# BUNG SEMINAR NOTES I: INTRODUCTION TO STACKS AND BUN G.

SPEAKER: NICK ROZENBLYUM
NOTES BY NIKOLAY GRANTCHAROV AND AARON SLIPPER

## 1. Functor of Points.

What is algebraic geometry? The classical answer is that it is the study of solutions to polynomial equations. The modern answer is that it is the study of schemes. Schemes have a rather technical and involved definition ("locally ringed spaces") but this formalism conceals the fact that really they are just a particular way of formalizing the notion of "solutions to polynomial equations." The functor of points philosophy brings this aspect of algebraic geometry once more to the fore.

Consider a scheme like $X = \operatorname{Spec} A[x,y]/(x^2+y^2-1)$. Then, for any $A$-algebra $B$, we see that

$$\begin{aligned}\operatorname{Maps}(\operatorname{Spec} B, X) =& \text{Ring homomorphisms } (A[x,y]/(x^2+y^2-1)), B) \\ =& \{(x,y) \in B \text{ such that } x^2+y^2=1\}.\end{aligned}$$

Thus the functor $\operatorname{Maps}(-, X) : \mathrm{Sch} \to \mathrm{Set}$, at least when restricted to affine schemes $B$, parameterizes solutions in $B$ to $x^2+y^2=1$. We write $X(S)$ for the set $\operatorname{Maps}(S,X)$, and we call these the $B$-points of $X$, or the points of $X$ with values in $B$. If, say $A=\mathbb{Q}$, $B=\mathbb{R}$, then the $B$-points of $X$ is our old friend the Euclidean circle. Indeed, the structures of geometric interest in algebraic geometry really lie in the sets $X(S)$ as $S$ varies.

So rather than talk about schemes as locally ringed spaces, it is often helpful to think about their functor of points; that is, what the sets $X(S)$ are for an arbitrary "test" scheme $S$. In this view it is often helpful to think of an element of $X(S)$ as a family of points of $X$ parameterized by $S$.

By the Yoneda embedding, we have a fully faithful embedding:

$$\mathrm{Schemes}/\operatorname{Spec}(A) \hookrightarrow \operatorname{Fun}(A\text{−alg}, \mathrm{Set})$$

The essential image of this embedding consists of those functors which form a sheaf in the Zariski topology, such that the functor is Zariski-locally representable by representable functors $\operatorname{Hom}(R, *)$. In some sense these definitions as functors are more natural than the definition given by locally ringed spaces. Other functors give other kinds of geometric objects: those that satisfy a sheaf condition in the fppf topology are called "algebraic spaces."

**Example 1.1.** *Consider the affine scheme* $\operatorname{Spec}\mathbb{Z}[x,y]/x^3-y^2-2$ *There is a* $\mathbb{Z}$*-point of this given by* $x=3$ *and* $y=5$*. Since we have canonical maps* $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{Z}$ *for each prime*

*Date*: January 4, 2023.

*$p$, we see that this also gives us solutions to $x^3 = y^2 + 2$ for each prime $p$. A solution of an equation in $\mathbb{Z}$ can be thought of as a family of solutions in $\mathbb{F}_p$ for each prime. They are "glued" together by virtue of the fact that they "come from" a solution in $\mathbb{Z}$.*[1]

## 2. Moduli Spaces.

The moral of the above is that we can understand schemes or other algebro-geometric objects by "what they parameterize" (e.g., solutions of systems of equations). From this perspective, a scheme $X$ is characterized by describing the sets $X(S)$, where $S = \mathrm{Spec}R$ is a "test" scheme.

We can just as easily reverse the process: start with natural functors, and ask what kind of geometric objects might "represent" these functors.

A nice example is a moduli functor. In general, a moduli functor parameterizes isomorphism classes of some kind of an object. Thus to each test scheme $S$, the moduli functor will produce the set of families of isomorphism classes of these objects over $S$.

**Example 2.1.** *Consider the moduli functor of curves*

$$\mathcal{M}(S) = \{\textit{relative curves } \mathcal{X}/S\}/\textit{iso}$$

*That is to say, an element of $\mathcal{M}(S)$ is a scheme $\mathcal{X}$ with a projective morphism $\pi : \mathcal{X} \to S$ such that the fiber of $\pi$ over each (geometric) point $s \in S$ is a smooth projective curve; here we identify two such schemes $\mathcal{X}$ and $\mathcal{X}'$ if there is an isomorphism of $S$-schemes $\phi : \mathcal{X} \to \mathcal{X}'$.*

The problem with the above moduli functor is that curves can have automorphisms, but the moduli functor forgets this.

### 2.1. Topological interlude.

The simplest topological space is the 1-point set with the discrete topology. (No, the empty set is not simpler than $\{*\}$.) However, the one-point set has no automorphisms and so will be of no pedagogical value in the present context.

After this, the next most complicated topological space is the two pointed set $\{a, b\}$. So let us consider the moduli functor of families of 2-pointed sets. What does this mean precisely? To be very technical:

**Definition 2.1.** *A map of topological spaces $p : \tilde{T} \to T$ is a family of 2-pointed sets over $T$ if for every point $t \in T$, there exists an open set $U$ containing $t$ such that there exists a homeomorphism*

$$\pi^{-1}(U) \to \{0, 1\} \times U.$$

In other words, $p : \tilde{T} \to T$ is a 2-sheeted covering map of $T$. Two such families $\tilde{T}$ and $\tilde{T}'$ over $T$ are isomorphic if there exists a homeomorphism $\tilde{T} \to \tilde{T}'$ respecting the projections to $T$. We define the *moduli functor of 2-pointed sets*, which we shall denote via $F$, as

$$F(X) = \{\text{Families of 2-pointed sets over } X\}/\text{iso}.$$

[1] As we know from the failure of the local-global principle, however, mere solutions of an equation at every prime does not guarantee that they "glue together" into a "global" solution over $\mathbb{Z}$. Failures of solutions to "glue" are, as one might expect, controlled by certain cohomology groups.

Consider, for example, $X = S^1$. We have the trivial family $S^1 \times \{a, b\} \to S^1$, and also the connected 2-sheeted cover $S^1 \to S^1$ given by $z \mapsto z^2$ (viewing $S^1$ as the unit circle in $\mathbb{C}$). In fact, $F(S^1)$ is a 2-point set consisting of only these two isomorphism classes. The two families (trivial, connected) over $S^1$ are indeed non-isomorphic: one cover is connected while the other has two disconnected components. Certainly they are not homeomorphic.

Now: we would like to know if there exists a topological space $\mathcal{M}$ representing this functor $F$. That is to say, is there a topological space $\mathcal{M}$ such that for all topological spaces $X$, the continuous maps from $X$ to $\mathcal{M}$, $\mathrm{Maps}(X, \mathcal{M})$, equal $F(X)$?

Well, for any topological space, we may recover the underlying set via $\mathrm{Maps}(*, -)$. In the case of our hypothetical moduli space $\mathcal{M}$, this would yield:

$$\mathrm{Maps}(*, \mathcal{M}) = F(*) = \{\text{Families of 2-pointed sets over } *\}/\text{iso},$$

which is just the set of one element, since there is only one family of two-pointed sets over $\{*\}$ up to isomorphism: the 2-point discrete space $\{a, b\}$. Thus we have found that $\mathrm{Maps}(*, \mathcal{M}) = \{\text{Points of } \mathcal{M}\}$ is just the singleton set; i.e., $\mathcal{M} = \{*\}$. But now we are led to a contradiction. For we know that $F(S^1)$ has at least two distinct elements; but if $\mathcal{M}$ is a point, $F(S^1) = \mathrm{Maps}(S^1, \mathcal{M}) = \mathrm{Maps}(S^1, *)$ has only one.

Alternatively: consider the diagram:

$$\{*\} \xrightarrow{p} S^1 \rightrightarrows \mathcal{M}$$

where the first arrow is the inclusion of $p \in S^1$, and the right two arrows are the trivial and connected 2-sheeted cover, respectively. However, there can only be one map $\{*\} \to \mathcal{M}$, since these are identified with isomorphism classes of families of 2-pointed sets over $\{*\}$, for which there is only one. So the two maps agree on every point; therefore they cannot be distinct!

Thus we see that moduli functors of objects for which there exist families that are isomorphic locally (or fiberwise!) but disagree globally *cannot* be represented by a topological space. This is somewhat discouraging: so many of the most natural families (vector bundles, and fibrations more generally) are all isomorphic (indeed “trivial”) locally.

It is worth noting that, in the example of families 2-point sets, topologists are ultimately able to skirt this difficulty. But this comes at the cost of changing the rules of the game! Namely, a moduli space exists if we allow it to live in the homotopy category rather than in the category of topological spaces itself. Indeed, we have that:

$$[X, \mathbb{R}P^\infty] \simeq \{\text{Families of 2-pointed sets over } X\}/\text{iso}$$

where $\mathbb{R}P^\infty$ is infinite-dimensional real projective space and

$$[X, \mathbb{R}P^\infty] = \mathrm{Maps}(X, \mathbb{R}P^\infty)/\text{homotopy equivalence}.$$

Topologists would thus say that $\mathbb{R}P^\infty$ has the homotopy-type of the classifying space $B\mathbb{Z}_2$, which is defined by the above functor $F$. Algebraic geometers are also able to get around the problem of the non-existence of moduli spaces by changing the rules of the game. Namely, we introduce the whole new notion of "stacks", which, like homotopy spaces, can "represent" these kinds of functors.[2]

But before we introduce stacks, we should address the role of symmetry in the above. We have seen that the existence of families that are locally but not globally isomorphic precludes the existence of moduli spaces. Such nontrivial families exist because the object whose isomoprhism classes we are trying to classify – e.g., the two-point set $\{a, b\}$ – has nontrivial symmetry – e.g., $a \mapsto b, b \mapsto a$.

Indeed, let us consider $S^1 = [0, 1]/\sim$ where $\sim$ identifies endpoints. Over $[0, 1]$, there is only one isomorphism-class of 2-point family, namely, the trivial family $\{a, b\} \times [0, 1]$. Over $S^1$, we can obtain 2-point families by identifying the fibers over 0 and 1,

$$\{a, b\} \times \{0\} \xrightarrow{\sim} \{a, b\} \times \{1\}.$$

If $\xrightarrow{\sim}$ sends $a$ to $a$ and $b$ to $b$, then the resulting family over $S^1$ is the trivial family $\{a, b\} \times S^1$. But if $\xrightarrow{\sim}$ swaps $a$ and $b$, the resulting family is nontrivial. Thus, symmetry lets us produce fiberwise isomorphic, but globally nonisomprhic families. Then, as we have seen, no moduli space can exist.

Let us think about this from the perspective of gluing over open sets rather than along boundaries. We first observe that the notion of a continuous family is local. That is to say, to check that $p : \tilde{T} \to T$ is a family, we need only check that $p|_{U_i} : \tilde{T}|_{U_i} \to T|_{U_i} = U_i$ is a family for each $U_i$ in an open cover $\{U_i\}_i$ of $T$.

Moreover, we may invert this process – that is to say, start with a family $\tilde{T}_i$ over each open set $U_i$ of an open cover and then "glue" them together over intersections. More specifically: we provide isomorphisms

$$\varphi_{i,j} : \tilde{T}_i|_{U_i \cap U_j} \to \tilde{T}_j|_{U_i \cap U_j}$$

for each pair $i, j$. For these isomorphisms to define a global family, they must satisfy a cocycle condition on triple intersections; moreover, two such collections $\{\tilde{T}_i, \varphi_{i,j}\}$ and $\{\tilde{T}'_i, \varphi_{i,j'}\}$ will glue to the same family if we can find isomorphisms $\psi_i : \tilde{T}_i \to \tilde{T}'_i$ over $U_i$ for each $i$ that intertwine $\varphi_{i,j}$ and $\varphi'_{i,j}$. In this way we see that isomorphism classes of families are controlled by some kind of $H^1$. Indeed, some older sources, e.g., Behrend, use the notation $\mathfrak{H}^1(X, G)$ for our protagonist $\mathrm{Bun}_G(X)$!

Our feeling is that extra symmetry on some overlap $U_0 \cap U_1$ can be used to "twist" the gluing data to give a nonisomorphic family. (I.e., we "reglue" after composing the gluing map with the symmetry.) Let us simplify by assuming that our open cover consists of only two open sets $U_0$ and $U_1$, with corresponding local families $\tilde{T}_0$ and $\tilde{T}_1$. (This loses no generality since we may glue all the $\{\tilde{T}_i\}_{i \neq 0}$ together into one family over $\bigcup_{i \neq 0} U_i$.) And say that $\phi$ : $\tilde{T}_0|_{U_0 \cap U_1}$ has a nontrivial symmetry; i.e., a non-identity homeomorphism $\tilde{T}_0|_{U_0 \cap U_1} \to \tilde{T}_0|_{U_0 \cap U_1}$ over $U_0 \cap U_1$. Then we may replace our gluing data $\{\tilde{T}_0, \tilde{T}_1, \varphi_{0,1}\}$ with $\{\tilde{T}_0, \tilde{T}_1, \varphi'_{0,1} := \phi \circ \varphi_{0,1}\}$. Because we have only two open subsets, the cocycle condition here is vacuous, and we may glue to a family $\tilde{T}'$.

[2]This analogy is not quite perfect as it is stated presently: the category of schemes embed fully faithfully into the category of stacks; on the other hand the category of topological spaces does not embed fully faithfully into the homotopy category.

Once more, the families $\tilde{T}$ and $\tilde{T}'$ are locally isomorphic (indeed, by construction, their restrictions are isomorphic over each $U_i$). But these will in general be nonisomorphic globally.

Let us return to the 2-pointed families over $S^1$ example. Let $U_0$ and $U_1$ two arcs of $S^1$ that together cover all of $S^1$, and let $\tilde{T}_0$ and $\tilde{T}_1$ be 2-point families over $U_0$ and $U_1$ (both are necessarily trivial since $U_0$ and $U_1$ are homeomorphic to $[0,1]$). We see that $U_0 \cap U_1$ is a disjoint union of two arcs. Thus $\tilde{T}_0|_{U_0 \cap U_1}$ has four symmetries, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the nontrivial automorphism of each disconnected component. If we we start with $\varphi_{0,1}$ as the trivial gluing, then we can obtain the nontrivial family by letting $\varphi'_{0,1} := \phi \circ \varphi_{0,1}$ where $\phi = (1,0) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\phi = (0,1)$. (Though notice that twisting by $\phi = (1,1)$ yields the trivial cover once more!)

2.2. **Groupoids.** To deal with the presence of symmetry in a moduli problem, we replace the functor from Rings $\to$ Sets with the functor Rings $\to$ Gpds. A groupoid can be thought of as a "set with symmetry". More precisely,

**Definition 2.2.** *A groupoid is a category where every morphism is invertible.*

There is also an algebraic definiton of a groupoid, but the categorical definiiton is more illuminating: it suggests that the correct notion for isomoprhism for groupoids is equivalence of categories. The algebraic definition does not make this clear.

**Example 2.2.** *Suppose the groupoid $G$ has 2 objects $x_1, x_2$ with one isomorphism between them: $Mor(x_i, x_j) = \{$a unique arrow $\varphi_{i,j}\}$ for all $i, j$. Then $G$ is equivalent to the trivial groupoid (aka trivial category), with one object $x$ and $Mor(x,x) = 1_x$.*

In fact, any groupoid is equivalent to a disjoint union of one-object categories with all endomorphisms invertible; i.e., to a union of groupoids of the form $*/G$, where $G$ is a group (see definition below). It is precisely this flexibility in the notion of equivalence of groupoids that makes the definition of a stack slightly subtle.

Notice that a set $S$ can always be viewed as a groupoid: the objects are elements of $S$, and the only morphisms are identity morphisms on each element $s \in S$. However, this groupoid may be equivalent to other groupoids with many more objects than elements of $S$. Indeed, we may freely proliferate the number of objects in any isomorphism class while maintaining an equivalence of groupoids.

Let us give some examples of groupoids. Firstly, say that we have a set $X$, with an action via a group $G$.

**Definition 2.3.** *We define the action groupoid, denoted $X/G$ to be the groupoid whose objects consist of elements of $X$ and whose morphisms, for $x, y \in X$, are*

$$hom(x,y) = \{g \in G : gx = y\}.$$

*In the particular case where $X$ is a point, we usually denote the corresponding action groupoid (with only one object) by $*/G$.*

Note that, as in any category, sets of morphisms between unequal objects are assumed to be disjoint; thus we distinguish the arrow labeled by $g : x \to g(x)$ and the arrow labeled by $g : y \to g(y)$ if $x \neq y$. The isomorphism classes of the action groupoid are in bijection with the orbits of $G$ on $X$. But the action groupoid remembers more data: for $x$ and $y$ in the same orbit, there may well multiple $g$ for which $gx = y$. And in the case where $x = y$, the action groupoid "remembers" the stabilizers; a.k.a, the automorphisms of $x$. Notice that the action groupoid is equivalent to a set if and only if $G$ acts freely on $X$; i.e., if and only if there is at most one arrow from an object $x$ to an object $y$.

We think of the action groupoid as the "true" quotient of $X$ by the group action $G$. It is is only equivalent to the orbit space if the action of $G$ is free.

Another natural example of a groupoid comes from topology:

**Definition 2.4.** *Given a topological space $T$, we have the fundamental groupoid $\pi_{\leq 1}(T)$ whose objects are points of $T$ and morphisms $hom(x, y)$ are homotopy classes of paths from $x$ to $y$, for $x, y \in T$.*

In fact, the fundamental groupoid and the action groupoid are intriguingly related. Let us begin with the action groupoid $X/G$. We construct a simplicial complex as follows: vertices are points of $X$. Edges between $x, y \in X$ are all $g \in G$ such that $gx = y$; i.e., morphisms of $X/G$. Now add a 2-simplex for every triple of paths $g_1 : x \to y$, $g_2 : y \to z$, and $g_3 : x \to z$ such that $g_2 g_1 = g_3$. We can consider a geometric realization of this simplicial complex. It turns out that the fundamental groupoid of this simplicial complex is equivalent to the action groupoid!

As a remark, if we did not attach 2-simplices to the $g_2 g_1 = g_3$, then the fundamental group of the resulting space would be a free group. And we do not have to insert higher-dimensional simplices for triple (and higher) composition equivalences, because simplices of dimension $\geq 3$ do not affect $\pi_{\leq 1}$.

Now, we upgrade the definition of moduli space:

**Definition 2.5.** *A moduli stack of families is a contravariant functor*

$$\textit{Spaces} \to \textit{Groupoids} : X \mapsto \textit{Groupoid of families over } X,$$

*where, if $Y \to X$ is a map of spaces, the corresponding functor of grupoids is given by pullback of families. In the case where our notion of "space" is that of a scheme, we have a covariant functor*

$$\textit{Rings} \to \textit{Groupoids} : A \mapsto \textit{Groupoid of families over } \mathrm{Spec} A.$$

So a moduli stack should be thought of as outputting families *with their automorphisms*. In a sense, we have solved the problem of the non-existence of moduli in the presence of symmetry in the cheapest way possible: we have simply defined our moduli functors to remember such symmetry. But this allows us to account for the full richness of family life.

There is a slight subtlety which we have neglected in the above definition. If $Z \to Y \to X$ are maps of spaces, then there will be corresponding pullback maps of groupoids: $F(X) \to F(Y) \to F(Z)$. Of course, "maps between groupoids" means functors. Meanwhile, the composition map, $Z \to X$, will induce a functor $F(X) \to F(Z)$. We would like to say that this functor "equals" the composition of functors $F(X) \to F(Y) \to F(Z)$. But in general it is very awkward to speak of two functors being "equal". Instead, we should insist upon there being a natural isomorphism between these two functors $F(Z) \rightrightarrows F(X)$, and that this isomorphism satisfies a natural cocycle condition over triple compositions. But let us transcend such pedantic detail.

We have been unable to avoid using the dreaded word "stack." All stacks that we shall treat here are moduli stacks of one form or another; thus the above conveys our main purposes when we use the word "stack." But let us give a provisional definition for a "stack" as such, avoiding all technical detail:

**Definition 2.6.** *A stack is a functor Rings $\to$ Groupoids that is a sheaf in the flat topology.*

The flat topology, and the notion of a sheaf on it, will be discussed next week. The main takeaways from today's talk are: 1) why we should consider functors in the first place, and 2) why these moduli functors should output groupoids rather than sets. And the benefit of describing functors to groupoids as moduli functors – i.e., functors that parameterize some kind of family – is that these functors are automatically sheaves. For general functors to groupoids, the sheaf property must be checked.

We can now give the "stacky" version of the moduli space of curves:

**Example 2.3.** *The moduli stack $\mathcal{M}$ of curves is the functor Sch $\to$ Groupoids whose objects*

$$\mathcal{M}(S) = \{\textit{relative curves } \mathfrak{X}/S\}$$

*and whose morphisms are $S$-isomoprhisms of such relative curves.*

**Example 2.4.** *The moduli stack* $\mathrm{Bun}_r$ *of rank-r vector bundles is the functor Sch $\to$ Groupoids whose objects*

$$\mathrm{Bun}_r(S) = \{\textit{Vector Bundles of rank } r/S\}$$

*and whose morphisms are $S$-isomorphisms of such vector bundles.*

Notice that the stacks defined above do not pay any attention to non-isomorphism maps of cruves $\mathfrak{X} \to \mathfrak{X}'$ over $S$ or non-isomoprhism maps of vector bundles (or sheaves, or schemes) $\mathfrak{V} \to \mathfrak{V}'$ over $S$. A more general notion of fibered category is required to account for these structures. But if we want to be able to glue these structures into families, then isomorphisms are all we will need to deal with; after all, the identification maps involved in gluing families should be invertible.

## 3. Some Examples of Stacks.

3.1. **BG.** The two most useful kinds of stacks are quotient stacks and mapping stacks. Our hero Bun$G$ involves both. In this section we shall discuss quotients.

Let us say that an algebraic group $G$ acts on a scheme $X$. We would like to describe the quotient $X/G$ as a stack. Of course, to do this, we must describe the groupoid of $(X/G)(T)$ of $T$-points of $X/G$. We hope to describe this as a moduli functor; that is to say, as a groupoid of some kind of family (i.e., {schemes over $X$ with isomorphisms}).

Before we do this, however, we ought to examine the simplest case, $G$ acting trivially on a point. To provide motivation, let us return to the topological setting and let our group be $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ with the discerete topology. We would like to know: for a topological space $T$, what is a $T$-point of $*/\mathbb{Z}_2$, i.e., a map from $T$ to $*/\mathbb{Z}_2$?

A naive but natural guess, which we shall call $(*/\mathbb{Z}_2)^{\text{naive}} : \text{Top} \to \text{Gpd}$, is given by $(*/\mathbb{Z}_2)^{\text{naive}}(T) = \text{Maps}(T,*)/\text{Maps}(T,\mathbb{Z}_2) = */\text{Maps}(\mathbb{Z}_2^{|\pi_0(T)|})$. This is almost right; however, this functor into groupoids does not actually define a sheaf.

To see this, let $T = S^1$. We see that $(*/\mathbb{Z}_2)^{\text{naive}}(S^1) = */\text{Maps}(S^1,\mathbb{Z}_2) = */\mathbb{Z}_2$, where the latter is thought of as just the groupoid. (Note: we are being somewhat abusive in our notation, using similar notation for the functor $(*/\mathbb{Z}_2)^{\text{naive}} : \text{Top} \to \text{Gpd}$ and the groupoid $*/\mathbb{Z}_2$.) Now, a sheaf is characterized by the property that compatible local sections patch uniquely into global sections. As we did above, we cover $S^1$ by two open arcs $U_0$ and $U_1$ such that $U_0 \cup U_1 = S^1$, and such that $U_0 \cap U_1$ consists of two disconnected arcs $A \sqcup B$.

We see that

$$(*/\mathbb{Z}_2)^{\text{naive}}(U_0) = */\mathbb{Z}_2,$$
$$(*/\mathbb{Z}_2)^{\text{naive}}(U_1) = */\mathbb{Z}_2$$

and

$$(*/\mathbb{Z}_2)^{\text{naive}}(U_0 \cap U_1) = */(\mathbb{Z}_2 \times \mathbb{Z}_2),$$

since $U_0 \cap U_1 = A \sqcup B$ has two disconnected components. The restrictions $\text{res}^{U_0}_{U_0 \cap U_1}$ and $\text{res}^{U_1}_{U_0 \cap U_1}$ both correspond to the diagonal embedding $\Delta : */\mathbb{Z}_2 \to */(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Recall from the theory of sheaves over sets that a section $s \in \Gamma(U_0)$ and a section $t \in \Gamma(U_1)$ should glue to a section over $\Gamma(U_0 \cup U_1))$ if $\text{res}^{U_0}_{U_0 \cap U_1}(s) = \text{res}^{U_1}_{U_0 \cap U_1}(t)$. Thus $\Gamma(U_0 \cup U_1))$ is a fiber product:

$$\begin{tikzcd}
\Gamma(U_0 \cup U_1)) \arrow[r] \arrow[d] & \Gamma(U_1) \arrow[d, "\text{res}^{U_1}_{U_0 \cap U_1}"] \\
\Gamma(U_0) \arrow[r, "\text{res}^{U_0}_{U_0 \cap U_1}"] & \Gamma(U_0 \cap U_1).
\end{tikzcd}$$

So, for $(*/\mathbb{Z}_2)^{\text{naive}}(S^1)$ to be a sheaf, then $(*/\mathbb{Z}_2)^{\text{naive}}(S^1)$ ought to agree with (i.e., be equivalent as groupoids to) the fiber product:

$$\begin{tikzcd}
(*/\mathbb{Z}_2) \times_{(*/(\mathbb{Z}_2 \times \mathbb{Z}_2))} (*/\mathbb{Z}_2) \arrow[r] \arrow[d] & */\mathbb{Z}_2 \arrow[d, "\Delta"] \\
*/\mathbb{Z}_2 \arrow[r, "\Delta"] & */(\mathbb{Z}_2 \times \mathbb{Z}_2).
\end{tikzcd}$$

We should now be a little more precise about what we mean by a "fiber product in groupoids" as it is pretty fundamental to the set-up of the theory of stacks. We will give an algebraic construction.

**Definition 3.1.** *If $A$, $B$, and $C$ are groupoids, $F : A \to C$ and $G : B \to C$ are functors, then we define $A \times_C B$ as a groupoid whose objects are $\{a, b, \phi\}$ where $a \in \text{Ob}(A)$, $b \in \text{Ob}(B)$, and $\phi : F(a) \xrightarrow{\sim} G(b)$ is an isomorphism in $C$. The morphisms $\{a, b, \phi\} \to \{a', b', \phi'\}$ are given by $\{(\varphi \in \text{Mor}(A) : a \xrightarrow{\sim} a', \psi \in \text{Mor}(B) : b \xrightarrow{\sim} b')\}$ such that*

$$\begin{tikzcd}
F(a) \arrow[r, "\phi"] \arrow[d, "F(\varphi)"] & F(b) \arrow[d, "G(\psi)"] \\
F(a') \arrow[r, "\phi'"] & F(b').
\end{tikzcd}$$

We can show that this groupoid satisfies the right kind of "2-Cartesian" universal property, so that, at least up to categorical equivalence, this is the "correct" notion of fiber product. But rather than get carried away with the details, let us return to the above fibered product.

**Example 3.1.** *The fiber product of $* \times_{(*/G)} *$, where $*$ is the trivial groupoid, is given by:*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & */G. \end{tikzcd}$$

*where $G$ represents the set of elements of the group $G$ thought of as a set in* Gpd. *This follows directly from the construction of fiber products described above.*

Let us return to our above example. We see that $(*/\mathbb{Z}_2) \times_{(*/(\mathbb{Z}_2 \times \mathbb{Z}_2))} (*/\mathbb{Z}_2)$ consists of 4 objects, given, essentially, by the 4 automorphisms in $*/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. But there are only two isomoprhism classes of objects, given, essentially, by $\mathbb{Z}_2 \times \mathbb{Z}_2/\Delta$ Thus the fiber product is equivalent to the groupoid $*/\mathbb{Z}_2 \sqcup */\mathbb{Z}_2$.

On the other hand, we saw that $(*/\mathbb{Z}_2)^{\text{naive}}(S^1) = */\mathbb{Z}_2$, which only has one isomorphism class of object. Thus $(*/\mathbb{Z}_2)^{\text{naive}}$ is not a sheaf. But we can correct this by "sheafifying" – or, since there is some confusion in the literature regarding this terminology,"stackifying." This regards the naive functor as true "only locally", and builds the general functor by gluing these local functors. We call the sheafified $(*/\mathbb{Z}_2)^{\text{naive}}$, somewhat abusively, $*/\mathbb{Z}_2$ or $B\mathbb{Z}_2$.

In the topological setting, we can define the stack $*/\mathbb{Z}_2$ directly. Recall that for any space $T$, we have the corresponding fundamental groupoid $\pi_{\leq 1}(T)$. Then we can define:

$$(*/\mathbb{Z}_2)(T) = \text{Fun}(\pi_{\leq 1}(T), */\mathbb{Z}_2),$$

where the isomorphisms are given by natural isomorphisms of functors. This is automatically a sheaf, and it tells us why the naive $*/\mathbb{Z}_2$ did not work for $S^1$: the fundamental group of $S^1$ is nontrivial. On the other hand, for simply connected test spaces $T$, the naive functor does indeed give the correct groupoid.

But rather than appeal to the general theory of stackification, or the topology-specific construction of the fundamental groupoid, a careful analysis of what it means to sheafify $(*/\mathbb{Z}_2)^{\text{naive}}$ will tell us precisely which *moduli* stack $*/\mathbb{Z}_2$ corresponds to.

Given a general space $X$, we use a so-called "good cover" of $X$; i.e., one for which all the open sets and finite intersections of the open sets in the cover are contractible. (In fact, we can relax this constraint: we need only have all single, double, and triple intersections in our open cover be *simply-connected*.)

Gluing two "sections" $\pi_{\leq 1}(U_i \cap U_j) \to */\mathbb{Z}_2$ and $\pi_{\leq 1}(U_j \cap U_j) \to */\mathbb{Z}_2$ (which we imagine to be coming from $\pi_{\leq 1}(U_i) \to */\mathbb{Z}_2$ and $\pi_{\leq 1}(U_j) \to */\mathbb{Z}_2$, respectively) is the same as providing a natural transformation between these two functors $\pi_{\leq 1}(U_i \cap U_j) \to */\mathbb{Z}_2$. Since $\pi_{\leq 1}(U_i \cap U_j)$ is equivalent to trivial category $*$, we see that this is the same as an isomorphism $* \to *$ in $*/\mathbb{Z}_2$; i.e., and element $\mathbb{Z}_2$, which we call $g_{ji}$. We see that the $g_{ji}$ must satisfy a cocycle condition, and that two cocycles correspond to the same family if the usual "coboundary" equivalence holds.

All this should look familiar: it is precisely the data required to glue a 2-element set $\{a, b\}$ over $X$. If $g_{ij}$ is the trivial element, it will glue $a$ to $a$ and $b$ to $b$ on $U_i \cap U_j$; if it is nontrivial it will glue glue $a$ to $b$ and $b$ to $a$. Thus we have:

$$(*/\mathbb{Z}_2)(X) = \text{Groupoid of 2-point families over } X;$$

i.e., the moduli stack of two-point families. In fact, applying exactly the same logic, we see that if $G$ is any finite discrete group, then $(*/G)(X)$ is the groupoid of $|G|$-point families over $X$ with a fiberwise, continuous, transitive $G$-action. Indeed, we can also permit $G$ to have topology; the naive $*/G$-functor is defined via $*/G(X) = */\{\text{Continuous maps } X \to G\}$, and the above reasoning still holds. So we have:

**Proposition 3.2.** *The stack $*/G$, defined as the sheafification of $(*/G)^{naive}$, represents the following moduli problem:*

$$(*/G)(X) = \textit{Groupoid of principal } G\textit{-torsors over } X.$$

Recall: the benefit of functors being described as a *moduli* stack is that these functors are automatically sheaves. So with this formulation we know automatically that $*/G$ is a sheaf.

And now we can return to the algebro-geometric setting, at least once we know what a principal $G$-bundle is.

**Definition 3.3.** *A principal $G$-bundle on a scheme $X$ is a scheme $\tilde{X}$ with a $G$ action such that the map $\tilde{X} \to X$ is faithfully flat and the $(pr, act) : G \times \tilde{X} \to \tilde{X} \times_X \tilde{X}$ is an isomorphism of schemes.*

This definition is economical because it minimizes the need for descent theory, which will be discussed next week. One could equivalently define a principal $G$-bundle as an fppf torsor; in particular, principal $G$-bundles are locally trivial in the fppf topology. Note: if $G$ is smooth, then $\tilde{X} \to X$ is smooth.

**Definition 3.4.** *Let $G$ be an algebraic group. The classifying stack $BG = */G$ is the stack whose $S$-points are $BG(S) =$ groupoid of principal $G$-bundles on $S$.*

For exactly the formal reasons outlined above (in the topological setting), this is the sheafification of

$$(*/G)^{\text{naive}} : S \mapsto */G(S).$$

We note that there is a canonical map of stacks

$$\begin{tikzcd} * \arrow[d] \\ */G. \end{tikzcd}$$

For an arbitrary test-scheme $S$, the composition of the map $S \to *$ with the vertical quotient map must provide us with a particular isomorphism class of $G$-torsor over $S$: this is simply the trivial $G$-torsor. And given

$$\begin{tikzcd} & * \arrow[d] \\ S \arrow[r, "P"] & */G, \end{tikzcd}$$

where $P$ is a torsor over $S$ and the bottom map is the corresponding map $S \to */G$, we have a (2-)Cartesian diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & * \arrow[d] \\
S \arrow[r, "P"] & */G.
\end{tikzcd}$$

Because any isomorphism class of torsor can thus be "pulled back" from the torsor $* \to */G$ along a map $S \to */G$, we say that $* \to */G$ is the "universal $G$-torsor."

3.2. **Quotient Stacks.** . We may now consider the setting of a group $G$ acting on a variety $X$. And we can define the functor $X/G : \mathrm{Sch} \to \mathrm{Gpd}$ given by:

$$X/G := \text{sheafification of } ((X/G)^{\text{naive}} : S \mapsto X(S)/G(S)).$$

Our goal is now to attempt to describe the sheafified functor (aka stack) explicitly as a moduli functor. Thus we ask: what family over $S$ is parameterized by $(X/G)(S)$ for a test scheme $S$?

We will actually bootstrap this from the description we already have for $*/G$, and some basic properties of so-called "associated bundles." The first thing we notice is that the map $X \to *$ should induce a canonical map $X/G \to */G$. Thus an $S$ point of $S \to X/G$ induces by composition an $S$-point $S \to */G$; i.e., a $G$-torsor $P$ over $S$.

Now, say we have a $G$-torsor $P$ over $S$. We can form the fiber product:

$$\begin{tikzcd}
X \times^G P \arrow[r] \arrow[d] & X/G \arrow[d] \\
S \arrow[r, "P"] & */G
\end{tikzcd}$$

We call the stack $X \times^G P$ the $X$-bundle associated to $P$, or the associated bundle of $P$ with fiber $X$. We have the diagram:

$$\begin{tikzcd}
& X \times^G P \arrow[rr] \arrow[dd] & & X/G \arrow[dd] \\
X \times P \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & X \arrow[ur] & \\
& S \arrow[rr] & & */G \\
P \arrow[ur] \arrow[rr] & & * \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in which all faces are Cartesian, and the maps into the page are $G$-torsors; thus the associated bundle $X \times^G P = (X \times P)/G$ where $G$ acts diagonally. It is possible to show that this quotient (which is certainly a functor to sets, since the aciton of $G$ on $X \times P$ is free) is representable by a scheme.

Now, if we have a map $S \to X/G$, and we let $P$ be the composed map to $*/G$, then by the Cartesianness of the back face we have a section $S \to X \times^G P$. In addition, it implies that there is a map $P \to X$ commuting with the diagram. We see in fact:

**Proposition 3.5.** *There is a canonical bijection between Maps$(S, X/G)$ such that the induced map $X \to */G$ corresponds to $P$, sections to the associated bundle $S \to X \times^G P$, and $G$-equivariant maps $P \to X$.*

In addition to telling us something useful about sections of the associated bundle and equivariant maps, this proposition tells us exactly the data required, in addition to the torsor $P$, to get a map from $S$ to $X/G$. So we have motivated:

**Definition 3.6.** *Let an algebraic group $G$ act on a scheme $X$. Then the quotient stack $X/G$ is the functor* $\mathrm{Sch} \to \mathrm{Gpd}$ *given by*

$$(X/G)(S) = \textit{Groupoid of principal } G\textit{-torsors } P \textit{ with a } G\textit{-equivariant map } P \to X.$$

We can once again verify that this functor is equivalent to the sheafification of $(X/G)^{\mathrm{naive}}$, though we will spare thew details. We can also make various sanity checks: 1) $*/G$ agrees with the definition given in the previous section for $BG$, 2) for a torsor $P$ over $S$ we have $P/G \simeq S$, 3) the associated bundle scheme $X \times^G P$ agrees with the stack quotient $(X \times P)/G$, etc.

Let us return briefly to the associated bundle construction. These often give rise to canonical equivalences that are freely assumed in the literature; e.g., the equivalence between vector bundles and $GL_n$-torsors.

**Example 3.2.** *For $G = GL_n$ and $V = k^n$ the standard $n$-dimensional representation, there is an equivalence of categories of prinicpal $GL_n$ bundles and rank $n$ vector bundles. On $S$-points, given a principal $GL_n$-bundle $P$ over $S$, we take the associated bundle, $V_P := (V \times P)/GL_n$, over $S$. This has rank $n$. The map in reverse is the "frame bundle":*

$$\mathcal{V} \mapsto P := \mathit{Isom}(\mathcal{V}, \underline{k}^n \times S)$$

*i.e., the sheaf of local isomorphisms with the trivial bundle.*

More generally, if $P$ is a $G$-torsor on $S$, then for any linear representaiton $V$ we can form the associated bundle $V \times^G P$, which is a vector bundle on $S$. This gives us a way of "linearizing" torsors on $S$. Vector bundles are usually much easier to work with than torsors – for example, the category of vector bundles over $S$ has a far richer collection of non-isomorphism maps than the category of $\mathrm{GL}_n$-torsors over $S$. Moreover, this begins to hint at how the representation theory of $G$ influences the geometry of $BG$.

Conversely, we can recover $P$ from its associated linear bundles. In fact, given a system of vector bundles $\{\mathcal{V}\}$ on $S$ corresponding to each finite-dimensional representation of $G$, such that the system of $\{\mathcal{V}\}$ possess natural tensorial properties (aka a Tannakian formalism), we can recover the torsor $P$ over $S$.

3.3. **Mapping stacks.** Let $X, Y \in \mathrm{Fun}(A-\mathrm{alg}, \mathrm{groupoid})$ be two stacks. Define

$$\mathrm{Maps}(X, Y)(S) := Y(X \times S)$$

It turns out to automatically be a stack, which we call the mapping stack. Thus Maps is an internal hom in in the category of stacks; something very much lacking in, e.g., the category of schemes.

Let us offer a few words on why this definition is natural. Firstly, for each point $s \in S$, we get a corresponding map $X \to Y$. So this stack does indeed send a test scheme $S$ to families of maps from $X$ to $Y$ paramterized by $S$. Another intuition is as follows. In the category of sets, we often write $Y^X$ to denote the set of functions from $X$ to $Y$. If $S$ is a test set, then the collection of maps from $S$ to $Y^X$ is simply $(Y^X)^S \simeq Y^{X\times S}$; i.e., $Y(X \times S)$. Thus we can think of this definition as categorifying the laws of exponentiation.

The main example for us is when $X$ is an algebraic curve and $Y = BG$. Then

**Definition 3.7.** *The moduli stack of principal $G$-bundles over a curve $X$ is the stack*

$$\mathrm{Bun}_G(X) := \mathrm{Maps}(X, BG).$$

Thus an object of $\mathrm{Bun}_G(X)(S)$ is a principal $G$-bundle on $X \times S$, which is a family of principal $G$-bundles on $X$ parameterized by $S$, i.e $BG(X \times S)$.

**Example 3.3.** *consider $G = \mathbb{G}_m$ the multiplicative group, $X$ a smooth projective curve, and $k$ an algebraically closed field. Then the moduli stack of line bundles on $X$ is*

$$\mathrm{Bun}_G(X) \cong \mathbb{Z} \times Pic^\circ(X) \times B\mathbb{G}_m.$$

Indeed, $\mathrm{Bun}_{\mathbb{G}_m}(X)(S)$ is the groupoid of isomorphism classes of $\mathbb{G}_m$-torsors over $X \times S$. By the associated bundle construction (see above!) every isomoprhism class of $\mathbb{G}_m$-torsor gives rise (bijectively and canonically!) to a line bundle. The underlying set of isomorphism classes of such line bundles is famously Pic; choosing a line bundle of $\mathcal{L}_1$ degree one gives us isomorphisms $\mathbb{Z} \times \mathrm{Pic}^\circ : (d, \mathcal{L}) \to \mathcal{L}_1^{\otimes d} \otimes \mathcal{L}$. But this moduli stack does not just output the isomorphism classes of $\mathbb{G}_m$-torsors; it also remembers their automorphisms. Each $\mathbb{G}_m$-torsor; or, equivalently, each line bundle, has a $\mathbb{G}_m$'s worth of symmetries. Hence the above decomposition. Note that it is very rare to have such an explicit description of $\mathrm{Bun}_G$ in general.

Leaving the $X$ =curve example for the moment, another interesting example to consider is when $X = \mathrm{Spec}(\mathbb{F}_q)$ and $G$ is a finite group. Then, we claim,

$$\underline{\mathrm{Maps}}(\mathrm{Spec}\,\mathbb{F}_q, BG) = G/G$$

where $G$ acts on $G$ by the adjoint action, and the RHS is the stacky quotient. The important thing here is that $\mathrm{Spec}\,\mathbb{F}_q$ – while merely a point in the Zariski topology – is nontrivial in the étale topology. Indeed, there are lot of connected étale covers: $\mathrm{Spec}\,\mathbb{F}_{q^k} \to \mathrm{Spec}\,\mathbb{F}_q$ for each $k \in \mathbb{Z}$. Each of these has $\mathbb{Z}_k$ symmetry. We can think of the map from the algebraic closure algebraic closure, $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{Spec}\,\mathbb{F}_q$, as a map akin to the map $EH \to BH$ in topology, where $H = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}}$. In fact, $\mathrm{Spec}(\mathbb{F}_q)$ really ought to be thought of as having the homotopy type of a circle $S^1$ (which also has a unique cyclic cover of degree $k$ for each $k \in \mathbb{Z}$), and $\mathrm{Spec}\,\overline{\mathbb{F}}_q$ ought to be thought of as its universal cover (which is, in fact, contractible, beng a geometric point).

In fact, note that, in the topological setting

$$(*/G)(S^1) = \mathrm{Fun}(\pi_{\leq 1} S^1, */G) = \mathrm{Fun}(*/\mathbb{Z}, */G).$$

Such a functor is uniquely determined by where it sends $1 \in \mathbb{Z}$. Thus the objects of the category are $G$. A natural transformation between two such functors, say $g_0, g_1$ in $G$, is given by an element of $\mathrm{Mor}(*/G)$; i.e., an element $g$, such that the following diagram commutes:

$$\begin{tikzcd} * \arrow[r, "g"] \arrow[d, "g_0"] & * \arrow[d, "g_1"] \\ * \arrow[r, "g"] & *. \end{tikzcd}$$

Thus the functor $g_1$ and $g_2$ are isomorphic if and only if they are conjugate, and the collection of such isomorphisms is given by the collection $g \in G$ such that $g_1 = g g_0 g^{-1}$. This is precisely the action groupoid $G/G$ under the adjoint action.

Returning to the algebro-geometric setting, we wish to work out expliccitly the groupoid of $G$-torsors over $\mathrm{Spec}\,\mathbb{F}_q$. Let $P$ be such a torsor. We first "trivialize" the torsor by pulling back to $\overline{\mathbb{F}}_q$: this gives us $P_{\overline{\mathbb{F}}_q}$, and we know it is isomorphic to the trivial torsor $G \times \mathrm{Spec}\,\overline{\mathbb{F}}_q$.

Remember that in the étale topology, this covering map is the analogue of an open set. The nontrivial aspect of étale covers is that their self-intersections can be nontrivial. So while we can usually cover a scheme by a single étale morphism (as indeed we have with $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{Spec}\,\mathbb{F}_q$), the analogue of the the "intersection" of $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{Spec}\,\mathbb{F}_q$ with itself is not simply $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{Spec}\,\mathbb{F}_q$ again. Indeed, "intersection" means fiber product, and we have the Cartesian diagram:

$$\begin{tikzcd} \hat{\mathbb{Z}} \times \mathrm{Spec}\,\overline{\mathbb{F}}_q \arrow[r, "\mathrm{act}"] \arrow[d, "\mathrm{pr}"] & \mathrm{Spec}\,\overline{\mathbb{F}}_q \arrow[d] \\ \mathrm{Spec}\,\overline{\mathbb{F}}_q \arrow[r] & \mathrm{Spec}\,\mathbb{F}_q, \end{tikzcd}$$

which is often written

$$\begin{tikzcd} \hat{\mathbb{Z}} \times \mathrm{Spec}\,\overline{\mathbb{F}}_q \arrow[d, shift right, "\mathrm{act}"'] \arrow[d, shift left, "\mathrm{pr}"] \\ \mathrm{Spec}\,\overline{\mathbb{F}}_q \arrow[d] \\ \mathrm{Spec}\,\mathbb{F}_q, \end{tikzcd}$$

where "act" means applying the corresponding Galois action (where $1 \in \hat{\mathbb{Z}}$ means Frobenius). This diagram is simply saying that $\mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathbb{F}_q$ is a $\hat{\mathbb{Z}}$-torsor: see our definition of torsor above!

So, to provide the data of a $G$-torsor over $\mathrm{Spec}\,\mathbb{F}_q$, we need to specify an isomorphism of $G$-torsors, $G \times \hat{\mathbb{Z}} \times \mathrm{Spec}\,\overline{\mathbb{F}}_q \xrightarrow{\sim} G \times \hat{\mathbb{Z}} \times \mathrm{Spec}\,\overline{\mathbb{F}}_q$, which satisfy the cocycle condition on triple overlaps. Unpacking this, we arrive at a computation essentially identical to that we did directly above with the circle: there are a $G$'s worth such isomorphisms, given by where we send $1 \in \hat{\mathbb{Z}}$. Two such isomorphism given by $g_0$ and $g_1$ will yield the same family if and only if they are conjugate in $G$; in this case, there is an isomorphism for each $h \in G$ such that $g_1 = hg_0h^{-1}$. Thus the resulting stack is indeed the adjoint quotient $G/G$.

Reflecting on the above, we see, once more, we see that we are dealing with an $H^1$: in this case the non-commutative $H^1_{\text{ét}}(\operatorname{Spec}\mathbb{F}_q, G)$ which is also the non-commutative group cohomology $H^1(\hat{\mathbb{Z}}, G)$. The only technical point here is that the group cohomology for any finite group $G$ satisfies $H^1(\hat{\mathbb{Z}}, G) = H^1(\mathbb{Z}, G)$.

## 4. Quasi-coherent sheaves on stacks

Let $G$ be a group acting on a $S$ scheme $X$. Consider the maps $\mathrm{pr}_2, \mathrm{act} : G \times_S X \to X$.

**Definition 4.1.** *A quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{O}_X$ is said to be $G$-equivariant if there is an isomorphism $pr^*(\mathcal{F}) \cong act^*\mathcal{F}$ plus some cocycle condition. More explicitly, the cocycle condition is: for any $S$ scheme, $T$, and $g, g' \in G(T)$, the following diagram of quasi-coherent sheaves on $T \times_S X$ commutes.*

$$\begin{tikzcd}
\rho_g^*\rho_{g'}(pr_2^*F) \arrow[r, "\rho_g^*\sigma_g"] \arrow[d, "\sim"] & \rho_g^*(pr_2^*F) \arrow[d, "\sigma_g"] \\
\rho_{g'g}^*(pr_2F) \arrow[r, "\sigma_{g'g}"] & pr_2^*F
\end{tikzcd}$$

*Here, $\rho = act$, $\rho_g := pr_1 \times (\rho \circ (g \times id_X)) : T \times_S X \to T \times_S X$ and $\sigma_g : \rho_g^* pr_2^* F \to pr_2^* F$ is the pull back of $\sigma$ via $g \times id_X$.*

Now, there is an equivalence of categories between $G$-equivariant quasi-coherent sheaves on $\mathrm{pt} := \operatorname{Spec}(k)$ and $G$-representations over $k$. Indeed, a quasi-coherent sheaf over a point is just a vector space, and the isomorphism of pullbacks induces the action map. The cocycle condition is precisely the condition this action map is a group homomorphism.

We also claim that such data characterizes a a quasi-coherent sheaf over $BG$. Indeed, a quasicoherent sheaf must be patched locally in the smooth topology. And $* \to BG$ is a smooth, surjective map. Recall that we have the diagram:

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & */G.
\end{tikzcd}$$

So to define a quasicoherent sheaf on $*/G$, we must define a sheaf on $*$, that is to say, a vector space $V$. Then we need to glue it over the double intersection (in this case $G$). That is to say, we must provide a $G$-isomorphism $\varphi : G \times V \to G \times V$. But this isomorphism must be compatible with itself (= satisfy a cocycle condition) over the triple-intersection:

$$\begin{tikzcd}
& G \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
G \times G \arrow[ur] \arrow[rr] \arrow[dd] & & G \arrow[ur] \arrow[dd] & \\
& * \arrow[rr] & & */G \\
G \arrow[ur] \arrow[rr] & & * \arrow[ur] &
\end{tikzcd}$$

This data is known as "descent data"; it allows us to "descend" the sheaf $V$ on $*$ to a quasicoherent sheaf on $*/G$. Unpacking definitions, we see that the cocycle condition is precisely the extra data making $V$ into an equivariant sheaf over $*$, that is to say a $G$-representation.

To summarize: the local patching data for a quasicoherent sheaf on $BG$ given by the cover $* \to */G$ provides us with an equivalence between quasicoherent sheaves on $*/G$ and $G$-equivariant quasicoherent sheaves over a point. We have shown:

**Proposition 4.2.** *There is an equivalence of categories:*

$$Qcoh(BG) \leftrightarrow Qcoh^G(pt) \leftrightarrow Rep(G).$$