Important: This exam has eight problems, each worth 15 points. Please solve at least seven of the eight problems. If you solve all eight, I will grade the seven most favorable to you.

Note that $7 \cdot 15 = 105$. If you score over 100 points I will round your score down to 100.

1. Negate the following statements.
   (a) The sky is blue and the grass is green.
   (b) There exists a real number $x$ such that $x > 2$ and $x < 3$.
   (c) For every real number $x$, if $x > 0$, then $x^2 > 0$.

2. Which of the following sets are compact? (The first four are subsets of $\mathbb{R}$ with the usual metric.)
   (a) $\mathbb{R}$
   (b) $[0, 1]$
   (c) $(0, 1)$
   (d) $\{1, 2, 3, 4, 5\}$
   (e) $B(X)$, where $X$ is the three-element set $\{0, 1, 2\}$.

3. Give an example of a metric space $X$ with the property that every subset $A$ of $X$ is bounded. (No proof required.)

4. (a) (5 points) Give an example of a set $A \subseteq \mathbb{R}$ and a sequence $(a_n)$ of elements of $A$, such that $(a_n)$ converges to some limit $L \notin A$. (No proof required.)
   (b) (10 points) Fill in the blank: Given an arbitrary subset $A \subseteq \mathbb{R}$, a sequence $(a_n)$ satisfying the conditions of part (i) exists if and only if $A$ is ...
5. Let \( X \) be an arbitrary metric space (in particular, \( X \) is not necessarily complete). Suppose \((a_n)\) is a sequence of elements of \( X \), and \((a_{n_k})\) a subsequence of \((a_n)\), with the following properties:

- The sequence \((a_n)\) is Cauchy.
- The subsequence \((a_{n_k})\) converges to some limit \( L \).

Prove that the sequence \((a_n)\) itself converges to \( L \).

6. In this problem we’re going to give one possible definition for the product of two metric spaces.

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. We’re going to define the product of these two spaces as follows.

As a set, \( X \times Y \) is the Cartesian product of \( X \) and \( Y \): it’s the set of ordered pairs \( X \times Y = \{(x, y) | x \in X \text{ and } y \in Y \} \).

For the metric, we’ll take the sum of the metrics on \( X \) and \( Y \):

\[
d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).
\]

(a) (5 points) Suppose we take \( X = \mathbb{R} \) and \( Y = \mathbb{R} \), both with the usual metric. What familiar metric do we get on \( X \times Y = \mathbb{R}^2 \)? (No proof required.)

(b) (10 points) Prove that \( d_{X\times Y} \) is a metric.

7. Prove that a metric space \( X \) is connected if and only if it satisfies the following property: the only subsets \( A \subseteq X \) that are both closed and open are \( \emptyset \) and \( X \) itself.

(Hint: In class we defined connectedness of a subset of a metric space. What does it mean for the metric space itself to be connected? Think of the space as a subset of itself.)

8. Let \( X \) be a nonempty set, and fix some \( x \in X \). Consider the function

\[
\text{Eval}_x : \mathcal{B}(X) \to \mathbb{R}
\]

defined by

\[
\text{Eval}_x(f) = f(x).
\]

Prove that \( \text{Eval}_x \) is continuous.