Smooth Dynamics 2
Problem Set Nr. 1
University of Chicago
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Problem 1

Let $M$ be a Riemannian manifold with metric $\langle \cdot, \cdot \rangle$ and Levi-Civita connection $\nabla$. Fix the standard identification of $TTM$ with $\otimes^3 TM$:

**Problem 1.1** If $\theta$ is the canonical 1-form on the cotangent bundle $T^*M$, and $I : TM \to T^*M$ is the isomorphism $v \to \langle v, \cdot \rangle$, then $I^* \theta = \alpha$, where $\alpha$ is given by

$$\alpha(u,v,w) = \langle u, v \rangle$$

**Solution:** Let $[\beta]$ be a tangent vector in $TTM$, given as an equivalence class of curves, with $\beta(0) = u$, $(\pi \circ \beta)'(0) = v$, $(\nabla_{(\pi \circ \beta)'(0)} \beta)(0) = w$. Then

$$I^* \theta(\beta'(0)) = \theta(I(\beta'(0))) = I((\beta'(0))(\pi \circ \beta)'(0)) = \langle \beta(0), (\pi \circ \beta)'(0) \rangle = \langle u, v \rangle$$

where we have used the fact that $\pi \circ \beta = \pi \circ I \circ \beta$, since $I$ is a fiber preserving isomorphism. \(\square\)

**Problem 1.2** Let $\xi_i = (u, v_i, w_i)$, $i = 1, 2$, then up to sign,

$$d\alpha(\xi_1, \xi_2) = \langle v_2, w_1 \rangle - \langle v_1, w_2 \rangle$$

**Solution:** We use the following formula: for a 1-form $\omega$ and smooth vector fields $X_i$, $i = 1, 2$,

$$d\omega(X_1, X_2) = \mathcal{L}_{X_1} \omega(X_2) - \mathcal{L}_{X_2} \omega(X_1) - \omega([X_1, X_2])$$

So let $X_i$ be any smooth vector fields on $TM$ with $X_i(u) = \xi_i$ for $i = 1, 2$. We can write

$$X_i = (U_i, V_i, W_i)$$

where $U_i = \pi(X_i)$, $V_i = D\pi(X_i)$, and $W_i = \nabla_{V_i} U_i$.

We now perform a pair of computations. Let $\beta_1 : (-\varepsilon, \varepsilon) \to TM$ be an integral curve of the vector field $X_1$ with $\beta_1(0) = u$. Observe that $U_2(\beta_1(t)) = \pi(X_2(\beta_1(t))) = \beta_1(t)$, and that $\beta_1$ and $V_2 \circ \beta_1 := V_2$ are vector fields along the curve $\pi \circ \beta_1$. Lastly, let $p = \pi(u)$. Then, since $\nabla$ is a Levi-Civita connection,

$$X_1(\alpha(X_2))(u) = \frac{d}{dt} \bigg|_{t=0} \langle \beta_1(t), Y_2(t) \rangle$$

$$= \langle \langle \nabla_{(\pi \circ \beta_1)'(0)} \beta_1 \rangle(0), Y_2(0) \rangle + \langle u, \langle \nabla_{(\pi \circ \beta_1)'(0)} Y_2 \rangle(0) \rangle$$

$$= \langle w_1, v_2 \rangle + \langle u, \langle \nabla w_2 \rangle(0) \rangle$$

Next,

$$X_1(X_2)(u) = \frac{d}{dt} \bigg|_{t=0} X_2(\beta_1(t))$$

and so we compute

$$\frac{d}{dt} \bigg|_{t=0} X_2(\beta_1(t)) = \frac{d}{dt} \bigg|_{t=0} \langle \beta_1(t), Y_2(t), (W_2 \circ \beta_1)(t) \rangle$$

$$= \langle u, \langle \nabla_{(\pi \circ \beta_1)'(0)} Y_2 \rangle(0), \langle \nabla_{(\pi \circ \beta_1)'(0)} (W_2 \circ \beta_1) \rangle(0) \rangle$$

$$= \langle u, \nabla_{v_1} Y_2(u), \nabla_{v_1} (W_2 \circ \beta_1)(0) \rangle$$
Analogous computations give the same results with $X_1$ and $X_2$ flipped (flipping all the numbers). Hence we compute

$$[X_1, X_2](u) = (u, \nabla_{v_1} Y_2(0), \nabla_{v_1} (W_2 \circ \beta_1)(0)) - (u, \nabla_{v_2} Y_1(0), \nabla_{v_2} (W_1 \circ \beta_2)(0))$$

$$= (u, \nabla_{v_1} Y_2(0) - \nabla_{v_2} Y_1(0), (\nabla_{V_1} W_2 - \nabla_{V_2} W_1)(u))$$

using the symmetry of the Levi-Civita connection. Therefore we conclude that

$$X_1(\alpha(X_2)) = \langle w_2, v_1 \rangle + \langle u, \nabla_{v_1} Y_2(0) \rangle$$

and therefore,

$$\text{Problem 1.3} \quad \text{The vector field } \dot{\varphi} \text{ on } TM \text{ given by}$$

$$\dot{\varphi}(u) = (u, u, 0)$$

generates the geodesic flow.

**Solution:** Let $u \in TM$. There is a unique geodesic $\gamma$ with $\dot{\gamma}(0) = u$. If $\varphi_t$ denotes the time $t$ map of the geodesic flow, then $\varphi_t(u) = \dot{\gamma}(t)$. Note $\gamma = \pi \circ \dot{\gamma}$. Then we have

$$\frac{d}{dt} \bigg|_{t=0} \varphi_t(u) = \frac{d}{dt} \bigg|_{t=0} \dot{\gamma}(t) = \langle \dot{\gamma}(0), \gamma'(0), (\nabla_{\dot{\gamma}} \dot{\gamma})(0) \rangle = (u, u, 0)$$

so that $\frac{d}{dt} \varphi_t(u) = \dot{\varphi}(u)$ and thus the vector field $\dot{\varphi}$ on $TM$ generates the geodesic flow.

**Problem 1.4** The restriction of the geodesic flow $\varphi$ to the unit tangent bundle $T^1M$ preserves the restriction of $\alpha$ to $T(T^1M)$.

**Solution:** Let $\bar{\alpha}$ denote the restriction of $\alpha$ to $T(T^1M)$. To show that $\bar{\alpha}$ is preserved by $\varphi$, it suffices to prove that $\mathcal{L}_{\dot{\varphi}} \bar{\alpha} = 0$. By Cartan’s formula,

$$\mathcal{L}_{\dot{\varphi}} \bar{\alpha} = d(\iota_{\dot{\varphi}} \bar{\alpha}) + \iota_{\dot{\varphi}} d\bar{\alpha}$$

We now show that each of the 1-forms on the right is zero. Observe that

$$\iota_{\dot{\varphi}} \bar{\alpha}(u) = \bar{\alpha}(\dot{\varphi}(u)) = \langle u, u \rangle = 1$$

for any $u \in T^1M$. Hence the function $\iota_{\dot{\varphi}} \bar{\alpha}$ is constant on $T^1M$ and therefore $d(\iota_{\dot{\varphi}} \bar{\alpha}) = 0$. 

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Now observe that $\tilde{\alpha} = i^*(\alpha)$, where $i : T^1 M \to TM$ is the inclusion. Hence $d\tilde{\alpha} = d(i^*\alpha) = i^*(d\alpha)$. Thus, for any vector $(u, v, w) \in T(T^1 M)$,

$$
i_\varphi d\tilde{\alpha}((u, v, w)) = d\alpha(\dot{\varphi}(u), (u, v, w)) = -\langle u, w \rangle = 0$$

since for $(u, v, w) \in T(T^1 M)$, we have $\langle u, w \rangle = 0$. Hence $\mathcal{L}_\varphi \tilde{\alpha} = 0$ and so $\tilde{\alpha}$ is preserved by the geodesic flow $\varphi$.

**Problem 1.5** $\alpha$ is a contact 1-form on $T^1 M$.

**Solution:** Let $n = \dim M$. We must prove that $|\alpha \wedge (d\alpha)^{n-1}|$ is a volume element on $T^1 M$, i.e. $\alpha \wedge (d\alpha)^{n-1}$ is nonvanishing on $T^1 M$.

Let $(x^1, \ldots, x^n)$ be local coordinates on $M$, and let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be local coordinates on $TM$ coming from a local trivialization. We can take this trivialization to be orthogonal, i.e., to be inner product preserving on the fibers. We will evaluate $\alpha$ in this local coordinates. For a curve $\beta : (-\varepsilon, \varepsilon) \to TM, \beta(t) = (a^1(t), \ldots, a^n(t), b^1(t), \ldots, b^n(t))$ in local coordinates, and

$$
\dot{\beta}(t) = \sum_{i=1}^n \dot{a}^i(t) \frac{\partial}{\partial x^i} + \sum_{i=1}^n \dot{b}^i(t) \frac{\partial}{\partial y^i}
$$

The horizontal component of $\dot{\beta}(t)$ is $(a^1(t), \ldots, a^n(t), \dot{a}^1(t), \ldots, \dot{a}^n(t))$. $\alpha$ evaluated on $\dot{\beta}(0)$ is the inner product in $TM$ of the horizontal component of $\dot{\beta}(t)$ at $t = 0$ with the tangent vector $\beta(0)$. Hence in local coordinates,

$$
\alpha(\dot{\beta}(0)) = \sum_{i=1}^n \dot{a}^i(0) b^i(0)
$$

Hence we conclude that in these coordinates,

$$
\alpha = \sum_{i=1}^n y^i dx^i
$$

and therefore

$$
d\alpha = \sum_{i=1}^n dy^i \wedge dx^i
$$

Hence, we easily compute that

$$
(d\alpha)^{n-1} = \sum_{j=1}^n \left( \bigwedge_{i \neq j} (dy^i \wedge dx^i) \right)
$$

Finally,

$$
\alpha \wedge (d\alpha)^{n-1} = \sum_{j=1}^n y^j dx^j \left( \bigwedge_{i \neq j} (dy^i \wedge dx^i) \right)
$$

This form is nonzero if and only if $y^j \neq 0$ for some $j$. If we restrict to $T^1 M$, this imposes the restriction that $\sum_{j=1}^n (y^j)^2 = 1$, which forces that $y^j \neq 0$ for some $j$. So the restriction
of $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing and therefore $\alpha$ is a contact 1-form on $T^1 M$. As a last note, by moving $dx^j$ to the right, we can write this form as

$$\alpha \wedge (d\alpha)^{n-1} = (-1)^{n-1} \left( \sum_{j=1}^{n} (-1)^{j+1} y^j \wedge dy^j \right) \wedge \left( \bigwedge_{i=1}^{n} dx^i \right)$$

Up to sign, this is the product of the standard volume form on $S^{n-1} \subset \mathbb{R}^n$ with the standard volume form on $\mathbb{R}^n$. Pulling back to $T^1 M$, this shows that $\alpha \wedge (d\alpha)^{n-1}$ is locally the product of the Riemannian volume on $M$ with the volume on $S^{n-1}$ in the tangent space given by the Riemannian structure.

\[ \square \]

**Problem 2**

**Problem 2.1** Let $M$ be compact without conjugate points. Prove the limits defining the Busemann functions on $\tilde{M}$ exist, and that

$$b^+_v (\pi(v)) = 0$$

$$|b^+_v (p) - b^+_v (q)| \leq d(p,q)$$

**Solution:** We will prove these properties for $b^+_v$, as exactly analogous proofs work for $b^-_v$. For $v \in T^1 M$, let

$$b^+_{v,t} (p) = d(\gamma_v(t), p) - t$$

where $\gamma_v$ is the unique parametrized geodesic with $\gamma_v(0) = v$ (note $\gamma_v$ is unit-speed). Since $M$ has no conjugate points, the metric universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$ via the exponential map at any point, where $n = \dim M$, and further, in $\tilde{M}$ any two points are connected by a unique geodesic which realizes the minimal distance between those two points. For $s \geq t \geq 0$, we have by the triangle inequality,

$$d(\gamma_v(s), p) - d(\gamma_v(t), p) \leq d(\gamma_v(s), \gamma_v(t)) = s - t$$

Rearranging this gives

$$d(\gamma_v(s), p) - s \leq d(\gamma_v(t), p) - t$$

which implies that $b^+_{v,s} \leq b^+_{v,t}$. Further, for any $t \geq 0$, we have

$$d(\gamma_v(t), p) - t = d(\gamma_v(t), p) - d(\gamma_v(t), \gamma_v(0)) \geq -d(\gamma_v(0), p)$$

again by the triangle inequality. So for each $p$, the function $t \to b^+_{v,t}(p)$ is decreasing and bounded below. Hence the limit $\lim_{t \to \infty} b^+_{v,t}(p)$ exists for each $p \in \tilde{M}$, and the Busemann function $b^+_v$ is thus well-defined.

For each $t$, we have

$$d(\gamma_v(t), \pi(v)) = d(\gamma_v(t), \gamma_v(0)) = t$$

and therefore $b^+_{v,t}(\pi(v)) = 0$ for every $t$, and hence $b^+_v (\pi(v)) = 0$. Also,

$$|b^+_{v,t}(p) - b^+_{v,t}(q)| = |d(\gamma_v(t), p) - d(\gamma_v(t), q)| \leq d(p,q)$$
for any $t, p, q \in \tilde{M}$, and therefore taking $t \to \infty$, we conclude that

$$|b^+_v(p) - b^+_v(q)| \leq d(p, q)$$

**Problem 2.2**

$$b^\pm_{\varphi_s(v)} = b^\pm_v - t$$

**Solution:** Let $v \in T^1\tilde{M}$. For any $p \in \tilde{M}$, we calculate

$$b^+_{\varphi_s(v)}(p) = \lim_{t \to \infty} d(\gamma_{\varphi_s(v)}(t), p) - t$$

$$= \lim_{t \to \infty} d(\gamma_v(t + s), p) - t$$

$$= (\lim_{t \to \infty} d(\gamma_v(t + s), p) - (t + s)) + s$$

$$= b^+_v(p) + s$$

and similarly,

$$b^-_{\varphi_s(v)}(p) = \lim_{t \to \infty} d(\gamma_{\varphi_s(v)}(-t), p) - t$$

$$= \lim_{t \to \infty} d(\gamma_v(-t + s), p) - t$$

$$= (\lim_{t \to \infty} d(\gamma_v(-t + s), p) - (s - t)) + s$$

$$= b^-_v(p) + s$$

(I am not sure how I am computing these signs incorrectly.)

**Problem 3**

I will skip the verification of the curvature of $\mathbb{H}$, the isometries of $\mathbb{H}$ and $\mathbb{D}$, and the geodesics of these models. I’ll do the identification of $\text{PSL}(2, \mathbb{R})$ with $T^1\mathbb{H}$.

**Problem 3.1** *The stabilizer of a point under the action of $\text{PSL}(2, \mathbb{R})$ is the compact subgroup $K = \text{SO}(2)/\{\pm I\}$, so there is an identification of $\mathbb{H}$ with the coset space of $K$, $\mathbb{H} = \text{PSL}(2, \mathbb{R})/K$.***

**Solution:** $\text{PSL}(2, \mathbb{R})$ consists of all orientation-preserving isometries of $\mathbb{H}$, acting by Mobius transformations. These isometries consist of 3 types: translations along a geodesic, limit rotations about a point on $\partial H$, and rotations about a point in $\mathbb{H}$. Of these, only rotations fix a point in $\mathbb{H}$. So the stabilizer of a point is the group of all rotations about that point, and since a rotation by any angle can be realized by a hyperbolic isometry, this subgroup is isomorphic to $\text{SO}(2)/\{\pm I\}$. (I am not sure how I am computing these signs incorrectly.)

**Problem 3.2** *The derivative action of $\text{PSL}(2, \mathbb{R})$ on $T^1\mathbb{H}$ is free and transitive and gives an analytic identification between $T^1\mathbb{H}$ and $\text{PSL}(2, \mathbb{R})$. Under this identification, the action of $\text{PSL}(2, \mathbb{R})$ on $T^1\mathbb{H}$ by isometries corresponds to left multiplication in $\text{PSL}(2, \mathbb{R})$.***

**Solution:** We first show the derivative action of $\text{PSL}(2, \mathbb{R})$ is transitive. $\text{PSL}(2, \mathbb{R})$ acts transitively on $\mathbb{H}$ by isometries, so it suffices to restrict to a single point (say $i$) and prove
that \( \text{PSL}(2, \mathbb{R}) \) acts transitively on \( T^1_i \mathbb{H} \), the fiber of \( T^1 \mathbb{H} \) over \( i \). But \( \text{PSL}(2, \mathbb{R}) \) has a subgroup which acts by rotations about \( i \), namely the collection of isometries

\[
z \mapsto \frac{\cos \theta z + \sin \theta}{-\sin \theta z + \cos \theta}
\]

for \( \theta \in \mathbb{R} \), corresponding to a rotation of angle \( \theta \). In particular, for \( \theta \in [0, 2\pi) \), this will rotate the vertical unit speed geodesic \( t \mapsto e^{t\theta} \) through \( i \) to the unit speed geodesic which makes an angle \( \theta \) with this vertical geodesic, measured counterclockwise. Since all angles \( \theta \) are realized, it follows that \( \text{PSL}(2, \mathbb{R}) \) acts transitively on \( T^1_i \mathbb{H} \).

We next show the derivative action of \( \text{PSL}(2, \mathbb{R}) \) is free on \( T^1 \mathbb{H} \). For suppose we have a vector \( v \in T^1 \mathbb{H} \) fixed by the derivative action of some element \( A \in \text{PSL}(2, \mathbb{R}) \). Since this action is transitive, by conjugating by an element of \( \text{PSL}(2, \mathbb{R}) \) we can assume that \( v \) is the vertical unit tangent vector corresponding to the geodesic \( \gamma \) given by \( \gamma(t) = e^{it} \). If \( A \) fixes \( v \), then since \( A \) is an isometry and therefore takes geodesics to geodesics, \( A \) must fix the entire geodesic \( \gamma \). Hence \( A \) fixes the line \( \text{Re}(z) = 0 \) in \( \mathbb{H} \). But by the classification of orientation-preserving isometries of \( \mathbb{H} \), any such isometry has at most one fixed point unless it is the identity. Hence we conclude that \( A = I \), and therefore the derivative action is free.

Fix \( v \) now to be the vertical unit tangent vector at \( i \). We identify \( T^1 \mathbb{H} \) with \( \text{PSL}(2, \mathbb{R}) \) by taking \( v = I \), the identity in \( \text{PSL}(2, \mathbb{R}) \), and then identifying \( A \) with \( dA(v) \). This identification makes sense since the derivative action of \( \text{PSL}(2, \mathbb{R}) \) is free and transitive, and is analytic since the derivative action of \( \text{PSL}(2, \mathbb{R}) \) on \( T^1 \mathbb{H} \) is clearly analytic. The fact that the derivative action corresponds to left multiplication in \( \text{PSL}(2, \mathbb{R}) \) is essentially by definition, since for \( w \in T^1 \mathbb{H}, A \in \text{PSL}(2, \mathbb{R}) \), we have that \( w = dB(v) \) for a unique \( B \in \text{PSL}(2, \mathbb{R}) \), and therefore

\[
dA(w) = dA(dB(v)) = d(A \circ B)(v)
\]

and composition of Möbius transformations corresponds to multiplication in \( \text{PSL}(2, \mathbb{R}) \).

**Problem 3.3** By endowing \( \text{PSL}(2, \mathbb{R}) \) with a suitable left-invariant metric, the identification \( \mathbb{H} \cong \text{PSL}(2, \mathbb{R})/K \) becomes an isometry. In this metric, called the Sasaki metric, geodesics in \( \mathbb{H} \) lift to geodesics in \( T^1 \mathbb{H} \) via \( \gamma \mapsto \dot{\gamma} \).

**Solution:** We can think of each vector \( v \in T^1 \mathbb{H} \) as a pair \( (z, \theta) \in \mathbb{H} \times S^1 \), where \( z = \pi(v) = a + bi \) and \( \theta \) is the angle that the unit speed geodesic through \( v \) makes with the vertical unit speed geodesic \( t \mapsto a + e^{t\theta}ib \) through \( z \), measured counterclockwise from the vertical. This gives a bundle isomorphism \( T^1 \mathbb{H} \cong \mathbb{H} \times S^1 \). We then put a metric on \( T^1 \mathbb{H} \) by giving the fiber \( (S^1)_z \) the metric it carries at each point \( z \) as the unit circle centered at \( z \) in the metric of \( \mathbb{H} \).

We claim that in this metric, \( \text{PSL}(2, \mathbb{R}) \) acts by isometries on \( T^1 \mathbb{H} \). This is trivial, since \( \text{PSL}(2, \mathbb{R}) \) acts by isometries on \( \mathbb{H} \), and therefore an element \( A \) takes the unit circle centered at \( z \) in \( \mathbb{H} \) isometrically onto the unit circle centered at \( A(z) \) in \( \mathbb{H} \). Call this metric \( d \).

We give \( \text{PSL}(2, \mathbb{R}) \) the metric induced from the metric \( d \) on \( T^1 \mathbb{H} \) by the identification of Problem 3.2. This metric is trivially left-invariant, since the derivative action of \( \text{PSL}(2, \mathbb{R}) \)
on $T^1\mathbb{H}$ corresponds to left multiplication in $\text{PSL}(2,\mathbb{R})$ under this identification, and we just verified above that the derivative action of $\text{PSL}(2,\mathbb{R})$ on $T^1\mathbb{H}$ is by isometries in the metric $d$ that we are pulling back.

To see that the identification $\text{PSL}(2,\mathbb{R})/K \leftrightarrow \mathbb{H}$ is an isometry, note that this corresponds to the identification in $T^1\mathbb{H}$ given by fixing an angle $\theta \in S^1$ and looking at points $(z,\theta)$ for $z \in \mathbb{H}$. This is isometric to $\mathbb{H}$ by definition.

Lastly, we should check that geodesics in $\mathbb{H}$ lift to geodesics in $T^1\mathbb{H}$ via the map $\gamma \rightarrow \dot{\gamma}$. Observe first that the geodesic $\gamma(t) = e^t i$ through $i$ with tangent vector the vertical through $i$ lifts to the curve $\dot{\gamma}(t) = (e^t i, 0)$ in $T^1\mathbb{H}$. This is a geodesic in $\mathbb{H}$ since for a fixed angle ($0$ here), the resulting submanifold of $T^1\mathbb{H}$ is canonically isometric to $\mathbb{H}$. Since $\text{PSL}(2,\mathbb{R})$ acts on $T^1\mathbb{H}$ by isometries, we can find other geodesics by applying elements $A$ of $\text{PSL}(2,\mathbb{R})$. But if $\pi: T^1\mathbb{H} \rightarrow \mathbb{H}$ denotes the projection, then $\pi(dA(\dot{\gamma}(t))) = A(\gamma(t))$ and it follows that the geodesic $dA(\dot{\gamma})$ is the lift of the geodesic $A(\gamma)$ in $\mathbb{H}$, which gives the claim, since $\text{PSL}(2,\mathbb{R})$ acts transitively on the geodesics of $\mathbb{H}$.

**Problem 4** The geodesic flow satisfies flip invariance: For $v \in TM$,

$$\varphi_{-t}(-v) = -\varphi_t(v)$$

**Solution:** Let $\gamma_v$ be a parametrized geodesic with $\dot{\gamma}_v(0) = v$. Observe that the curve $\beta_v(t) = \gamma_v(-t)$ is also a geodesic, with $\dot{\beta}_v(t) = -\dot{\gamma}_v(-t)$. Then

$$\varphi_{-t}(-v) = \dot{\beta}_v(-t) = -\dot{\gamma}_v(t) = -\varphi_t(v)$$

**Problem 5** Show that on $T^1\mathbb{H} = \text{PSL}(2,\mathbb{R})$, the geodesic flow is given by right multiplication by the 1-parameter subgroup:

$$G = \left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}$$

*(renamed to avoid notation conflict)*

**Solution:** We consider first the geodesic $\gamma(t) = e^t i$ in $\mathbb{H}$. Observe that

$$a_t(z) = e^t z$$

for $z \in \mathbb{H}$. In particular, $a_t(i) = \gamma(t)$. Letting $\varphi$ denote the geodesic flow on $T^1\mathbb{H}$, and putting $\dot{\gamma}(0) = I \in \text{PSL}(2,\mathbb{R})$ (under our identification) we conclude that the time $t$ map of the geodesic flow applied to this point is

$$\varphi_t(I) = a_t = I \cdot a_t$$

The geodesic flow is preserved by isometries. Since elements of $\text{PSL}(2,\mathbb{R})$ acting by left multiplication act by isometries on $\text{PSL}(2,\mathbb{R})$, we conclude that for any $B \in \text{PSL}(2,\mathbb{R})$,

$$\varphi_t(B) = B \cdot \varphi_t(I) = B \cdot a_t$$

Thus the time $t$ map of the geodesic flow is given by right multiplication by $a_t$. 

\[\square\]
Problem 6

Problem 6.1 Verify that the horocyclic foliations $H^\pm$ are the foliations of $\text{PSL}(2, \mathbb{R})$ by cosets of the horocyclic subgroups

$$P^+ = \left\{ h^+_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$P^- = \left\{ h^-_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Solution: We will begin by computing the Busemann functions for $H^\pm$. We need the following general fact: for any $v \in T^1 \tilde{M}$, $p \in \tilde{M}$, where $\tilde{M}$ is a simply connected nonpositively curved manifold, the Busemann function $b^+_v$ (and similarly $b^-_v$), we have

$$\nabla b^+_v(p) = -\dot{c}_v(0),$$

where $c_v$ is the unique geodesic in $\tilde{M}$ with $c_v(0) = p$ and $c_v(\infty) = \gamma_v(\infty)$ (the two geodesics are asymptotic at infinity).

We also compute how the Busemann functions transform under an isometry $f : \tilde{M} \to \tilde{M}$ and switching from $+$ to $−$. We have

$$b^+_{df(v)}(f(p)) = \lim_{t \to \infty} d(\gamma_v(t), f(p)) - t$$

$$= \lim_{t \to \infty} d(f(\gamma_v(t)), f(p)) - t$$

$$= \lim_{t \to \infty} d(\gamma_v(t), p) - t$$

$$= b^+_v(p)$$

from which we see that $b^+_{df(v)} \circ f = b^+_v$.

To switch directions, we note that $b^+_v = b^-_{-v}$, which is clear by the flip invariance of the geodesic flow.

With all this in mind, we first compute $b^+_I$, where $I \in \text{PSL}(2, \mathbb{R})$ is identified with the vertical unit tangent vector at $i$. Of course $\gamma_I(t) = e^{ti}$, so $\gamma_I(\infty) = \infty \in \partial H$. The geodesics $\gamma$ with $\gamma(\infty) = \infty$ are the vertical lines in $\mathbb{H}$, oriented toward $\infty$. Hence we conclude that the integral curves of the gradient field $\nabla b^+_I$ are all geodesics of the form

$$t \to a + e^{-t}bi, \ a + bi \in \mathbb{H}$$

It follows that the gradient field consists entirely of vertical vectors of norm 1 pointing toward the real axis. Hence the vectors orthogonal to this vector field (which form the tangent space of the level sets of $b^+_I$) are all vectors parallel to the real axis. It follows that

$$b^+_I(bi) = b^+_I(e^{\log b}i) = -\log b$$

and it follows that

$$b^+_I(a + bi) = -\log b$$
for any $a + bi \in \mathbb{H}$. Hence it follows, for any $A \in \text{PSL}(2, \mathbb{R})$, that
\[ b_A^+(z) = (b_A^+ \circ A^{-1})(z) = -\log(\text{Im}(A^{-1}z)) \]

If we put $J$ to be the counterclockwise rotation by $\pi$ about the point $i$ (so $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$) then
\[ J(z) = -\frac{1}{z} \]
and $J = J^{-1}$, and it’s easy to see that
\[ b_J^-(z) = b_J^+(z) = b_J^+(J(z)) = -\log(\text{Im}(-z^{-1})) = -\log(|z|^{-2}\text{Im}(z)) \]
and therefore
\[ b_A^-(z) = -\log(|A^{-1}(z)|^2\text{Im}(A^{-1}z)) \]

In particular, this verifies that the Busemann functions are $C^\infty$. The dependence on $A$ here is also $C^\infty$, incidentally.

The Busemann function $b_J^+$ has level sets $\text{Im}(z) = \text{const}$. These level sets are the orbits of points on the imaginary axis under the isometry $t \to z + t$, for $t \in \mathbb{R}$. This corresponds to $h_t^+$. Since $b_J^+$ is easily seen to be invariant under precomposition with $h_t$, $h_t^+$ takes gradient vectors to gradient vectors. We thus see that
\[ \nabla b_J^+(t + i) = h_t^+ \]
so that
\[ \mathcal{H}^+(I) = \{ h_t^+ : t \in \mathbb{R} \} \]

For an isometry $f$, its effect on Busemann functions computed earlier implies that
\[ \nabla b_{df(v)}^+ \circ df = \nabla b_v^+ \]
and therefore
\[ \mathcal{H}^+(df(v)) = df(\mathcal{H}^+(v)) \]

Hence we immediately see that
\[ \mathcal{H}^+(A) = \{ A \cdot h_t^+ : t \in \mathbb{R} \} \]

since the derivative action is left multiplication. Thus the horocyclic foliation $\mathcal{H}^+$ coincides with the foliation by cosets of the 1-parameter subgroup $P^+$.

It’s easily checked that $b_I^- \circ h_I^- = b_I^-$, by direct calculation, where
\[ h_t(z) = \frac{1}{tz + 1} \]
for $t \in \mathbb{R}$. Then in the same way as before, we conclude that
\[ \mathcal{H}^-(A) = \{ A \cdot h_I^- : t \in \mathbb{R} \} \]
so that the horocyclic foliation $\mathcal{H}^-$ coincides with the foliation by cosets of the 1-parameter subgroup $P^-$. \qed
Problem 6.2  Verify that the geodesic flow on $T^1 \mathbb{H}$ is Anosov.

Solution:  We identify $T^1 \mathbb{H}$ with $\text{PSL}(2, \mathbb{R})$ in the usual way. We have three foliations of $T^1 \mathbb{H}$: the horocyclic foliations $\mathcal{H}^+$ and $\mathcal{H}^-$, as well as the foliation by flow lines $\mathbb{R}_\varphi$. In $\text{PSL}(2, \mathbb{R})$, these become the 1-parameter subgroups generated by $h_t^+$, $h_t^-$, and $a_t$ respectively. From the relation
\[ b^\pm_{\varphi_t(v)} = b^\pm_v - t \]
one sees immediately that the horocyclic foliations are preserved by the geodesic flow, i.e.
\[ \mathcal{H}^\pm(\varphi_t(v)) = \varphi_t(\mathcal{H}^\pm(v)) \]
for any $t \in \mathbb{R}$.

Our first goal is to prove that these foliations are everywhere transverse to one another. $\text{SL}(2, \mathbb{R})$ is a 2-fold cover of $\text{PSL}(2, \mathbb{R})$ by the map $A \rightarrow -A$, so the Lie algebra of $\text{PSL}(2, \mathbb{R})$ can be identified with the Lie algebra of $\text{SL}(2, \mathbb{R})$,
\[ \mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \]
Left multiplication acts as an isometry on $\text{PSL}(2, \mathbb{R})$, so it suffices to verify these three foliations are transverse at $I$, i.e. that their tangent vectors at $I$ together span $\mathfrak{sl}(2, \mathbb{R})$. We compute
\[ T_I G = \mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ T_I P^+ = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]
\[ T_I P^- = \mathbb{R} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]
and it is evident that these three subspaces together span $\mathfrak{sl}(2, \mathbb{R})$.

It follows that we have a splitting $\mathfrak{sl}(2, \mathbb{R}) = T_I G \oplus T_I P^+ \oplus T_I P^-$ which gives rise via the left multiplication isometry action to a splitting of this type at every point into three subspaces, one of which, $G$, is the flow direction, and the other two, corresponding to $P^+$ and $P^-$, remain invariant under the flow.

To prove that the geodesic flow $\varphi$ is Anosov, we will show that the leaves of the foliation $P^+$ are exponentially contracted by the geodesic flowing forward in time, and the leaves of the foliation $P^-$ are exponentially contracted flowing backward in time. Since the left multiplication isometry action is transitive, it suffices to prove these assertions for the leaves through $I$. First we consider $P^+$. Take two points $h^+_i$ and $h^+_s$ in $P^+$. We will flow these points forward by a time $t$, taking
\[ h^+_i \rightarrow h^+_i a_t \]
for $i = s, t$. Let $d$ denote the distance on $\text{PSL}(2, \mathbb{R})$. Observe that on the horocycle $\text{Im}(z) = 1$ which lifts to $P^+$, the lift is an isometry onto its image since all of the gradient
vectors of $B^+_I$ along $\mathcal{H}^+(I)$ are vertical. Further, distance on $\text{Im}(z) = 1$ coincides with Euclidean distance. We conclude that

$$d(h^+_s, h^+_r) = |s - r|$$

By the fact that left multiplication is an isometry, it follows that for any $A \in \text{PSL}(2, \mathbb{R})$,

$$d(A \cdot h^+_s, A \cdot h^+_r) = |s - r|$$

Now observe the following commutation relation: for any $s, t \in \mathbb{R}$,

$$h^+_s a_t = a_t h^+_e t_s$$

It follows that

$$d(h^+_s a_t, h^+_r a_t) = d(a_t h^{-t_s}, a_t h^{-t_r}) = d(h^{-t_s}, h^{-t_r}) = e^{-t}|s - r| = e^{-t}d(h^+_s, h^+_r)$$

where we have used the fact that left multiplication is an isometry. So we have exponential contraction along $P^+$ going forward in time.

Now we calculate that $h^-_s = Jh^-_{-s}J^{-1}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence

$$d(h^-_s, h^-_r) = |s - r|$$

as well, since conjugation by $J$ is an isometry at the level of $\mathbb{H}$, which takes the horocycles $\text{Im}(z) = \text{const}$ to the horocycles corresponding to limit rotations about $0 \in \partial \mathbb{H}$.

We then compute the commutation relation

$$h^-_s a_t = a_t h^e t_s$$

which leads in an analogous way to the + case to the formula

$$d(h^-_s a_t, h^-_r a_t) = e^t d(h^-_s, h^-_r)$$

which proves that the leaves of $P^-$ are exponentially contracted as we flow backward in time. This completes the verification that the geodesic flow is Anosov.