4. Metric spaces, open and closed sets
Math 20300, Winter 2019

§3.1 - §3.3 (through 3.3.22)

Read sections 3.1 and 3.2, and:

• Learn the definition of metric space, and understand why \( \mathbb{R} \) is a metric space (with the usual metric). As further examples, learn the definition of discrete metric, and inherited metric on a subspace.

• Learn the definition of the metrics \( d_1, d_2, d_\infty \) on \( \mathbb{R}^n \).

• Look at the definition of \( d_p \) in general. Don't worry about Hölder's inequality, which is used to prove that \( d_p \) is a metric.

• If it’s written in French, you’re not expected to read it.

Practice problems for 3.1-3.2:

1. Prove that \( d_1 \) and \( d_\infty \) are metrics on \( \mathbb{R}^n \). (The metric \( d_1 \) is sometimes called “Manhattan distance.”)

2. Prove that \( d_{1/2} \) is not a metric on \( \mathbb{R}^n \).

3. Consider the “zero metric”: \( d(x, y) = 0 \) for all \( x, y \). Is this a metric?

4. How about the “one metric”: \( d(x, y) = 1 \) for all \( x, y \)?

5. How about the “discrete metric”: \( d(x, x) = 0 \), but if \( x \neq y \), then \( d(x, y) = 1 \)?

6. How about the “distance-squared metric” on \( \mathbb{R} \), given by \( d(x, y) = (x - y)^2 \)?

7. Consider the “Google maps time metric”: for every two points \( x \) and \( y \) in the continental US that can be reached by car, \( d(x, y) \) is the amount of time it takes to get from \( x \) to \( y \), by the fastest route possible. Is this a metric?
8. Consider the “Hamming distance.” Fix a positive integer \( n \), and let \( X \) be the set of all strings of zeroes and ones, of length \( n \). (The typical element of \( X \) looks like “0100101”. Biologists may change 0 and 1 to the letters A, C, G, T.) Given two strings \( x \) and \( y \), define \( d(x, y) \) to be the number of positions where the strings have different symbols. (For example, \( d(00000, 00001) = 1 \), because the strings differ only in the last place. Note that all strings in \( X \) have the same length, and we use this “same length” property to align the strings for comparison.) Is this a metric?

9. Consider the edit distance, aka Levenshtein distance. Let \( S \) be the set of all (finite-length) strings (of 0’s and 1’s, or of A’s, C’s, G’s and T’s, as you please). Given such a string, a “single editing step” is one of the following three operations: delete a character, insert a character, or change a character to a different character. Given two strings \( x \) and \( y \), let \( d(x, y) \) be the smallest number of single editing steps needed to turn string \( x \) into string \( y \). Is this a metric?

Read the first half of §3.3, up to and including 3.3.22, and:

- Learn the definition of open ball, closed ball, open set, and closed set.

Problems for §3.3:

10. Show that open balls are open and closed balls are closed. (“Show” means “prove.”)

11. Consider Hamming distance with \( n = 5 \), and let \( x \) be the string “00000”. What is the open ball of radius 0.5 about \( x \)? Radius 2.5? Radius 6?

12. Consider Levenshtein distance, and let \( x \) be the alphabet (the string “ABCD...Z” of length 26). What is the open ball of radius 0.5 about \( x \)?

13. For each of \( d_1, d_2, d_\infty \), draw the open ball of radius 1 about the origin in \( \mathbb{R}^2 \).

14. Consider the triangle inequality \( d(a, c) \leq d(a, b) + d(b, c) \). Consider the metrics \( d_1, d_2, d_\infty \) on \( \mathbb{R}^2 \). Let \( a = (0, 0) \) and \( c = (2, 3) \), but let \( b \) vary. For each of these metrics, find all points \( b \) for which equality holds in the triangle inequality. (In other words: “\( b \) is on the way from \( a \) to \( c \),” or better, “\( b \) is on some best path from \( a \) to \( c \).”)

15. Consider Hamming distance with \( n = 5 \). Which sets are open? Which sets are closed?

16. Consider Manhattan distance \( d_1 \) on \( \mathbb{R}^2 \). Which sets are open? Which sets are closed?
17. Prove that the union of two open sets is open (in an arbitrary metric space).

18. Prove that the intersection of two open sets is open (in an arbitrary metric space).

19. What can you say about the union of an open set and a closed set?

20. Suppose \( O \) is an open set and \( C \) is a closed set. What can you say about the difference set \( O - C \)? What about \( C - O \)?

21. Which of the following sets is open? Closed?
   (a) \((0, 1)\) in \( \mathbb{R} \) with the usual metric
   (b) \([0, 1]\) in \( \mathbb{R} \) with the usual metric
   (c) The unit circle in \( \mathbb{R} \) with the usual metric
   (d) The set \( \{x\} \) consisting of a single point, in an arbitrary metric space
   (e) \([0, 1]\) in \( \mathbb{R} \) with the usual metric
   (f) \([0, 1]\) in \([0, 2]\) with the metric induced from the usual metric on \( \mathbb{R} \)
   (g) The set of points \((x, y)\) such that \(x^2 + y^2 < 1\), in \( \mathbb{R}^2 \) with Manhattan metric \(d_1\)
   (h) The “open first quadrant” (aka the set of points \((x, y)\) such that \(x > 0\) and \(y > 0\)), in \( \mathbb{R}^2 \) with Manhattan metric \(d_1\)
   (i) The “open first quadrant” (aka the set of points \((x, y)\) such that \(x > 0\) and \(y > 0\)), in \( \mathbb{R}^2 \) with Euclidean metric \(d_2\)
   (j) The “open first quadrant” (aka the set of points \((x, y)\) such that \(x > 0\) and \(y > 0\)), in \( \mathbb{R}^2 \) with metric \(d_\infty\)
   (k) \(\emptyset\) in an arbitrary metric space
   (l) \(X\) (the whole space) in an arbitrary metric space \(X\)