A Harder-Narasimhan theory for Kisin modules

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Theorem (Wiles, Taylor-Wiles, BCDT)

Any elliptic curve $E/\mathbb{Q}$ is modular.

A key input into Wiles’ method and subsequent improvements is a good understanding of Galois deformation spaces at $\ell = p$. 
Local deformation problem

Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $\Gamma_K$ be the absolute Galois group of $K$. Fix

$$\bar{\rho} : \Gamma_K \to \text{GL}_n(\mathcal{O}_p).$$

There is a universal deformation space $D_{\bar{\rho}}$ represented by a quotient of a power-series ring $R_{\bar{\rho}}$ over $\mathbb{Z}_p$ (when $\bar{\rho}$ is absolutely irreducible).

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HN-theory and Kisin varieties

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If $E$ is an elliptic curve over $K$ with good reduction, then $E[p^n]$ is a finite flat group scheme over $\mathcal{O}_K$ for all $n$. The representation of $\Gamma_K$ on the $p^n$-torsion points is called flat.

The flat deformation space $D_{\rho}^{\text{fl}}$ is the subspace of $D_{\rho}$ of representations that come from finite flat group schemes over $\mathcal{O}_K$. 
What are the connected components of $D_{\rho}^{\text{fl}}[1/p]$?

- (Ramakrishna) When $n = 2$ and $K = \mathbb{Q}_p$, then $D_{\rho}^{\text{fl}}[1/p]$ is connected.
- When $n = 2$, we have full description of connected components for any $K$ by work of Kisin, Imai, Gee, and Hellmann.
- When $n > 2$, the question is open in general (unless $K$ is mildly ramified).
Theorem (Kisin)

There is a projective variety $X_{\overline{\rho}}$ over $\mathbb{F}_p$ such that $X_{\overline{\rho}}(\mathbb{F})$ is the set of finite flat group schemes $G$ over $\mathcal{O}_K$ such that $G(K) \cong \overline{\rho} \otimes_{\mathbb{F}_p} \mathbb{F}$.

Application

Connected components of $D_{\overline{\rho}}^{[1/p]}$ are related to the connected components of $X_{\overline{\rho}}$. 
**Kisin modules**

**Definition**

Assume $K/\mathbb{Q}_p$ is totally ramified of degree $e$ and $F$ is a finite field. Let $\varphi : F[[u]] \to F[[u]]$ be the homomorphism sending $u \mapsto u^p$. A *Kisin module* of rank $n$ and height $\leq h$ over $F$ is a finite free $F[[u]]$-module $M_F$ with a semilinear map

$$\phi_{M_F} : M_F \to M_F$$

such that the cokernel (of the linearization) is killed by $u^{eh}$.

**Theorem (Kisin)**

The category of Kisin modules over $F$ of height $\leq 1$ is anti-equivalent to the category of finite flat group schemes over $O_K$ with an $F$-action.
The generic fiber of \((M, \phi)\) is \((M[1/u], \phi_M[1/u])\). (This is an étale \(\mathbb{F}((u))\)-module).

The degree of \((M, \phi_M)\) is \(\frac{1}{e} \dim_{\mathbb{F}} \ker(\phi_M)\).

The slope is \(\mu(M) := \deg(M)/\text{rank}(M)\).

This was inspired by Fargues’ (2010) theory of Harder-Narasimhan filtrations for finite flat group schemes.
Examples

Let $(\mathcal{M}, \phi_{\mathcal{M}})$ have height $\leq 1$ and let $\mathcal{G}_\mathcal{M}$ be corresponding finite flat group scheme over $\mathcal{O}_K$.

- If $\mathcal{M}$ has slope 1, then $\mathcal{G}_\mathcal{M}$ is étale.
- If $\mathcal{M}$ has slope 0, then $\mathcal{G}_\mathcal{M}$ is multiplicative, i.e., Cartier dual to étale.

Remark

The HN-filtration generalizes the connected-etale sequence for finite flat group schemes.

$$1 \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{G}^{\text{et}} \to 1$$
Theorem (L.-W. E.)

The function $\mu$ defines an HN-theory on the category of Kisin modules. In particular, any $M$ has a canonical HN-filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$$

by strict subobjects such that $M_{i+1}/M_i$ is semi-stable and $\mu(M_i/M_{i-1}) < \mu(M_{i+1}/M_i)$.

Definition

The HN-polygon is the concave polygon with breakpoints given by $(\text{rank}(M_i), \text{deg}(M_i))$. In particular, it starts at $(0, 0)$ and ends at $(\text{rank}(M), \text{deg}(M))$. 
Kisin varieties

**Definition**

For $\nu = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{Z}$ and $a_{i+1} \geq a_i$, a Kisin module $(\mathcal{M}, \phi_{\mathcal{M}})$ over $\mathbb{F}$ of rank $n$ has **Hodge type** $\nu$ if there exists a basis $\{e_i\}$ of $\mathcal{M}$ such that $u^{a_i}e_i$ generates the image of $\phi_{\mathcal{M}}$.

**Definition**

Let $(\mathcal{M}_{\overline{\rho}}, \phi)$ be the étale $\mathbb{F}_p((u))$-module of rank $n$ attached to $\overline{\rho}$. The closed **Kisin variety** has points given by

$$X_{\overline{\rho}}^\nu = \{ \mathcal{M}[1/u] \cong \mathcal{M}_{\overline{\rho}} | \mathcal{M} \text{ has Hodge type } \leq \nu \}.$$ 

It is a projective scheme over $\mathbb{F}_p$. 
Stratification

Theorem (L.-W. E.)

There is a stratification

$$\bigcup_{P} X_{\rho}^{\nu, P} = X_{\rho}^{\nu}$$

by locally closed subschemes indexed by concave polygons $P$ such that the points of $X_{\rho}^{\nu, P}$ are the Kisin modules with HN-polygon $P$.

Remark

For any point in $X_{\rho}^{\nu}$, the HN-polygon lies above the Hodge polygon $\nu$ with the same endpoints. Hence, there are a finite number of such strata.
In the following slides, for different Hodge polygons \( \nu \), we draw the set of possible HN-polygons.

- For any \( \overline{\rho} \) of the appropriate dimension, the strata of \( X^\nu_{\overline{\rho}} \) will be indexed by this finite set of polygons.
- The Hodge polygon \( \nu \) appears in black.
- We color the polygons the same if they share the same segments in common with the Hodge polygon.
- Only strata with the same color can occur on the same connected component (i.e., the union of the strata with same color is open and closed in \( X^\nu_{\overline{\rho}} \)).

**Remark**

*For any particular \( \overline{\rho} \), many of the strata could be empty. For example, if \( \overline{\rho} \) is irreducible, then only the constant slope stratum will appear.*
Components for $GL_2$, $\nu = (0, 3)$
Components for $\GL_3$, $\nu = (0, 0, 1)$
Components for $GL_3$, $\nu = (-1, 0, 1)$
Components for $\text{GSp}_4, \nu = (-2, -1, 1, 2)$
Tensor product theorem

**Expected Theorem (L.-W. E.)**

Let $M$ and $N$ be Kisin modules over $\mathbb{F}$. If $M$ and $N$ are semistable, then

$$M \otimes_{\mathbb{F}} N$$

is semistable of slope $\mu(M) + \mu(N)$.

**Application**

Study Kisin varieties for reductive groups $G$ and $G$-valued flat deformation rings.