

where

$$\begin{cases} \Box v + v = 0 \\ v|_{t=0} = v_0, \ \partial_t v|_{t=0} = v_1 \end{cases}$$

7) $T_{max} = \infty$. Conversely if u scatters, then

$\|u\|_{L^5([0,\infty), L^{10})} < \infty$

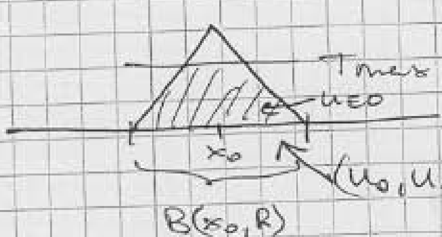
8) If $\|(u_0, u_1)\|_{H^1 \times L^2(\mathbb{R}^3)} \leq \varepsilon_0$ (fixed small number)

$\Rightarrow (CP)_\pm$ has global solution ($T_{max} = \infty$)

that scatters, ie, $\|u\|_{L^5((-\infty,\infty), L^{10}(\mathbb{R}^3))} \leq$

$\leq C \|(u_0, u_1)\|_{H^1 \times L^2(\mathbb{R}^3)}$

9) We have finite propagation speed:



Clearly, for (3) we need to first ask if

$F = F(\cancel{t,x})$

$\Box v + v = 0$

$(v, \partial_t v)|_{t=0} = (v_0, v_1) \in H^1 \times L^2$

Satisfies $\|v\|_{L^3(\mathbb{R}, L^6(\mathbb{R}^3))} \leq C\|(v_0, v_1)\|_{H^1 \times L^2} + \|F\|_{L^1_t L^2_x}$

Strichartz estimates,

Clearly: this involves dispersion i.e. of $v(t) \to 0$ as $t \to \pm\infty$

To prove 8), we write again Duhamel formula.

$$u(t) = \cos(t\omega) u_0 + \frac{\sin(t\omega)}{\omega} u_1 + \int_0^t \frac{\sin((t-s)\omega)}{\omega} (\mp u^3(s))\, ds$$

$$\cancel{\sup_{0 \leq t < T} \|u(t)\|_{H^1}} + \|u\|_{L^3([0,T], L^6(\mathbb{R}^3))} \leq C\Big(\underbrace{\|u_0, u_1\|_{H^1 \times L^2}}_{\leq \varepsilon_0} + \|u\|^3_{L^3([0,T], L^6)} \Big)$$

true but useless

Strichartz dispersion

note: ~~Tmax~~ for all $t \in [0, T_{max})$ set

$\varphi(t) = \|u\|_{L^3([0,t), L^6(\mathbb{R}^3))}$

$\varphi \geq 0, \ \varphi(0) = 0, \ \varphi \in C([0, T_{max}))$

Claim: $\varphi(t) \leq C\varepsilon_0 \ \forall t \in [0, T_{max})$

If so $\Rightarrow T_{max} = \infty$

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if $K \geq 0$ then you scattering i.e.

$\|u\|_{L^3(\mathbb{R}, L^6)} < \infty$ (Concentration compactness)

Where does P-S theorem come from?

What is K?

$\varphi \mapsto e^{\lambda}\varphi \qquad \lambda \in \mathbb{R} \qquad J(\varphi) = \int_{\mathbb{R}^3} \frac{1}{2}\left(|\nabla\varphi|^2 + \varphi^2\right) - \frac{1}{4}\varphi^4$ ← Stationary energy

$J : H^1 \to \mathbb{R}$ continuous

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J(\varphi + \varepsilon\psi) = \int \left(\nabla\varphi\nabla\psi + \psi\varphi - \varphi^3\psi\right)dx = 0 \ \forall \psi$$

$$\Leftrightarrow \ -\Delta\varphi + \varphi = \varphi^3 \text{ in } H^1\text{-sense}$$

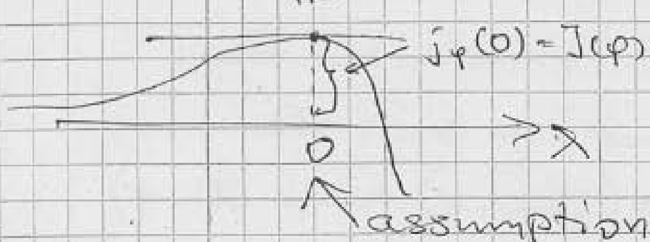
Note $J'(\varphi) = 0 \Leftrightarrow -\Delta\varphi + \varphi = \varphi^3$

$$K(\varphi) = \left(J'(\varphi), \varphi\right) = \int\left(-\Delta\varphi + \varphi - \varphi^3\right)\varphi$$

fix $\varphi \neq 0$

$$\frac{d}{d\lambda}\Big|_{\lambda=0} \underbrace{J\left(e^{\lambda}\varphi\right)}_{j_\varphi(\lambda)} = \left(J'(\varphi), \varphi\right) = K(\varphi)$$

$$j_\varphi(\lambda) = \int_{\mathbb{R}^3} \frac{1}{2}e^{2\lambda}\left(|\nabla\varphi|^2 + \varphi^2\right) - \frac{1}{4}e^{4\lambda}\varphi^4\,dx$$



$j_\varphi'(\lambda) = K(e^{\lambda}\varphi)$

$K(\varphi) = 0$

Q: Determine the least height $J(\varphi)$

$K(\varphi) = 0, \quad \varphi \not\equiv 0$

$\inf\{J(\varphi) \mid \varphi \in H^1_{radial}, K(\varphi) = 0 \quad \varphi \not\equiv 0\}$

Answer: $J(Q)$

The open problem:

$$\left(\begin{array}{ll} -\varphi''(r) - \frac{2}{r}\varphi' + \varphi = \varphi^3(r) & r > 0 \\ \varphi'(0) = 0 & \int_0^\infty (\varphi')^2 + \varphi^2 \, dr < \infty \\ \varphi(0) = h & \end{array} \right) \quad \text{Uniqueness}$$

Theorem: Variational problem (above) uniquely attained by $\varphi = \pm Q$

<u>Corollary</u>: amongst all (weak) H^1-solutions $\varphi \not\equiv 0$ of

$-\Delta\varphi + \varphi = \varphi^3$

$\pm Q$ uniquely attains $J(\varphi)$

Proof: $\circledast \Leftrightarrow J'(\varphi) = 0 \Rightarrow K(\varphi) = \langle J'(\varphi), \varphi \rangle = 0$

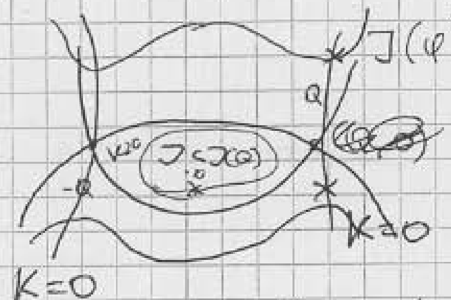
assume $\exists \varphi \in H^1_{radial}$, $\varphi \not\equiv 0$ which is a solution of $\circledast$

Recall from calculus: minimize $f(x,y)$ subject to a constant $g(x,y) = 0$

$J'(\varphi) = \lambda K'(\varphi) \qquad 0 = K(\varphi) = \langle J'(\varphi), \varphi \rangle = \lambda \langle K'(\varphi), \varphi \rangle$

$$= \lambda \int 2\left(|\nabla\varphi|^2 + \varphi^2\right) - 4\varphi^4\Big) - 2\varphi^4 = -2\lambda \int \varphi^4 \neq 0 \quad \lambda \neq 0$$

$$J'(\varphi) = \lambda K'(\varphi) = 0$$



H^1 $\quad J'(\pm Q) = 0$

$* \; K<0$

Cannot go inside or outside

if $K(u(0)) \geq 0 \qquad \xi(u(0), \partial_t u(0)) < \xi(Q, 0) = J(Q)$

$\Rightarrow$ then $K(u(t)) \geq 0 \quad \forall t$

$$\Rightarrow \int \frac{1}{4}\left(|\nabla u(t)|^2 + u(t)^2 + u_t^2\right) dx = \underbrace{\xi(u, u_t)}_{\text{const}} - \underbrace{\frac{K}{4}(u(t))}_{\geq 0}$$

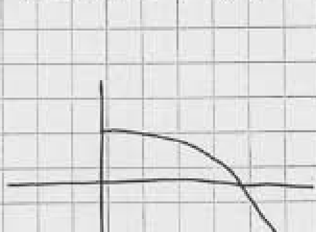
$$\leq \xi(u_0, u_1)$$

To prove FTB if $K<0$

Consider $y(t) = \frac{1}{2}\|u(t)\|_2^2 \geq 0$

You then check that $y''(t) \leq -\delta$

$0 < t < T_{max} = \infty$



GE_+ and scatter to a free wave

$H^1_{rad} \times L^2_{rad}$ $\varepsilon < \varepsilon(Q,0)$ $(+\delta)$ ↑ open problem

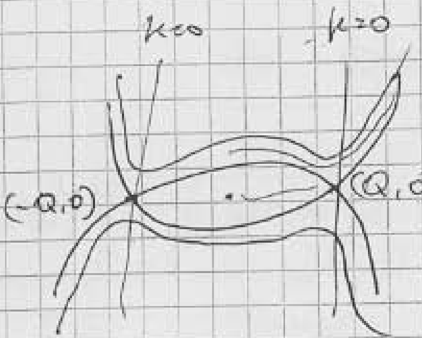
$\# FTB_+$

∃ smooth codimension

1 ~~dim~~ manifold

Start on manifold GE_+ scatter to the manifold

$$(u, u_t)(t) = (Q, 0) + S_0(t)(v_0, v_1) + \underset{H^1 \times L^2}{o(1)} \text{ as } t \to \infty$$



$H^1 \times L^2$

$$u_{tt} - \Delta u + u - u^3 = 0$$

$$u = \varphi + \eta$$

$$\eta_{tt} - \Delta \eta + \eta - 3Q^2 \eta = 0$$

$$H = -\Delta + 1 - 3Q^2$$

$$\begin{cases} \eta_{tt} + H\eta = 0 \\ Hg = -k^2 g \\ \eta|_{t=0} = g, \ \dot\eta|_{t=0} = 0 \end{cases} \quad \Rightarrow \quad \eta(t) = \cosh(t) g$$

$$u_{tt} - \Delta u + u + 2\alpha u_t - u^3 = 0 \qquad \alpha > 0$$

Burq - Raugel - Schlag

— FTB

— Converge in $H^1 \times L^2$ to one of the φ's

$$-\Delta\varphi + \varphi = \varphi^3$$

$u_{tt} - \Delta u + u - u^3 = 0$ $\qquad \mathbb{R}^{1+3}$

$u(0,\cdot) = u_0 \in H^1, \; \partial_t u(0,\cdot) \in L^2$

Q: We can have — $FTB_\pm$ / — GE or GE+scattering

Can we decide apriori (form data) what will happen?

In fact, what is the possible dynamics of a global solution as $t \to \infty$?

Claim: $\exists$ solutions $u = u(x) \not\equiv 0$

Berestycki - Lions - 83

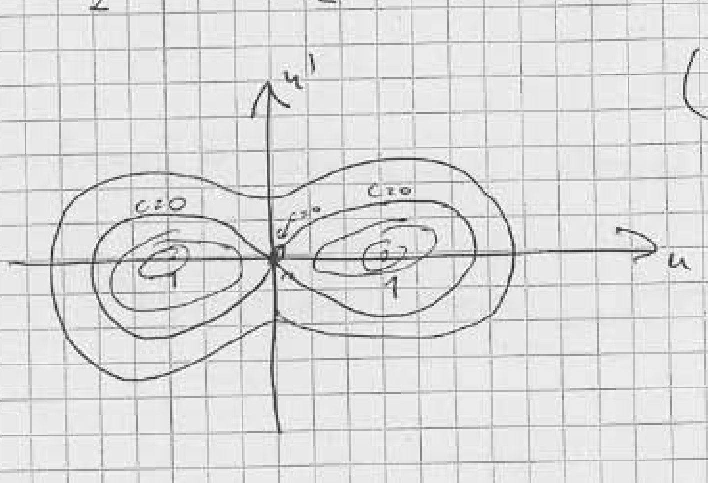
$d=1$ $\quad -u''(x) + u - u^3 = 0$

$u = 0, \pm 1$ $\quad$ we want $u \neq 0$, $u(x) \to 0$ as $x \to \pm\infty$

$-u'u'' + u'u - u'u^3 = 0$

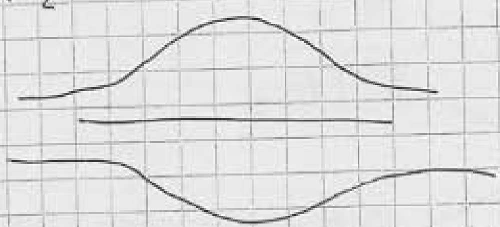
$-\frac{1}{2}(u')^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4 = \text{const} = C = 0 \qquad x \in \mathbb{R}$

$$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} u' \\ u - u^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix} - \begin{pmatrix} 0 \\ u^3 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \qquad \pm i\sqrt{2}$$

$$u' = \frac{\pm 1}{\sqrt{\frac{1}{2}u^4 - u^2}} \qquad u(x) = \frac{\sqrt{2}}{\cosh(x)}$$

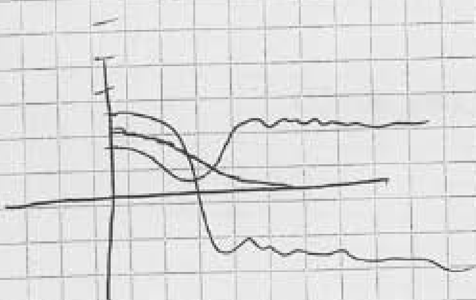


Theorem ~~Fact~~: 1) $\exists\ Q > 0$ radial

$Q \in H^1(\mathbb{R}^3)$ with $-\Delta Q + Q = Q^3$.

In fact, $Q \in C^\infty$, $Q(r) \sim c\,\frac{e^{-r}}{r}$ as $r \to \infty$

2) Q is unique



$\leftarrow$ Open probem uniqueness

Theorem (Payne-Sattinger ~75)

Consider $\begin{cases} u_{tt} - \Delta u + u - u^3 = 0 \\ u|_0 = u_0,\ \partial_t u|_0 = u_1 \end{cases}$ solve this for $0 \le t < T_{max}$

Assume $E(u_0, u_1) < E(Q, 0)$

If $K(u_0) = \int_{\mathbb{R}^3} \left(|\nabla u_0|^2 + u_0^2 - u_0^4\right)(x)\,dx \ge 0$

Global existence

$\Rightarrow GE_\pm$ ($T_{max} = \infty$ both forward & backwards) is true

If $K(u_0) < 0 \Rightarrow T_{max} < \infty$ for both $t > 0$ and $t < 0$

Defocusing equation

$$u_{tt} - \Delta u + u + u^3 = 0$$

$$E = \int_{\mathbb{R}^3} \frac{1}{2}\left(|\nabla u|^2 + |u|^2 + |u_t|^2\right) + \frac{1}{4} u^4 \, dx$$

$T_{max} = \infty$, we also know that

$$\|(u_0, u_1)\|_{H^1 \times L^2} \ll 1 \Rightarrow u \text{ scatters}$$

Q: Do we have scattering $\forall$ data?

A: Yes, Concentration compactness

$$u_{tt} - \Delta u + u - u^3 = 0$$

$$E = \int_{\mathbb{R}^3} \frac{1}{2}\left(|\nabla u|^2 + u^2 + u_t^2\right) - \frac{1}{4} u^4 \, dx$$

for small data even without Strichartz we obtain global existence for (FNLKG)

$$\int \underbrace{u^4}_{u^3 u^1} dx \leq \left(\int u^6\right)^{1/2} \left(\int u^2\right)^{1/2} \leq C\|u\|_{H^1}^3 \|u\|_2 \leq C\|u\|_{H^1}^4$$

$$E = \frac{1}{2}\left(\|u\|_{H^1}^2 + \|u_t\|_2^2\right) - C\|u\|_{H^1}^4 = \text{const}$$

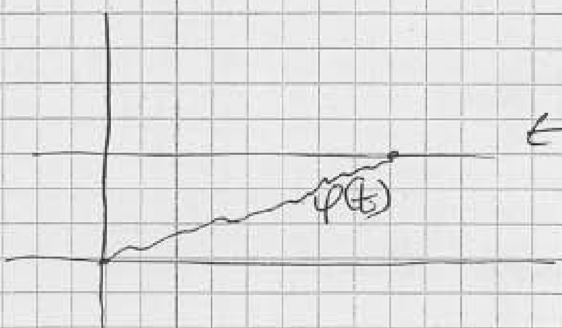
If $\|u(t)\|_{H^1}^2 + \|u_t(t)\|_2^2 = \varphi(t)$

$$\varphi < \varepsilon \Rightarrow T_{max} = \infty$$

Section 2.5 [NS]

$2C\varepsilon_0$

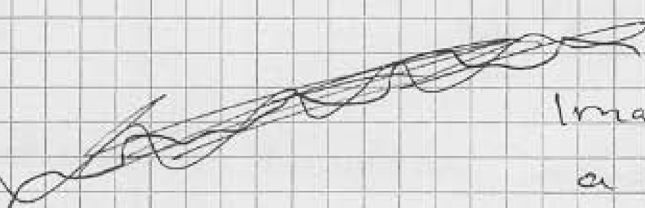
from Strichartz estimate



$$2C\varepsilon_0 \le C\varepsilon_0 + \underbrace{(2C\varepsilon_0)^3}_{< C\varepsilon_0} < 2C\varepsilon_0 \quad \text{— contradiction}$$

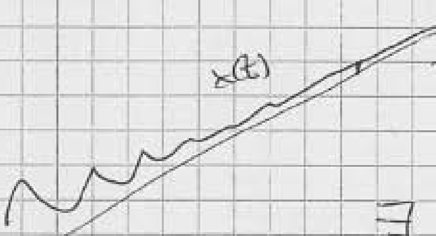
Classical scattering:

$\ddot{x} = -\nabla V(x)$ Newton's equation for 1 particle in a potential $V(x)$



Imagine $V \equiv 0$ outside a ball $|x| \ge R$

What if $V \neq 0$ outside a ball ball but $V(x) \to 0$ as $|x| \to \infty$



$\tilde{x}(t) = vt + x_0$

$\exists\ \dot{\tilde{x}}(0), \tilde{x}(0)$ so that $\ddot{\tilde{x}} = 0$, then

$|(x(t), \dot{x}(t)) - (\tilde{x}(t), \dot{\tilde{x}}(t))| \to 0$ as $t \to \infty$

Recall that for $f \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz functions) the Fourier transform satisfies

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\, dx$$

and its inverse is

$$(\hat{f})^\vee(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} \hat{f}(\xi)\, d\xi = f(x).$$

Under symmetries the Fourier transform behaves as follows:
- if $(T_y f)(x) := f(x-y)$ is the translation by $y \in \mathbb{R}^d$, then

$$\widehat{T_y f}(\xi) = e^{-iy\xi} \hat{f}(\xi)$$

- if $R \in SO(d, \mathbb{R})$ is a rigid motion about the origin, then $(T_R f)(x) := f(Rx)$ satisfies

$$\widehat{T_R f}(\xi) = \hat{f}(R^{-T}\xi)$$

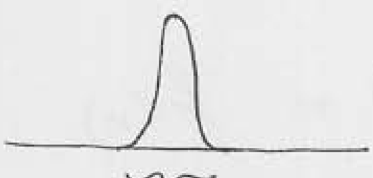
(note that this is how one turns $x \mapsto Rx$ into a <u>symplectic</u> transformation $(x,\xi) \mapsto (Rx, R^{-T}\xi)$)
- if $(D_\lambda f)(x) := f(\lambda x)$, $\lambda > 0$ is a <u>dilation</u>, then $\widehat{D_\lambda f}(x) = \lambda^{-d} \hat{f}(\lambda^{-1}\xi)$

or equivalently,

$$\widehat{\lambda^d f(\lambda\cdot)}(\xi) = \hat{f}(\lambda^{-1}\xi)$$

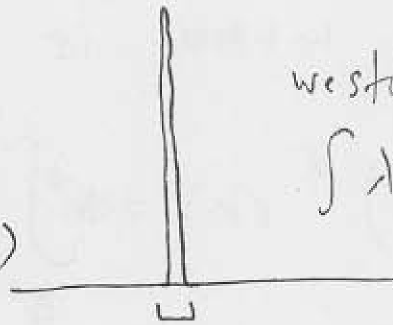
let us illustrate this by taking f to be a nice bump function, say $f \geq 0$, $\int f = 1$.

f



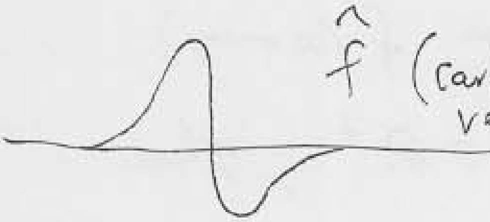
width about ~1

$\rightsquigarrow$ $\lambda^d f(\lambda\cdot)$ (λ large)

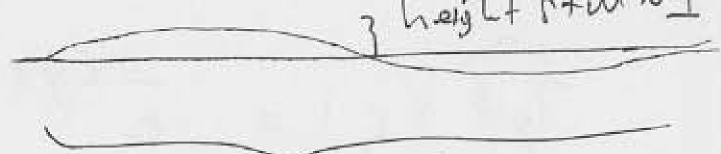


we still have $\int \lambda^d f(\lambda x)\,dx = 1$

width $\sim \frac{1}{\lambda}$



$\hat{f}$ (cartoon version) $\rightsquigarrow$ $\widehat{\lambda^d f(\lambda\cdot)}(\xi) = \hat{f}\left(\frac{\xi}{\lambda}\right)$



height still ~1

width λ (large)

$\hat{f}$ is a nice Schwartz function again living at scale 1

So we see that the L^1-normalized "peak" $\lambda^d f(\lambda x)$ leads to an L^∞-normalized Fourier transform of width λ. Not surprising, since it takes very <u>large</u> frequencies (of size λ) to resolve a thin peak (of size $\frac{1}{\lambda}$).

This is a basic intuition of the Fourier transform.

Conversely, if a function f has the property that $\hat{f}$ only lives on frequencies of size $|\xi| \leq R$ (ie, support of $\hat{f}$ lies in the ball $B(0,R)$), then we would expect f not to be able to form very tall thin peaks. This is the meaning of Bernstein's inequality which we did in class. Recall

Lemma 1: If $f \in L^2(\mathbb{R}^d)$, $\operatorname{supp}(\hat{f}) \subset B(0,R)$

$\Rightarrow$ $f \in L^\infty(\mathbb{R}^d)$ and

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq C R^{\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^d)}$$

where C is a dimensional constant.

Proof:
$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} \hat{f}(\xi)\, d\xi \quad (\text{since } \hat{f} \in L^1)$$

by Cauchy Schwarz

$$|f(x)| \leq (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{f}(\xi)|\, d\xi \leq (2\pi)^{-d} (CR)^{d/2} \|f\|_2 \quad \blacksquare$$

Suppose we now take $f \in L^2_{(\mathbb{R}^d)}$ and write (4)

$$\hat{f}(\xi) := \sum_{j=1}^{\infty} \hat{f}(\xi) \mathbb{1}_{[2^{j-1} \le |\xi| < 2^j]} + \hat{f}(\xi) \mathbb{1}_{[|\xi| < 1]} \qquad (1)$$

were $\mathbb{1}_E$ is the <u>indicator</u> of the set E.

Let
$$f_0 := \left(\hat{f}(\xi) \mathbb{1}_{[|\xi|<1]} \right)^{\vee}$$

$j \geq 1$,
$$f_j := \left(\hat{f}(\xi) \mathbb{1}_{[2^{j-1} \le |\xi| < 2^j]} \right)^{\vee}$$

Then
$$\|f_0\|_\infty \overset{\text{lemma}}{\le} C \|f_0\|_2 = C \|\hat{f}_0\|_2 \left(\le C \|\hat{f}\|_2 \right)$$

and
$$\|f_j\|_\infty \underset{\text{Lemma}}{\le} C\, 2^{\frac{jd}{2}} \|f_j\|_{L^2(\mathbb{R}^d)} \le C \|f_j\|_{H^{d/2}(\mathbb{R}^d)}$$

since we can convert $2^{j\frac{d}{2}}$ into $|\xi|^{d/2}$ hitting $\hat{f}_j(\xi)$.

But then (5)

(2) $$\|f\|_{L^\infty} \le \sum_{j=0}^{\infty} \|f_j\|_\infty \le C \sum_{j=0}^{\infty} \|f_j\|_{H^{\frac{d}{2}}(\mathbb{R}^d)}$$

we could have also let $j \in \mathbb{Z}$ and set

(★) $$f_j = \hat{f}\, \mathbb{1}_{[2^{j-1} \le |\xi| < 2^j]} \quad \forall j \in \mathbb{Z}$$

rather than bunch small frequencies together in f_0 as in (1) above. Then instead of (1):

(2') $$\|f\|_\infty \le C \sum_{j=-\infty}^{\infty} \|f_j\|_{H^{\frac{d}{2}}(\mathbb{R}^d)}$$

Continuing with (2) we compure away $2^{j\varepsilon/2}$ for some fixed $\varepsilon > 0$ and apply Cauchy–Schwartz:

$$\|f\|_\infty \le C \left(\sum_{j=0}^{\infty} \left(2^{j\varepsilon/2} \|f_j\|_{H^{d/2}} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} 2^{-j\varepsilon} \right)^{\frac{1}{2}}$$

$$= C \|f\|_{H^{\frac{d}{2}+\frac{\varepsilon}{2}}}$$

This is the estimate $\|f\|_{L^\infty(\mathbb{R}^d)} \le C \|f\|_{H^s(\mathbb{R}^d)}$ (3)

for class where $s > \frac{d}{2}$.

Need here to say (2), (and 2') is sharper and (6)
is an example of a <u>BESOV space</u> bound,
where as (3) is a <u>Sobolev</u> estimate.
The advantage of (2') is that it is
<u>scaling invariant</u> (if you replace
$f(\cdot)$ by $f(2\cdot)$ nothing changes).

There are many variants of Bernstein's lemma from above. For example

<u>lemma 2</u>: If $f \in L^2(\mathbb{R}^d)$, $\mathrm{supp}(\hat{f}) \subset B(0,R)$
then $f \in L^p(\mathbb{R}^d)$ for all $2 \le p \le \infty$
and

$$(4) \quad \|f\|_{L^p(\mathbb{R}^d)} \le C R^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^2(\mathbb{R}^d)}$$

<u>Proof:</u>

$$\|f\|_p^p = \int |f|^p \le \|f\|_\infty^{p-2} \|f\|_2$$

$$\le C R^{\frac{d}{2}(p-2)} \|f\|_2^p \quad \text{so}$$

lemma 1

(4) follows $\square$

In particular, let us take $p = 6$, $d = 3$. 7

Then

$$\|f\|_{L^6(\mathbb{R}^3)} \le C R \|f\|_{L^2}$$

(5)
$$= C R \|\hat{f}\|_{L^2}$$
$$\le C \|f\|_{\dot{H}^1(\mathbb{R}^3)}$$

Now suppose we write f_j as in $\star$ with $f = \sum_j f_j$. Then (5) $\Rightarrow$

$$\|f_j\|_{L^6(\mathbb{R}^3)} \le C \|f_j\|_{\dot{H}^1(\mathbb{R}^3)}$$

and therefore

$$\sum_{j \in \mathbb{Z}} \|f_j\|^2_{L^6(\mathbb{R}^3)} \le C \underbrace{\sum_{j \in \mathbb{Z}} \|f_j\|^2_{\dot{H}^1(\mathbb{R}^3)}}_{\sum_{j \in \mathbb{Z}} \int_{2^{j-1} \le |\xi| < 2^j} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi}$$

(6)

$$= \|f\|^2_{\dot{H}^1}$$

But what is $\sum_{j\in\mathbb{Z}} \|f_j\|^2_{L^6}$ on the left-hand side of (6)?

In fact, one has the following:

$$(7)\quad \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2\right)^{\frac{1}{2}} \right\|_{L^6} \le \left(\sum_{j\in\mathbb{Z}} \|f_j\|^2_{L^6}\right)^{\frac{1}{2}}$$

or more generally, $\forall\ 1 \le p \le 6$

$$\left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^p\right)^{\frac{1}{p}} \right\|_{L^6} \le \left(\sum_{j\in\mathbb{Z}} \|f_j\|^p_{L^6}\right)^{\frac{1}{p}}$$

For $p=1$ this is the triangle inequality (Minkowski's inequality), whereas for $p=6$ it is evident. The cases $1 \le p \le 6$ then follow by <u>interpolation</u>. What is the advantage of (7)?

The left-hand side here is (9)
an example of a famous square
function as used in harmonic analysis.
They are part of Calderon-Zygmund
theory (see my books with
MUSCALU in Cambridge U-press,
for example volume 1). It turns out

that for $f_j = \left(\chi(2^{-j}\xi)\hat{f}(\xi)\right)^{\vee}$
where $\chi(\xi) \in C^\infty$ with compact support
in $\frac{1}{2} < |\xi| < 2$ and such that

$$\sum_{j\in\mathbb{Z}} \chi(2^{-j}\xi) = 1 \quad \forall\ \xi \neq 0$$

(partition of unity) one does
have

$$(8) \qquad \|f\|_6 \simeq \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^6}$$

In other words there is a (deep) 10
~~orthogonality~~ property of the
f_i which we conseq not only
on L^2 (where it is obvious) but
also on L^p! However, not
~~on~~ for $p=1,\infty$. This is not so
easy and goes by the name of
Littlewood - Paley theory (see Vol 1
above)

If we accept this, then from (8), (7)
(6) we conclude that

$$(9) \quad \|f\|_{L^6(\mathbb{R}^3)} \leq C \|f\|_{\dot{H}^1(\mathbb{R}^3)}$$

as desired.

We remark that (this is deep) one
must use smooth cut off functions in
defining f_i, for (8) to work.

Indeed, it is false that (11)

$$\circledast \quad \| (\chi_B \hat{f})^{\vee} \|_{L^p(\mathbb{R}^d)} \le C \|f\|_{L^p(\mathbb{R}^d)}$$

where $p \neq 2$ and B is the ball if $d \geq 2$. Clearly it is true for $p=2$.
The fact that $\circledast$ is false for $p \neq 2$ is a famous result of Charles Fefferman from about 1971. It is due to the existence of Kakeya sets in $\mathbb{R}^2$ (and higher).
So Littlewood-Paley theory requires smooth partitions of Unity on the Fourier side.
Finally, there are other ways of deriving (9) namely via Riesz potentials and Calderon-Zygmund theory (instead of using Bernstein and orthogonality)

①

In class, ⊗ is false that

⊗ $\| Q \otimes_B \hat{\mathbb{Q}}^\times \| = C(H) / L_\infty$ [illegible]

where $q \neq 2$ and 3 on the tail

of [illegible] Counting of a [illegible]

The fact that ⊗ is false for $p \neq 2$

is [illegible] [illegible] of Cassels

[illegible] [illegible] about 1975

It is due to the existence of [illegible]

only for $p = 2$ [illegible]

So [illegible] Tate [illegible]

[illegible] [illegible] [illegible]

[illegible] [illegible] [illegible]

Finally there are other [illegible] ②

[illegible] [illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] [illegible]

([illegible] [illegible] [illegible])

Problem A: In class we proved the apriori estimate
$$\|f\|_{L^{3/2}(\mathbb{R}^3)} \le C\|\nabla f\|_{L^1(\mathbb{R}^3)}$$
for all $f \in C^1(\mathbb{R}^3)$ with compact support (denote this space by $C^1_{comp}(\mathbb{R}^3)$).

Let X be the closure (or completion) of C^1_{comp} under the norm $\|\nabla f\|_1$.

a) Show that (where L^1_{loc} are measurable functions absolutely integrable on every compact set)
$$X = \left\{ f \in L^1_{loc}(\mathbb{R}^3) \;\middle|\; \nabla f \in L^1(\mathbb{R}^3) \right\}$$
where ∇f is the distributional (or weak) gradient: This means there exists $\vec{v} \in L^1(\mathbb{R}^3, \mathbb{R}^3)$
$$(*) \quad \langle \vec{v}, \vec{\varphi} \rangle = -\langle f, \operatorname{div} \vec{\varphi} \rangle$$
for all smooth vectorfields $\vec{\varphi}$ of compact support.

b) Show that $X \subset L^{3/2}(\mathbb{R}^3)$ and $\|f\|_{L^{3/2}} \le C\|\nabla f\|_1$ $\forall f \in X$

c) Let $\chi(x)$ be a standard cutoff function.

Decide for which $\alpha > 0$ the function

$$f(x) = |x|^{-\alpha} \chi(x)$$

is in X and for which α it is in $L^{3/2}(\mathbb{R}^3)$. Do you expect to find different α in these two cases?

d) Give an example (or several) of $f \in L^{3/2}(\mathbb{R}^3) \setminus X$

In the literature, $X = \mathring{W}^{1,1}(\mathbb{R}^3)$.

Problem B: a) Show that the space $H^1(\mathbb{R}^d)$ consists of all functions $f \in L^2(\mathbb{R}^d)$ which admit a weak gradient (as in Problem A ⊛) $\nabla f \in L^2(\mathbb{R}^d)$.

b) Find all $1 \le p \le \infty$ so that

$$H^1(\mathbb{R}^d) \underset{\text{embedding}}{\overset{\text{continuous}}{\hookrightarrow}} L^p(\mathbb{R}^d) \quad \text{(embeds)}$$

for every choice of $d = 1, 2, \ldots$

Problem C: Let $F : X \times Y \to X$ where (X, d) is a complete metric space and (Y, d_Y) is some other metric space. Assume that

i) $d\big(F(x,y), F(\tilde{x},y)\big) \le \gamma\, d(x,\tilde{x}) \quad \forall y \in Y,\ \forall x, \tilde{x} \in X$

where $0 < \gamma < 1$ is fixed

ii) $d\big(F(x,y), F(x,\tilde{y})\big) \le M\, d_Y(y,\tilde{y}) \quad \forall x \in X,\ y, \tilde{y} \in Y$

where M is a fixed positive constant.

Show that the unique fixed point $F(x(y), y) = x(y)$ satisfies

$$d(x(y), x(\tilde{y})) \leq K d_Y(y, \tilde{y})$$

What is the optimal choice of K?

---

would like to now show that (5)

$$\| \underbrace{\partial_t \left(\tfrac{1}{4\pi t} \sigma_{tS^2} * f \right)}_{u} \|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t} \|D^2 f\|_{L^1(\mathbb{R}^3)} \quad \forall t > 0$$

ere, $\|D^2 f\|_{L^1(\mathbb{R}^3)} = \sum_{j,k=1}^{3} \|\partial^2_{x_j x_k} f\|_{L^1(\mathbb{R}^3)}$

e write

$$\partial_t u(t,x) = \partial_t \left(\frac{t}{4\pi} \int_{S^2} f(x+t\omega)\, \sigma_{S^2}(d\omega) \right)$$

$$= \underbrace{\frac{1}{4\pi t^2} \int_{tS^2} f(x+y)\, \sigma_{tS^2}(dy)}_{=: F(t,x)} + \underbrace{\frac{1}{4\pi t} \int_{x+tS^2} \nabla f(y) \cdot \frac{y-x}{t}\, \sigma_{tS^2+x}(dy)}_{=: G(t,x)}$$

e convert F into a volume integral as in class:

$$|F(t,x)| = \frac{1}{4\pi t^2} \left| \int_{x+tS^2} f(y) \frac{y-x}{t} \cdot \underbrace{\vec{n}(y)}_{\substack{\text{unit normal}\\ \text{vector to } x+tS^2}} \sigma(dy) \right|$$

$$\leq \frac{1}{4\pi t^2} \int_{|y-x|\leq t} \left| \operatorname{div}_y \left(f(y) \frac{y-x}{t} \right) \right| dy$$

$$\leq \frac{1}{4\pi t^2} \int_{|y-x|\leq t} \left(|\nabla f| + \frac{3}{t} |f(y)| \right) dy$$

$$\leq \frac{1}{4\pi t^2}\left(\int_{\mathbb{R}^3} |\nabla f|^{3/2}\right)^{2/3}\left(\frac{4}{3}\pi t^3\right)^{\frac{1}{3}} +$$

$$+ \frac{3}{4\pi t^3}\left(\int_{\mathbb{R}^3} |f|^3\right)^{1/3}\left(\frac{4\pi}{3} t^3\right)^{2/3}$$

$$\leq \frac{C}{t}\,\|D^2 f\|_1$$ where we

used the Sobolev embedding estimates

$$\|\varphi\|_{L^{3/2}(\mathbb{R}^3)} \leq C\|\nabla\varphi\|_{L^1(\mathbb{R}^3)}$$

$$\|\varphi\|_{L^3(\mathbb{R}^3)} \leq C\|D^2\varphi\|_{L^1(\mathbb{R}^3)}$$

valid for Schwartz functions φ in $\mathbb{R}^3$ (although much less is needed on φ).

As for $G(t,x)$, we write

$$G(t,x) = \frac{1}{4\pi t}\int_{x+tS^2} \tilde{f}(y)\,\sigma(dy)$$

where $\tilde{f}(y) := \nabla f(y)\cdot\frac{y-x}{t}$ (t,x frozen)

Then as in class,

$$|G(t,x)| \le \frac{1}{4\pi t} \int_{|x-y|\le t} \left| \operatorname{div}\left(\tilde{f}(y)\frac{y-x}{t}\right)\right| dy$$

$$\le \frac{1}{4\pi t} \int_{|y-x|\le t} \left[|\nabla \tilde{f}(y)| + \frac{3}{t} |\tilde{f}(y)| \right] dy$$

$$\le \frac{C}{t} \int_{|y-x|\le t} \Big(\underbrace{|D^2 f(y)|}_{=: \sum_{i,j=1}^{3} |\partial_{ij}^2 f(y)|} + \frac{1}{t} |\nabla f(y)| \Big) dy$$

$$\le \frac{C}{t} \|D^2 f\|_{L^1} + \frac{C}{t^2} \|\nabla f\|_{L^{3/2}} (t^3)^{1/3}$$

$$\le \frac{C}{t} \|D^2 f\|_{L^1(\mathbb{R}^3)}$$

as claimed. $\square$

Computation of Fourier transform of σ_{S^2}, the surface measure on S^2. ①

$$\int_{S^2} e^{-ix\cdot\xi}\,\sigma_{S^2}(d\xi) \overset{\text{by rotation}}{=} \text{invariance}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$

$$\int_{S^2} e^{-i|x|\xi_3}\,\sigma(d\xi) \quad \underset{\text{coordinates}}{\overset{\text{spherical}}{=}}$$

$$= \int_0^{2\pi}\int_0^{\pi} e^{-i|x|\cos\theta}\,\sin\theta\,d\theta\,d\varphi$$

$$= \frac{2\pi}{i|x|}\int_0^{\pi} \frac{d}{d\theta}\left(e^{-i|x|\cos\theta}\right)d\theta \qquad (x \neq 0)$$

$$= \frac{2\pi}{i|x|}\left(e^{i|x|} - e^{-i|x|}\right) = 4\pi\,\frac{\sin|x|}{|x|}$$

So $\widehat{\sigma}_{S^2}(x) = 4\pi\,\frac{\sin|x|}{|x|}$ in fact for all $x \in \mathbb{R}^3$

Let σ_{tS^2} be the surface measure on tS^2 where $t>0$.

Then from $\int_{tS^2} f(\omega)\,\sigma_{tS^2}(d\omega) = t^2 \int_{S^2} f(t\omega)\,\sigma_{S^2}(\omega)$

for any continuous function f on $\mathbb{R}^3$ it follows that

$$\widehat{\sigma_{tS^2}}(x) = t^2\, \widehat{\sigma_{S^2}}(tx)$$

$$= 4\pi t \,\frac{\sin(t|x|)}{|x|}$$

It follows that

$$\left(\frac{\sin(t|\xi|)}{|\xi|}\right)^{\vee}(x) = \frac{1}{4\pi t}\,\sigma_{tS^2}$$

as claimed.

(3)

Decay of solutions of $\Box u = 0$ in $\mathbb{R}^3$
where $u(0,x) = f(x)$, $\partial_t u(0,x) = g(x)$
and f, g Schwartz functions in $\mathbb{R}^3$.

First, the previous two pages show that
if $f=0$, then
$$u(t,x) = \frac{1}{4\pi t}\left(\sigma_{tS^2} * g\right)(x)$$
$\forall\ t>0$. Indeed: Clearly $u(t,x) \to 0$ ~~$g(x)$~~ as $t\to 0$
<u>uniformly</u> in $x \in \mathbb{R}^3$ (<u>prove it</u>!)

Further we have
$$u(t,x) = \frac{t}{4\pi}\int_{S^2} g(x+t\omega)\,\sigma_{S^2}(d\omega)$$
and so
$$\partial_t u(t,x) = \frac{1}{4\pi}\int_{S^2} g(x+t\omega)\,\sigma_{S^2}(d\omega)$$
$$+ \frac{t}{4\pi}\int_{S^2} \nabla g(x+t\omega)\cdot\omega\,\sigma_{S^2}(d\omega)$$
$$\xrightarrow{\text{as } t\to 0} g(x) \text{ uniformly in } x\in\mathbb{R}^3.$$
(prove it!)

Finally, that $\Box u = 0$ is due to
$$\widehat{\frac{1}{4\pi t}\sigma_{tS^2}}(\xi) = \frac{\sin t|\xi|}{|\xi|}.$$

If $g=0$, but $f\neq 0$ then we claimed in class that

$$u(t,x) = \partial_t \underbrace{\left(\frac{1}{4\pi t} \sigma_{t\delta^2} * f \right)}_{=:v}$$

First, $\Box u = \partial_t (\Box v) = 0$

(note: v is smooth in both variables and Schwartz in space, by the Fourier representation).

Second, by page (3), $u(t,x) \to f(x)$ uniformly in x as $t \to 0$. On the other hand, $\partial_t u(t,x) = \partial_t^2 v \overset{\Box v = 0}{=} \Delta v(t,x)$,

and $\Delta v(t,x) = \frac{1}{4\pi t}(\sigma_{t\delta^2} * \Delta f)(x) \to 0$ as $t \to 0$ uniformly in $x \in \mathbb{R}^3$,

So indeed, u above solves $\Box u = 0$

$u|_{t=0} = f$, $\partial_t u|_{t=0} = 0$.

In class we proved that

$$\left\| \frac{1}{4\pi t}(\sigma_{t\delta^2} * g) \right\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t} \|\nabla g\|_{L^1}$$

Fundamental solutions

For the wave equation $\Box u = 0$ in $\mathbb{R}^{1+3}_{t,x}$ we saw that

$$u(x,t) = (K_t * g)(x)$$

where $g \in \mathcal{S}(\mathbb{R}^3)$ satisfies $\Box u = 0$,

$u|_{t=0} = 0$, $\partial_t u|_{t=0} = g$. Recall

that $$K_t(x) = \frac{1}{4\pi t}\sigma_{tS^2}, \quad t>0$$

We would like to find this FUNDAMENTAL SOLUTION K_t for other dimensions and for the Klein–Gordon equation $\Box u + u = 0$.

Exercise: For the $1+1$-dimensional wave equation $u_{tt} - u_{xx} = 0$ show that

$$K_t(x) = \frac{1}{2}\chi_{[-t,t]}(x), \quad t>0 \text{ is}$$

the fundamental solution.

Now we would like to find the fundamental solution of $\Box u = 0$ in $1+2$ dimensions.

We could do this by inverting the Fourier transform:

$$K_t(x) = \int_{\mathbb{R}^2} \frac{\sin(t|\xi|)}{|\xi|} e^{ix\cdot\xi}\, d\xi$$

However, this is delicate since this integral per se is not well-defined. In fact, writing it in polar coordinates we find

$$K_t(x) = \int_0^\infty \sin(rt)\left(\int_0^{2\pi} e^{i|x|r\cos\theta}\, d\theta\right) dr$$

which is not absolutely convergent. One way around this is to use a summation method such as Abel's:

$$\circledast \quad K_t(x) = \lim_{\varepsilon \to 0+} \int_0^\infty e^{-\varepsilon r^2} \sin(rt)\left(\int_0^{\pi} e^{i|x|r\cos\theta}\, d\theta\right) dr$$

You can look up that $\int_0^{\pi} e^{i|x|r\cos\theta}\, d\theta = 2\pi J_0(r|x|)$.

Where $J_0(\cdot)$ is the standard Bessel function.

So then one can actually compute the integrals in $\circledast$, using property of Bessel functions. I leave this to you and instead present a different route which avoids the Fourier transform.

FS3

This alternative approach is based on a change of variables:

$$\rho = \sqrt{t^2 - r^2}, \quad t > 0, \quad 0 \le r < t$$

where $r = |x|$. Then $\rho_t = \frac{t}{\rho}$, $\rho_r = -\frac{r}{\rho}$

$$\rho_{tt} = \frac{1}{\rho} - \frac{t^2}{\rho^3} = -\frac{r^2}{\rho^3}, \quad \rho_{rr} = \frac{-1}{\rho} + \frac{r}{\rho^2}\rho_r = \frac{-1}{\rho} - \frac{r^2}{\rho^3}$$
$$= -\frac{t^2}{\rho^3}$$

Now we seek solutions of $\square u = 0$ in the region $0 \le r < t$ of the form

$$u(r, t) = \varphi(\rho)$$

In other words, these a <u>radial</u> solutions, they only depend on $r = |x|$.

One has $u_r = \varphi'(\rho)\rho_r$, $u_{rr} = \varphi''(\rho)\rho_r^2 + \varphi'(\rho)\rho_{rr}$

$$u_{tt} = \varphi''(\rho)\rho_t^2 + \varphi'(\rho)\rho_{tt}$$

$$u_{rr} = \varphi''(\rho)\frac{r^2}{\rho^2} + \varphi'(\rho)\left(-\frac{t^2}{\rho^3}\right)$$

$$u_{tt} = \varphi''(\rho)\frac{t^2}{\rho^2} - \frac{r^2}{\rho^3}\varphi'(\rho)$$

$$u_r = -\frac{r}{\rho}\varphi'(\rho)$$

FS4

So then

$$\Box u \overset{\text{radial Laplacian}}{=} u_{tt} - u_{rr} - \frac{d-1}{r} u_r$$

$$= \varphi''(\rho) + \frac{d}{\rho}\varphi'(\rho) \qquad (1)$$

This is remarkable since we obtain an equation of ρ alone. The deeper reason for this lies with the Lorentz invariance of the wave equation. Recall that the group of Lorentz transformations is defined as precisely those linear transformations in $\mathbb{R}^{1+d}_{t,x}$ which leave the quadratic form $t^2 - |x|^2$ invariant, i.e., all matrices $A \in GL(d+1, \mathbb{R})$ with

$$\langle L A\binom{t}{x}, A\binom{t}{x}\rangle = \underbrace{\langle L\binom{t}{x}, \binom{t}{x}\rangle}_{t^2-|x|^2 \,=\, \text{MINKOWSKI METRIC}}$$

where

$$L = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & & \ddots & \\ & 0 & & -1 \end{pmatrix}$$

with the -1 block of size d.

For example, in $\mathbb{R}^{1+3}_{t,x}$ any A of the form

FSJ

$$A_\alpha = \begin{pmatrix} \cosh\alpha & \sinh\alpha & & \\ \sinh\alpha & \cosh\alpha & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

is a Lorentz transform. In fact the group of Lorentz transforms is generated by these $\{A_\alpha \mid \alpha \in \mathbb{R}\}$ and $SO(3, \mathbb{R})$.

Returning to the wave equation you will immediately verify that for any smooth f with $\Box f = 0$ one has $\Box (f(A\cdot)) = 0$ with A Lorentz. This is the analogue of $SO(d, \mathbb{R})$ keeping $\Delta_{\mathbb{R}^d}$ invariant ($\Delta(f(R\cdot)) = (\Delta f)(R\cdot)$ for any rotation R)

FS6

Coming back to our computation involving $\rho = \sqrt{t^2 - r^2}$ we see that ρ is nothing but the root of the Minkowski distance.

Let us now solve equation (1) in dimension $d = 2$. Thus $\varphi''(\rho) + \frac{2}{\rho}\varphi'(\rho) = 0$
with $\psi := \varphi'$ becomes $\psi'(\rho) + \frac{2}{\rho}\psi(\rho) = 0$
which is separable: $\int \frac{d\psi}{\psi} = -\int \frac{2}{\rho}\, d\rho$

or $\log \psi = -2 \log \rho + C$

or $\psi(\rho) = k \cdot \rho^{-2} = \varphi'(\rho)$

$\Rightarrow \quad \varphi(\rho) = \frac{c}{\rho}$ with some $c = \text{const.}$

What does this mean for $u(t,r)$?
It means we are looking for solutions of the form
$$u(t,r) = \frac{c}{\sqrt{t^2 - r^2}}$$

While it is not clear a priori that this procedure is successful (in dim = 3 it does not work, for example) it does work for dim = 2. FS7

If we set

$$(2) \quad K_t(x) = \frac{1}{2\pi} \frac{1}{\sqrt{t^2-|x|^2}} \quad \left\{ \begin{array}{l} t>0 \\ x \in \mathbb{R}^2 \end{array} \right.$$

then $u(t,x) := (K_t * g)(x)$,

for any $g \in \mathcal{S}(\mathbb{R}^2)$ (Schwartz, say)

satisfies i) $\Box u = 0$, $u \in C^\infty(\mathbb{R}^{1+2}_{t,x})$

ii) $u|_{t=0} = 0$, $\partial_t u|_{t=0} = g$

As for i), we can write $(t>0)$

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2 : |y|<t} \frac{f(x+y)}{\sqrt{t^2-|y|^2}} \, dy$$

$$= \frac{t}{2\pi} \int_{|y|<1} \frac{f(x+ty)}{\sqrt{1-|y|^2}} \, dy \quad (3)$$

whence $u \in C^\infty$

Furthermore, $\Box u = 0$ by construction (equation (1), check it!). Clearly, from (3), $u(0,x)=0$.

And $$\partial_t u(t,x) = \frac{1}{2\pi}\int_{|y|<1} \frac{f(x+ty)}{\sqrt{1-|y|^2}}\,dy$$

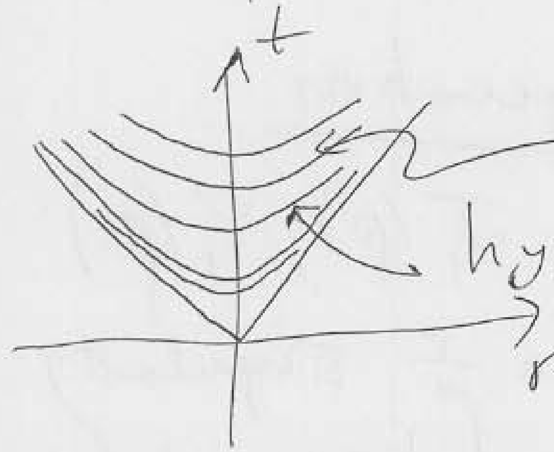
$$+ \frac{t}{2\pi}\int_{|y|<1} \frac{\nabla f(x+ty)\cdot y}{\sqrt{1+|y|^2}}\,dy$$

So $$\partial_t u(0,x) = \frac{1}{2\pi}\int_{|y|<1} \frac{f(x)}{\sqrt{1-|y|^2}}\,dy$$

$$\underset{\text{coordinates}}{\overset{\text{polar}}{=}} f(x)\,\frac{2\pi}{2\pi}\int_0^1 \frac{r\,dr}{\sqrt{1-r^2}} = f(x)$$

as claimed. So, indeed, (2) does define the fundamental solution of $\Box u = 0$ in $\mathbb{R}^2$. There are other ways of obtaining this, next to the Fourier transform alluded to in the beginning. Most notably, the METHOD OF DESCENT (see Fritz John or Evans books)

We should not stop here, but (F59)
try to use the "foliation of
hyperboloids" given by $\rho = \sqrt{t^2 - r^2}$
also for other problems. This
picture explains
the foliation.



Lorentz transforms
leave these sheets
invariant, but of course
move points on the sheets.

We will try to obtain the fundamental
solution of $\Box u + u = 0$ by this
method. Instead of equation (1)
we obtain

(4) $\qquad \varphi''(\rho) + \frac{d}{\rho}\varphi'(\rho) + \varphi(\rho) = 0$

note we have a singularity at $\rho = 0$.
Set $d = 3$ and transform $\varphi(\rho) = \frac{h(\rho)}{\rho}$

Then you can check that (4) (FS10)

becomes the equation

(5) $h''(\rho) + \frac{1}{\rho} h'(\rho) + \left(1 - \frac{1}{\rho^2}\right) h(\rho) = 0$

This is a <u>Bessel equation</u>

with fundamental system $J_1(\rho), Y_1(\rho)$.

The function $Y_1(\rho)$ has a $\frac{1}{\rho}$ singularity near $\rho = 0$, which is too strong. Indeed, reverting back to r, t this would mean that

$$u(t,r) = \frac{c\, Y_1(\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} \sim \frac{1}{t^2-r^2}$$

near $r = t$, but this is <u>not</u> integrable due to a $\frac{1}{t-r}$ singularity.

So we need to discard Y_1 and only use $J_1(\rho)$ which starts with a ρ in its Taylor expansion. So

$$K_t(x) = K(t,x) = \frac{c_0 J_1(\sqrt{t^2-|x|^2})}{\sqrt{t^2-|x|^2}} 1\!\!1_{\{|x|<t\}}$$

(7511)

is our candidate for the fundamental solution in dimensions $\mathbb{R}^{1+3}$ for the equation $\begin{cases} \Box u + u = 0 \\ u|_{t=0} = 0, \ \partial_t u|_{t=0} = g \end{cases}$

Thus $$u(t,x) = (\tilde{K}_t * g)(x) =$$

$$= c_0 t^2 \int_{|y|<1} \frac{g(x+ty)}{\sqrt{1-|y|^2}} J_1\left(t\sqrt{1-|y|^2}\right) dy$$

Clearly, as $t \to 0+$ this $\to 0$ uniformly in x (taking $g \in \mathcal{S}(\mathbb{R}^3)$)

And $\partial_t u(t,x)\big|_{t=0} = 0$, as is easy to see!

So something is wrong, we obtained a solution $\tilde{K}_t$ which is not singular enough! This is not so surprising if we remember that

The fundamental solution of (Fs12)
the wave equation in $\mathbb{R}^{1+3}$
lives on the light cone $t=r$
(it is the measure $\frac{1}{4\pi t}\sigma_{tS^2}$).
It is completely unreasonable to
expect that going from $\Box u=0$
to $\Box u+u=0$ would completely
remove this singularity on the light cone.
In fact, the correct fundamental
solution of $\Box u+u=0$ in $\mathbb{R}^{1+3}_{t,x}$
is

$$(6)\quad K_t(x)=\begin{cases}\frac{1}{4\pi t}\sigma_{tS^2}-\frac{1}{4\pi}\frac{J_1(\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} & \text{for } r=|x|\le t\\ 0 & \text{if } |x|>t\end{cases}$$

In particular, we lose the sharp Huygens principle.
Problem: Check that this is so.

Let us look now for something (F513) easier, namely the fundamental solution of $\Box u + u = 0$ in $\mathbb{R}^{1+1}_{t,x}$ (in other words, on the line).

For wave this was $K_t(x) = \frac{1}{2}\chi_{[|x|<t]}$ which is not too singular.

In fact, setting $d=1$ in (4) we obtain
$$\varphi''(\rho) + \frac{1}{\rho}\varphi'(\rho) + \varphi(\rho) = 0$$
which is a Bessel equation with fundamental system $J_0(\rho), Y_0(\rho)$.

Y_0 is singular at $\rho = 0$ so we look for the fundamental solution in the form

$$(7) \quad K_t(x) = c_0 J_0\left(\sqrt{t^2 - r^2}\right)\chi_{[r<t]}$$

indicator function.

F514

Problem: Show that $c_0 = \frac{1}{2}$ in (7) indeed gives you the correct fundamental solution for $\Box u + u = 0$ on the line