ALGEBRAICITY OF BG AND Bun_G

ZHILIN LUO

Abstract. We will first review the definition of stacks and algebraic stacks, then introduce the classifying stack BG and show that it is an algebraic stack. We will also introduce the stack $\mathrm{Bun}_{G,X}$ and focus on the case when $G = \mathrm{GL}_n$ and X is a smooth projective curve over an algebraically closed field k. We will recall Weil's automorphic interpretation of $\mathrm{Bun}_{n,X}(k)$. Finally we will sketch a proof for the algebraicity of $\mathrm{Bun}_{n,X}$.

Contents

1. Prestacks and stacks 1
2. G-bundles 3
3. Algebraicity of BG 4
4. Automorphic interpretation of Bun_n d'après Weil 6
5. Algebraicity of Bun_n 6
References 8

Any corrections and comments are welcome.
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1. Prestacks and stacks

We recall the definition of prestacks and stacks from Justin's talk.

Let k be a commutative ring, and CAlg be the category of commutative k-algebras. By definition, an affine k-scheme is a representable functor from CAlg to the category of Sets. Via Yoneda's lemma we will freely identify $\mathrm{CAlg} \simeq \mathrm{Aff}^{\mathrm{op}}$.

Recall that PreStk:=Fun(CAlg, Grpd). To make it precise, for $\mathfrak{X} \in \mathrm{PreStk}$, any $A \in \mathrm{Calg}$, $\mathfrak{X}(A)$ is a groupoid, and for any morphism $f : A \to B \in \mathrm{Hom}_{\mathrm{CAlg}}(A, B)$, there exists a functor $\mathfrak{X}(f) : \mathfrak{X}(A) \to \mathfrak{X}(B)$ between groupoids satisfying various compatibilities:

(1) There exists a natural isomorphism $\epsilon_A : \mathfrak{X}(\mathrm{id}_A) \simeq \mathrm{id}_{\mathfrak{X}(A)}$;
(2) For a diagram in CAlg,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there exists a natural isomorphism between functors $\eta_{f,g} : \mathfrak{X}(g \circ f) \simeq \mathfrak{X}(g) \circ \mathfrak{X}(f)$ satisfying "composition relation", i.e. whenever we have a diagram in CAlg,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{X}(h\circ g\circ f) & \xrightarrow{\eta_{h,g\circ f}} & \mathfrak{X}(h)\circ\mathfrak{X}(g\circ f) \\ \downarrow{\scriptstyle \eta_{h\circ g,f}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{X}(h)}*\eta_{g\circ f}} \\ \mathfrak{X}(h\circ g)\circ\mathfrak{X}(f) & \xrightarrow{\eta_{h,g}*\mathrm{id}_{\mathfrak{X}(f)}} & \mathfrak{X}(h)\circ\mathfrak{X}(g)\circ\mathfrak{X}(f) \end{array}$$

(3) For any $f:U\to V\in\mathrm{Hom}_{\mathrm{CAlg}}(U,V)$, and $\xi\in\mathrm{Ob}(\mathfrak{X}(U))$,

$$\eta_{\mathrm{id}_U,f}(\xi)=\epsilon_V(\mathfrak{X}(f)(\xi)),\quad \eta_{f,\mathrm{id}_V}(\xi)=\mathfrak{X}(f)(\epsilon_U(\xi)).$$

In literature $\mathfrak{X}$ is usually called a pseudo-functor.

A stack is just a prestack with gluing conditions whenever we equip Aff with suitable Grothendieck topology. Notice that we need to glue both morphisms and objects. In the following, we will freely interchange CAlg and $\mathrm{Aff}^{\mathrm{op}}$ when necessary.

Remark 1.0.1. In the following, we will always use the étale topology on Aff, which consists of collections of morphisms of affine schemes $\{U_i\to U\}_{i\in I}$ with I finite, $U_i\to U$ étale and $\bigsqcup_i U_i\to U$ fully faithful.

Definition 1.0.2 (Stack)**.** $\mathfrak{X}\in\mathrm{PreStk}$ is called a Zariski/étale/smooth/fppf/fpqc stack if for any covering $\{U_i\to U\}$

(1) (Gluing objects) Given $x_i\in\mathrm{Ob}\big(\mathfrak{X}(u_i)\big)$ and morphisms $\phi_{i,j}:x_i|_{U_{i,j}}\to x_j|_{U_{i,j}}$ satisfying the cocycle condition

$$\phi_{i,j}|_{U_{i,j,k}}\circ\phi_{j,k}|_{U_{i,j,k}}=\phi_{i,k}|_{U_{i,j,k}}$$

(This is ususally called a descent datumn for $\mathfrak{X}$ w.r.t. the covering $\{U_i\to U\}$) there exists an object $x\in\mathfrak{X}(U)$ and isomorphisms $\phi_i:x|_{U_i}\simeq x_i\in\mathfrak{X}(U_i)$ such that

$$\phi_{j,i}\circ\phi_i|_{U_{i,j}}=\phi_j|_{U_{i,j}},\quad\forall i,j\in I$$

(Ususally call the descent datumn is effective);

(2) (Gluing morphisms) For any $x,y\in\mathfrak{X}(U)$, the presheaf

$$\underline{\mathrm{Isom}}_U(x,y):(U_i\to U)\mapsto\mathrm{Hom}_{\mathfrak{X}(U_i)}(x|_{U_i},y|_{U_i})$$

is a sheaf. Equivalently,

a) For $x,y\in\mathrm{Ob}\big(\mathfrak{X}(U)\big)$ and $\phi_i:x|_{U_i}\to y|_{U_i}$ such that

$$\phi_i|_{U_{i,j}}=\phi_j|_{U_{i,j}}$$

there exists a unique morphism $\eta:x\to y$ such that $\eta|_{U_i}=\phi_i$.

b) Given $x,y\in\mathrm{Ob}\big(\mathfrak{X}(U)\big)$ and morphisms $\phi:x\to y,\psi:x\to y$, such that

$$\phi|_{U_i}=\psi|_{U_i}$$

then $\phi=\psi$.

Let us see some examples.

Example 1.0.3. For $S\in\mathrm{Aff}$, let $\mathrm{Bun}_{n,S}$ be the prestack of rank n vector bundles over Aff, which sends $X\in\mathrm{Aff}$ to $\mathrm{Bun}_{n,S}(X)$ the groupoid of rank n-vector bundles over $X_S:=X\times S$ with vector bundle isomorphisms between them. A morphism between two vector bundles $\mathcal{V}_1\to X_{S_1}$ and $\mathcal{V}_2\to X_{S_2}$ is just a morphism $\phi:S_1\to S_2$ with an isomorphism $\mathcal{V}_1\simeq(\mathrm{id}_X\times\phi)^*\mathcal{V}_2$, i.e. an isomorphism between $\mathcal{V}_1$ and the pull-back of $\mathcal{V}_2$ along the map $\mathrm{id}_X\times\phi$.

Notice that the prestack structure is clear because the natural transformations attached to $\phi : S_1 \to S_2$ is given by the pull-back of vector bundles, while the pull-back of vector bundles is functorial respecting various compositions.

For the stacky structure, property 1) (Descent is effective) says that the vector bundle $(\mathcal{V}_i \to U_i)$ an an open cover $\{U_i \to U\}$ can be glued to a vector bundle on U whenever there are isomorphisms $\alpha_{i,j} : \mathcal{V}_i|_{U_{i,j}} \simeq \mathcal{V}_j|_{U_{i,j}}$ satisfying the cocycle conditions. But this is exactly the definition of vector bundle we have seen in differential/algebraic geometry. Property 2) (Gluing morphisms) says that isomorphisms of vector bundles over the same base scheme can be defined locally on an open cover and glued in a unique way if they agree on the overlap, but this is just the definition.

It turns out that for vector bundles, *=Zariski/étale/fppf locally trivial are all equivalent. As one can see, the cocycle condition shows that rank n-vector bundles on X are classified by the Cech cohomology $\check{H}^1_*(X, \mathrm{GL}_n)$. It turns out that $\check{H}^1_*(X, \mathrm{GL}_n)$ are all isomorphicm, which follows from the fact that any finite projective module are locally free in Zariski topology. Here we view GL_n as a representable sheaf on the corresponding site (indeed, by Grothendieck, any scheme is a fpqc, and hence * sheaf).

When $X = \mathrm{Spec}(k)$, we denote the stack by B_n.

2. G-bundles

You must have already heard that vector bundles are equivalent to GL_n-bundles. In general, let G be a smooth affine group scheme of finite type over k (Equivalently, we can view G as a representable sheaf of groups in Zariski topology), for instance you may take $G = \mathrm{GL}_n, \mathrm{Sp}_{2n}$ and let X be a scheme. A G-bundle (torsor) over X, by definition, is a sheaf $\mathcal{P}$ on $\mathrm{Aff}_{\mathrm{et}}$ with a G-action $G \times \mathcal{P} \to \mathcal{P}$, such that there exists an étale cover $\{U_i \to X\}$ with $\mathcal{P}|_{U_i} \simeq U_i \times G$ (as trivial G-bundle over U_i) and the G-action is locally trivial. Here for a G-action we mean that as a sheaf of sets, for the above cover $\{U_i \to U\}$, the map $G(U_i) \times \mathcal{P}(U_i) \to \mathcal{P}(U_i)$ is the usual group action in set-theoretic sense.

Notice that in Justin's talk, a scheme is defined to be a Zariski sheaf on Aff with surjective open immersion affine covers. Hence every schematic issues can be pull-back to the affine cover. But let us not worry about it and imagine we are in the usual world of schemes.

Here we assume that G is smooth, hence a G-bundle in étale topology is the same as a G-bundle in fppf topology. If G is not smooth (e.g. in positive characteristic) one need to work with fppf topology.

Using fpqc descent for affine morphisms, one can show that $\mathcal{P}$ is represented by an affine scheme over X. On the other hand, if G is not affine, as shown in Justin's talk, it can happen that a G-bundle is not representable by a scheme.

For a scheme X, let $\mathrm{Bun}_{G,X}$ be the stack sending $S \in \mathrm{Aff}$ to the groupoid of G-bundles on X_S with the morphisms being G-equivariant morphisms. In particular, any G-equivariant morphisms between two G-bundles with the same base scheme $(\mathcal{P}_1 \to S) \mapsto (\mathcal{P}_2 \to S)$ is necessarily an isomorphism, simply because they are locally trivial after pulling back to an étale cover, and the G-equivariant morphism on trivial bundles must be an isomorphism. In general, for two G-bundles $\mathcal{P}_1 \to S_1$ and $\mathcal{P}_2 \to S_2$, a morphism between them is a

G-equivariant commutative diagram

$$\begin{array}{ccc} \mathcal{P}_1 & \longrightarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

Notice that such a commutative diagram must be Cartesian, simply because one has a G-equivariant map $(\mathcal{P}_1 \to S_1) \mapsto (\mathcal{P}_2 \times_{S_2} S_1 \to S_1)$ which is necessarily an isomorphism by the above discussion.

Indeed $\mathrm{Bun}_{G,X}$ is a stack. The prestack structure is clear by the functoriality of pullback, and the gluing condition follows from descent theory for fppf sheaves. When $X = \mathrm{Spec}k$ we denote it as $\mathrm{B}G$ and will also write it as $[\cdot/G]$.

Remark 2.0.1. One can show that $\mathrm{Bun}_{n,X}$ is isomorphic to $\mathrm{Bun}_{\mathrm{GL}_n,X}$. to make it precise, for a GL_n-bundle $\mathcal{P}_n \to X_S$ with desecnt datum $\{(\mathrm{GL}_n \times U_i \simeq \underline{\mathrm{Isom}}(\mathcal{O}_i^n, \mathcal{O}_i^n), g_{i,j})\}$, using the defining representation one can construct a vector bundle $\left(\mathcal{P}_n \times \mathcal{O}_{X_S}^n\right)/\mathrm{GL}_n$ on X_S with descent datum $\{(\mathcal{O}_i^n \times T_i, g_{i,j})\}$. Conversely, given a vector bundle $\mathcal{V}_n \to X_S$, one can consider the sheaf $\underline{\mathrm{Isom}}(\mathcal{O}_{X_S}^n, \mathcal{V})$ which has a natural GL_n-torsor structure with descent datum $\{(\mathrm{GL}_n \times U_i, g_{i,j})\}$. Indeed the constructions give an isomorphism between the two stacks.

3. Algebraicity of BG

We show that $\mathrm{B}G$ is an algebraic stack.

First let us recall the definition of algebraic spaces and algebraic stacks from Justin's talk.

Definition 3.0.1. Let $\mathcal{F}$ be a sheaf on $\mathrm{Aff}_{\mathrm{et}}$. $\mathcal{F}$ is called an algebraic space if

(1) For any $U \in \mathrm{Ob}(\mathrm{Aff})$, $U \to \mathcal{F}$ is representable by schemes $\Longleftrightarrow \Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by schemes;
(2) There exists an étale surjective covering $\{U_i \to \mathcal{F}\}$ with $U_i \in \mathrm{Ob}(\mathrm{Aff})$.

Let $\mathfrak{X}$ be a stack on $\mathrm{Aff}_{\mathrm{et}}$. $\mathfrak{X}$ is called an algebraic stack if

(1) For any $U \in \mathrm{Ob}(\mathrm{Aff})$, $U \to \mathcal{X}$ is representable by algebraic spaces $\Longleftrightarrow \Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces;
(2) There exists an smooth surjective covering $\{U_i \to \mathcal{X}\}$ with $U_i \in \mathrm{Ob}(\mathrm{Aff})$;

Remark 3.0.2. Actually we can relax the definitions above by only requiring the existence of a representable by schemes/algebraic spaces surjective étale/smooth atlas $\{U_i \to *\}$.

In the following let us show that $\mathrm{B}G$ is an algebraic stack.

To do so, following the definition, we need to find a smooth atlas for $\mathrm{B}G = [\cdot/G]$. A natural candidate is given by the trivial bundle on $\mathrm{Spec}(k) = \cdot$:

$$\cdot \xrightarrow{\mathrm{tr}} \mathrm{B}G.$$

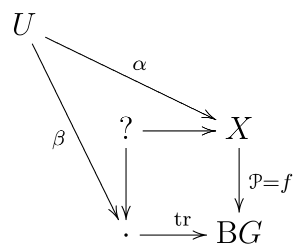
Let us show that indeed it is a smooth surjective atlas. For any $X \in \mathrm{Aff}$ with a G-torsor $\mathcal{P} \to U$ corresponding to $f : X \to \mathrm{B}G$, consider the fiber product

$$\begin{array}{ccc} ? & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f=\mathcal{P}} \\ \cdot & \xrightarrow{\mathrm{tr}} & \mathrm{B}G \end{array}$$

In general, recall that given two morphisms (i.e. a 2-natural transformation with various compatibilities) between prestacks $i_1 : \mathfrak{X}_1 \to \mathcal{Y}$ and $i_2 : \mathfrak{X}_2 \to \mathcal{Y}$, their fiber product $\mathfrak{X}_1 \times_{\mathcal{Y}} \mathfrak{X}_2$ is defined to be the prestack whose evaluation at $U \in \mathrm{Aff}$ is given by the groupoid

$$\big(\mathfrak{X}_1 \times_{\mathcal{Y}} \mathfrak{X}_2\big)(U) = \{(x_1, x_2, \alpha) \mid x_1 \in \mathrm{Ob}(\mathfrak{X}_1(U)), x_2 \in \mathrm{Ob}(\mathfrak{X}_2(U)), \alpha : i_1(x_1) \simeq i_2(x_2)\}$$

Returning to our situation, a $V \in \mathrm{Ob}(\mathrm{Aff})$-point of the fiber product is given by



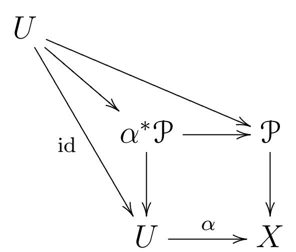
which consists of

$$\begin{aligned}\big(\cdot \times_{\mathrm{B}G} X\big)(U) =& \{(\alpha, \beta, \phi) \mid \phi : f \circ \alpha \simeq \mathrm{tr} \circ \beta\} \\ =& \{(\alpha, \phi) \mid \phi : \alpha^*\mathcal{P} \simeq G \times U\}\end{aligned}$$

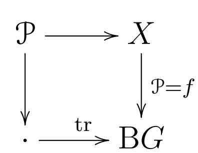
Notice that a G-bundle is trivial if and only if it has a global section, hence

$$= \{(\alpha, s) \mid s : U \to \alpha^*\mathcal{P} \text{ is a section}\} = \mathcal{P}(U)$$

using the pull-back diagram



It follows that $\big(\cdot \times_{\mathrm{B}G} X\big)(U) = \mathcal{P}(U)$ and hence is representable by the G-torsor, i.e. we have the following Cartesian diagram



It follows that $\mathrm{tr} : \cdot \to \mathrm{B}G$ is a smooth surjective atlas since we assume G is smooth and hence $\mathcal{P} \to X$ is smooth.

For the representability of the diagonal, the idea is similar. By definition, for any $X \in \mathrm{Aff}$ with two G-bundles $\mathcal{P}_1, \mathcal{P}_2$ mapping to $\mathrm{B}G \times \mathrm{B}G$, for any $U \in \mathrm{Aff}$, we look at

$$\big(X \times_{\mathrm{B}G \times \mathrm{B}G} \mathrm{B}G\big)(U) = \{(f, g, \phi) \mid \phi : \Delta \circ g \simeq (\mathcal{P}_1, \mathcal{P}_2)\}$$

which can be identified as

$$\{(f, g, \phi) \mid \phi : (f^*\mathcal{P}_1, f^*\mathcal{P}_2) \simeq (\mathcal{P}_g, \mathcal{P}_g)\} \simeq \{(f, \phi) \mid \phi : f^*\mathcal{P}_1 \simeq f^*\mathcal{P}_2\} = \mathrm{Isom}(\mathcal{P}_1|_U, \mathcal{P}_2|_U)$$

which, through passing to a locally étale trivialization cover, checked that it is still a G-torsor, and hence a scheme.

It follows that we complete the proof that $\mathrm{B}G$ is an algebraic stack.

Similarly one can define the quotient stack $[Z/G]$ for Z a scheme acted by G, whose evaluation at $U \in \mathrm{Aff}$ consisting of G-bundles $\mathcal{P} \to U$ and G-equivariant morphisms $\mathcal{P} \to Z$. One can show that $[Z/G]$ is an algebraic stack.

4. Automorphic interpretation of Bun_n d'après Weil

In the following, for convenience let us assume that $k = \overline{k}$ is an algebraically closed field and X is a smooth projective curve over k. Let us restrict to $G = \mathrm{GL}_n$ and hence vector bundles of rank n in the following discussion.

Before proving the algebraicity of $\mathrm{Bun}_{n,X}$ we would like to give an automorphic interpretation of the groupoid $\mathrm{Bun}_{n,X}(k)$ following Weil. This can be viewed as a motivation for number theoriest (it is the case for me!) to study $\mathrm{Bun}_{n,X}$ and more generally $\mathrm{Bun}_{G,X}$.

Let $F = k(X)$, $\eta = \mathrm{Spec}(F)$ and $|X|=$ set of closed points of X. For any $x \in |X|$, set $F_x \simeq k((t)) \supset \mathcal{O}_x \simeq k[[t]]$. Set

$$F \subset \mathbb{A} = \prod_{x\in|X|}{}' F_x = \{(a_x)_{x\in|X|} \mid x \in \mathcal{O}_x \text{ a.a. } x \in |X|\} \supset \mathcal{O} = \prod_{x\in|X|} \mathcal{O}_x$$

We claim that there is an isomorphism of groupoids

$$\mathrm{Bun}_{n,X}(k) \simeq \mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$$

By definition $\mathrm{Bun}_{n,X}(k)$ classifies rank n-vector bundles on X. Let $\mathcal{V} \to X$ be a vector bundle of rank n. Then by (Zariski) local triviality $V|_\eta$ is a vector space of dimension n, and for any $x \in |X|$, $V|_{X_x\simeq\mathrm{Spec}(\mathcal{O}_x)}$ is a free $\mathcal{O}_x$-module of rank n.

To define $\mathcal{V} \to X$, we need gluing datum on their intersection $X_x^\bullet = X_x \cap \eta = \mathrm{Spec}(k((t)))$: $V|_{X_x} \otimes_{\mathcal{O}_x} F_x\big) \simeq V|_\eta \otimes_F F_x$. After fixing trivialization $\xi_\eta : F^n \simeq V$ and $\xi_x : \mathcal{O}_x^n \simeq V|_{X_x}$, set $g_x = \xi_\eta^{-1} \circ \xi_x|_{X_x^\bullet} \in \mathrm{GL}_n(F_x)$. Set $g = (g_x)_{x\in|X|}$. Notice that indeed $g \in \mathrm{GL}_n(\mathbb{A})$ since the trivialization ξ_η is defined over a Zariski open dense subset $U \subset X$. In particular for $x \in |U|$, $g_x \in \mathrm{GL}_n(\mathcal{O}_x)$.

It follows that a tuple $(\xi_\eta, (\xi_x)_{x\in|X|})$ which is determined by a vector bundle $\mathcal{V} \to X \in \mathrm{Bun}_{n,X}(k)$ provides an element in $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$ where the quotients by $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(\mathcal{O})$ comes from the isomorphisms for the datum $(\xi_\eta, (\xi_x)_{x\in|X|})$. Conversely from Beauville–Laszlo's patching theorem (or fpqc descent is enough), given a point in the double quotient $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$ it provides a vector bundle on X. Moreover, it is an isomorphism of groupoid. For any coset $[g] = \mathrm{GL}_n(F) \cdot g \cdot \mathrm{GL}_n(\mathcal{O})$ the corresponding G-bundle has centralizer $\mathrm{GL}_n(F) \cap g \cdot \mathrm{GL}_n(\mathcal{O}) \cdot g^{-1}$.

The same discussion applies to other groups as long as we can trivialize any G-bundles over the generic point. For instance it works for connected reductive groups (also see the work of Drinfeld-Simpson).

5. Algebraicity of Bun_n

In the following we will still assume that X is a smooth projective curve over an algebraically closed field $k = \overline{k}$.

The algebraicity of $\mathrm{Bun}_{G,X}$ can be reduced to $\mathrm{Bun}_{n,X}$. To make it precise, fix a group embedding $G \hookrightarrow \mathrm{GL}_n$. This induces a morphism of algebraic stacks $\mathrm{B}G \to \mathrm{BGL}_n$ sending $\mathcal{P}$ to $\big(\mathcal{P} \times \mathrm{GL}_n\big)/G$. It turns out that for any $\mathcal{P}_n : U \in \mathrm{Aff} \to \mathrm{BGL}_n$, we have $\mathcal{P}_n \times_{\mathrm{BGL}_n} \mathrm{B}G \simeq \mathcal{P}_n/G \simeq \big(\mathcal{P}_n \times \mathrm{GL}_n/G\big)/\mathrm{GL}_n$. Hence this morphism is schematic and quasi-projective, from which one can deduce that $\mathrm{Bun}_{G,X} \to \mathrm{Bun}_{n,X}$ is schematic. In particular the diagonal of $\mathrm{Bun}_{n,X}$ is schematic.

It remains to construct a smooth atlas for $\mathrm{Bun}_{n,X}$.

From Justin's talk, we have seen that for any $0 < k < n$, the following functor

$$\begin{aligned}\mathrm{Gr}(k,n) : \mathrm{Aff} &\to \mathrm{Sets}\\ S &\mapsto \{\mathcal{O}_S^n \twoheadrightarrow Q \mid \ Q \text{ locally free of rank } r\}\end{aligned}$$

is represented by a projective scheme which is ususally called the Grassmannian. When $k = 1$ it is just the projective space $\mathbb{P}^n$.

It turns out that every vector bundle can arise in the above way in a suitable sense. To make it more precise, by Serre vanishing, for a projective morphism $p : X_U \to U$ with $U \in \mathrm{Aff}$, fix a relative ample line bundle $\mathcal{O}_{X_U}(1)$, then for any coherent sheaf $\mathcal{E}_U$ on X_U, there exists an integer r_0 such that for any $r \geq r_0$, $H^1(X_U, \mathcal{E}_U(r) \simeq \mathcal{E}_U \otimes \mathcal{O}_{X_U}(1)^{\otimes r}) = 0$ and $H^0(X_U, \mathcal{E}_U(r)) \otimes \mathcal{O}_{X_U} \twoheadrightarrow \mathcal{E}_U(r)$ is surjective, i.e. $\mathcal{E}_U(r)$ is generated by global sections for $r \geq r_0$. In other words, $\mathcal{O}_{X_U}(-r)^{\dim H^0(X_U,\mathcal{E}_U(r))} \twoheadrightarrow \mathcal{E}_U$ for $r \geq r_0$.

In general, Grothendieck introduces the following generalization called the Quot scheme. To make it precise, let $\mathcal{E}$ be a coherent sheaf on X, consider the functor

$$\begin{aligned}\mathrm{Quot}_{\mathcal{E}/X/k} : \mathrm{Aff} &\to \mathrm{Sets}\\ U &\mapsto \{(\mathcal{F}, q) \mid \mathcal{F} \in \mathrm{QCoh}_{\mathrm{fin.pre}}(X_U), \mathcal{F} \text{ flat over } U, q : \mathcal{E}_U \twoheadrightarrow \mathcal{F}\}\end{aligned}$$

Then Grothendieck shows that $\mathrm{Quot}_{\mathcal{E}/X/k}$ is a disjoint union of projective schemes stratified by the Hilbert polynomial of $\mathcal{F}$. To make it precise, fix an ample line bundle $\mathcal{O}_X(1)$ on X. With the above notation, for any $s \in U$ a closed point, set $P_{\mathcal{F}_s}(r) = \chi(X_s, \mathcal{F}_s(r)) = \sum_{i=0}^{1}(-1)^i \cdot \dim H^i(X_s, \mathcal{F}_s \otimes \mathcal{O}_X(1)^{\otimes r})$. By Serre vanishing and Riemann-Roch, for $r \gg 0$, $P_{\mathcal{F}_s}(r) = \chi(X_s, \mathcal{F}_s(r)) = \dim H^0(X_s, \mathcal{F}_s \otimes \mathcal{O}_X(1)^{\otimes r})$ is a polynomial in r. Set $\mathrm{Quot}^P_{\mathcal{E}/X/k}$ consisting $(\mathcal{F}, q)$ such that $P_{\mathcal{F}_s}(r) = P(r)$ for any $s \in U$. Then $\mathrm{Quot}^P_{\mathcal{E}/X/k}$ is represented by a projective scheme and $\bigsqcup_P \mathrm{Quot}^P_{\mathcal{E}/X/k} = \mathrm{Quot}_{\mathcal{E}/X/k}$.

For our purpose, we consider the sub-functor when $\mathcal{E}$ is a vector bundle and the quotient is also a vector bundle

$$\begin{aligned}\mathrm{Quot}^\circ_{\mathcal{E}/X/k} : \mathrm{Aff} &\to \mathrm{Sets}\\ U &\mapsto \{(\mathcal{F}, q) \mid \mathcal{F} \text{ locally free } q : \mathcal{E}_U \twoheadrightarrow \mathcal{F}\}\end{aligned}$$

It turns out that $\mathrm{Quot}^\circ_{\mathcal{E}/X/k}$ is representable by an open subscheme of $\mathrm{Quot}_{\mathcal{E}/X/k}$ which can also be stratified by the Hilbert polynomial.

Returning to our problem, for any $U \in \mathrm{Aff}$, $\mathrm{Bun}_{n,X}(U)$ classifies vector bundles $\mathcal{E}_U$ of rank n over X_U. By Serre vanishing, there exists $r_0 \in \mathbb{N}$ such that $\mathcal{O}_{X_U}^{\dim H^0(X_U,\mathcal{E}_U(r))}(-r) \twoheadrightarrow \mathcal{E}_U$ and $H^1(X_U, \mathcal{E}_U(r)) = 0$ for any $r \geq r_0$. In particular there is a surjection $H^0(X_U, \mathcal{E}_U(r)) \otimes \mathcal{O}_{X_U} \twoheadrightarrow \mathcal{E}_U(r)$. Therefore any vector bundle $\mathcal{E}_U$ over X_U is a quotient of $\mathcal{O}_{X_U}^{\dim H^0(X_U,\mathcal{E}_U(r))}(-r)$. Furthermore, for any closed point $s \in U$, by Riemann-Roch $\chi(X_s, \mathcal{E}_s(r)) = \dim H^0(X_s, \mathcal{E}_s(r)) = \deg(\mathcal{E}_s(r)) + \mathrm{rk}(\mathcal{E}_s(r))(g-1) = n \cdot \deg(\mathcal{O}_X(r)) + \deg(\mathcal{E}_s) + n \cdot (g-1)$ depends only on the fiberwise degree of $\mathcal{E}_U$. Notice that since $\mathcal{E}_U$ is a flat family, the Hilbert polynomial and hence the fiberwise degree is fixed. It follows that we have a morphism

$$\bigsqcup_{d \in \mathbb{Z}} \bigsqcup_{r \geq 0} \mathrm{Quot}^\circ_{\mathcal{O}(-r)^{n \cdot \deg(\mathcal{O}_X(r)) + d + n\cdot(g-1)}/X/k} \to \bigsqcup_{d\in\mathbb{Z}} \mathrm{Bun}^d_{n,X} = \mathrm{Bun}_{n,X}$$

where $\mathrm{Bun}_{n,X}^d$ is the substack of vector bundles of fiberwise degree d. Now based on the above discussion, the above morphism is a surjection. Actually, for any fixed r, one can show that this is a $\mathrm{GL}_{n\cdot\deg(\mathcal{O}_X(r))+d+n\cdot(g-1)}$-bundle, and hence is a desired smooth atlas.

Once we have a smooth atlas, we can define the notion of tangent and cotangent complex, which will be discussed next time by Prof. Ginzburg.

References

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
Email address: zhilinchicago@uchicago.edu