Stationary measures on vector spaces

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Abstract

We study the classification of stationary measures for linear actions on vector spaces. A large part is a self-contained write-up of a known result by Bougerol [Bou].

1 Introduction

Let μ be a Borel probability measure on $G = GL(V)$, and let $\Gamma_\mu := \overline{\langle \text{supp } \mu \rangle} \subset G$ be the (topological) closure of the semigroup generated by the support of μ.

In this note, we are interested in studying the μ-stationary measures on the vector space V with respect to the Γ_μ-action on V by left multiplication.

Definition. We say that a Borel probability measure ν on V is *μ-stationary* if $\mu * \nu = \nu$, i.e.

$$\nu = \int_{GL(V)} g_* \nu \, d\mu(g).$$

Clearly if ν is Γ_μ-invariant then it is μ-stationary. Also note that since supp μ acts linearly on V, the origin of V is a fixed point, so the delta mass δ_0 at the origin of V is always a μ-stationary probability measure on V. We would like to understand when there are other μ-stationary probability measures on V, and if so whether we can classify all of them. In the rest of this note, we say that a μ-stationary measure ν on V is *nontrivial* if $\nu \neq \delta_0$.

In order to state our main classification result, we need the following two notions.

Definition. A Borel probability measure μ on $GL(V)$ has *finite first moment* if

$$\int_{GL(V)} \log \max(\|g\|, \|g^{-1}\|) d\mu(g) < \infty.$$

Here $\|\cdot\| := \|\cdot\|_{GL(V)}$ is the operator norm on $GL(V)$ with respect to a fixed norm on V.

Definition. We define the *top Lyapunov exponent* of μ on a Γ_μ-invariant subspace $W \subset V$ as

$$\lambda_{1,W} = \lambda_{1,W}(\mu) := \lim_{n\to\infty} \frac{1}{n} \int_{GL(V)} \log \|g\|_{GL(W)} d\mu^{(n)}(g),$$

where $\mu^{(n)} := \mu * \mu * \cdots * \mu$ is the n-th convolution power of μ, and for $g \in GL(V)$, $\|g\|_{GL(W)}$ denotes the operator norm of the restriction $g|_W$ in $GL(W)$.

The following result gives a necessary and sufficient condition for the existence of a nontrivial μ-stationary measure on V.

Theorem 1.1. Let μ be a Borel probability measure on $GL(V)$ with finite first moment. Then there exists a nontrivial μ-stationary measure ν on V if and only if there exist Γ_μ-invariant subspaces $W' \subsetneq W \subset V$ such that

(i) Γ_μ acts compactly on W/W', i.e. the image of $\rho_{W/W'} : \Gamma_\mu \to GL(W/W')$ is compact,

(ii) either $W' = 0$, or the top Lyapunov exponent of μ on W' is negative,

(iii) the support of every μ-stationary probability measure on V is in W.

The author only knew afterwards that the main proposition (Proposition 5.5) was already proved in the necessity direction of [Bou, Thm. 5.1]. Theorem 1.1 follows directly from Proposition 5.5 (see Section 6) (can be shown that (i) in [Bou, Thm. 5.1] can be improved to ensure $d_2 > 0$ if $d > 0$).

The following result classifies the stationary measures on V in terms of the compact Γ_μ-orbits on W/W'.

Theorem 1.2. Suppose there is a nontrivial μ-stationary measure on V and let $W' \subsetneq W \subset V$ be the Γ_μ-invariant subspaces from Theorem 1.1. Then the map $\nu \mapsto \operatorname{supp} \pi_*\nu$ gives a one-to-one correspondence between

$$\{\text{ergodic } \mu\text{-stationary measure on } V\} \qquad \leftrightarrow \qquad \{\text{compact } \Gamma_\mu\text{-orbit in } W/W'\},$$

where $\pi : W \to W/W'$ is the quotient map.

We can describe the inverse map in a more explicit way in terms of the asymptotic behavior in law of the random walk on V induced by μ.

Theorem 1.3. For any compact Γ_μ-orbit $\mathcal{C}$ in W/W', let $m_\mathcal{C}$ be the Haar (probability) measure supported on $\mathcal{C}$. Let $s : W/W' \to W$ be a linear section, i.e. a linear map such that $\pi \circ s = \mathrm{id}$. Then the weak-$*$ limit

$$\nu_\mathcal{C} := \lim_{n\to\infty} \mu^{(n)} * (s_* m_\mathcal{C})$$

exists and does not depend on the choice of the section s. Moreover, the map $\mathcal{C} \mapsto \nu_\mathcal{C}$ is the inverse map of the bijection in Theorem 1.2.

Using the classification of stationary measures, we can obtain the following equidistribution result.

Theorem 1.4. For all $x \in W$, let $\mathcal{C}$ is the compact Γ_μ-orbit of $x + W'$ in W/W'. Then

1. we have the weak-$*$ convergence
$$\frac{1}{n}\sum_{i=0}^{n-1} \mu^{(i)} * \delta_x \to \nu_\mathcal{C}.$$

2. For $\mu^\mathbb{N}$-almost every word $b = (b_1, b_2, \ldots) \in GL(V)^\mathbb{N}$, we have the convergence of the empirical measures
$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{b_i b_{i-1}\ldots b_1 x} \to \nu_\mathcal{C} \qquad \text{as } n \to \infty.$$

The following definition is standard when considering stationary measures.

Proposition 1.5. [BL, Lem. II.2.1] Let μ be a Borel probability measure on $G = GL(V)$ and ν be a μ-stationary measure on V. Then for $\mu^\mathbb{N}$-almost every $b = (b_1, b_2, \ldots) \in G^\mathbb{N}$, there exists a probability measure ν_b on V such that for all $g \in \Gamma_\mu$,

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \ldots b_n g)_* \nu.$$

Moreover, we have

$$\nu = \int_{G^\mathbb{N}} \nu_b \, d\mu^\mathbb{N}(b).$$

The measure ν_b is sometimes called the *limit measure* of ν with respect to the word b.

We can describe the limit measures of any stationary measures on V.

Theorem 1.6. For each compact Γ_μ-orbit $\mathcal{C}$ in W/W', for $\mu^\mathbb{N}$-almost every word $b \in GL(V)^\mathbb{N}$, the limit measure

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \ldots b_n)_* \nu_\mathcal{C}$$

is supported on the compact subset $p_b(\mathcal{C}) \subset W$ for some linear section $p_b : W/W' \to W$. In particular, ν_b is compactly supported on W.

If Γ_μ acts trivially on W/W', then ν_b is a delta mass $\delta_{\xi(b)}$ for $\mu^\mathbb{N}$-almost every word b, and thus ν is μ-proximal (cf. [BQ1, Sect. 2.7]).

The note is structured as follows.

1. In section 2, we recall a few preliminary facts about stationary measures and top exponents.
2. In section 3, we recall the situation when the action is irreducible, which will form the building blocks of the general case.
3. In section 4, we list a few properties of Γ_μ-actions that satisfy (i) and (ii) of Theorem 1.1. In particular most of Theorem 1.2, 1.3, 1.4 and 1.6 will be proved in this section.
4. In section 5, we study properties of the action on the span of the support of any given stationary measure on V. The main result in this section is Proposition 5.5, when we show that the action on this span satisfies (i) and (ii) of Theorem 1.1.
5. In section 6, we conclude by proving Theorem 1.1 using results from the previous sections.

2 Preliminary facts

We first recall that, in the case of a compact action, we have the standard fact that any stationary measure is invariant.

Proposition 2.1. [BQ3, Lem. 8.4] Let μ be a Borel probability measure on $G = GL(V)$ and ν be a μ-stationary measure on V. If Γ_μ acts compactly on V, then ν is Γ_μ-invariant. Moreover, if ν is ergodic, then the support of ν is a single compact Γ_μ-orbit in V, and ν is the unique μ-stationary measure supported on this orbit.

We recall the following general theorem by Furstenberg and Kesten, which follows from Kingman's subadditive ergodic theorem and the ergodicity of the Bernoulli shift.

Theorem 2.2. [FK, Thm. 2], see also [BQ1, Lem. 4.27].
Let μ be a Borel probability measure on $GL(V)$ with finite first moment. For $\mu^{\mathbb{N}}$-a.e. $b = (b_1, b_2, \ldots) \in G^{\mathbb{N}}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|b_n \cdots b_1\| = \lim_{n\to\infty} \frac{1}{n} \log \|b_1 \cdots b_n\| = \lambda_{1,V}(\mu).$$

In particular, if $\lambda_{1,V} < 0$, then $\|b_1 \cdots b_n\| \to 0$ for $\mu^{\mathbb{N}}$-almost every word b.

To simplify notation, given a vector space V' with a homomorphism $\rho_{V'} : \Gamma_\mu \to GL(V')$, we say that μ has *negative top exponent on* V' if the top Lyapunov exponent $\lambda_{1,V'}$ of $\rho_{V'}$ with respect to μ is negative.

We need the following two lemmas that allow us to carry certain properties to invariant subspaces and quotients.

Lemma 2.3. Let μ be a Borel probability measure on $GL(V)$ with finite first moment. Let $W \subset V$ be a Γ_μ-invariant subspace of V. Then the following are equivalent:

(i) μ has negative top exponent on V.

(ii) μ has negative top exponent on W and V/W.

Proof. In fact the top exponent on V is the maximum of the top exponents on W and V/W. This is standard. See, for instance, [FKi, Lem. 3.6]. $\square$

We also need the following elementary result about boundedness.

Lemma 2.4. Let μ be a Borel probability measure on $GL(V)$. Let $W \subset V$ be a Γ_μ-invariant subspace of V. Given a subset $B \subset \Gamma_\mu$, if B is bounded from above in $GL(V)$, then B is bounded from above in $GL(W)$ and $GL(V/W)$.

3 The irreducible case

We first recall the classification of stationary measures for *irreducible* Γ_μ-actions, i.e. the only Γ_μ-invariant subspaces of V are 0 and V.

Proposition 3.1. Let μ be a Borel probability measure on $GL(V)$. Suppose that Γ_μ acts irreducibly on V. Then there exists a nontrivial μ-stationary probability measure ν on V if and only if Γ_μ is compact in $GL(V)$.

Proof. If Γ_μ is compact in $GL(V)$ then clearly there is a nontrivial Γ_μ-invariant measure on V (by averaging via the finite Haar measure on Γ_μ), hence in particular μ-stationary.

The opposite direction was proved in [BL, Prop. V.8.1]. $\square$

We will also need another proposition that shows that for irreducible actions, assuming a boundedness condition, the only two options are negative top exponent and compact action.

Proposition 3.2. Let μ be a Borel probability measure on $G = GL(V)$ with finite first moment. Assume that Γ_μ is irreducible. If for $\mu^{\mathbb{N}}$-almost every $b = (b_1, b_2, \ldots) \in G^{\mathbb{N}}$, the sequence

$$\{b_n b_{n-1} \ldots b_1 \mid n \geq 1\}$$

is bounded from above (with respect to the operator norm on $GL(V)$), then either μ has negative top exponent on V, or Γ_μ is compact in $GL(V)$.

Proof. The assumption implies that the top exponent is nonpositive by Theorem 2.2. Hence it suffices to consider the case when $\lambda_{1,V} = 0$.

Let $C : G^{\mathbb{N}} \to \mathbb{R}_+ \cup \{\infty\}$ be a measurable function such that

$$\|b_n b_{n-1} \ldots b_1\| \leq C(b) \qquad \text{for all } n.$$

Then by assumption, we can take C to be finite $\mu^{\mathbb{N}}$-almost surely. If we take C' large enough, there is a subset $\mathcal{B} \subset G^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\mathcal{B}) > 1/2$ such that $C(b) < C'$ for all $b \in \mathcal{B}$. Now fix a μ-stationary measure $\nu_{\mathbb{P}}$ on $\mathbb{P}(V)$, and consider the dynamical system on $G^{\mathbb{N}} \times \mathbb{P}(V)$ with the map

$$T(b, v) := \left(\sigma(b), \log \frac{\|b_1 v\|}{\|v\|}\right),$$

where $\sigma : G^{\mathbb{N}} \to G^{\mathbb{N}}$ is the left shift map. Note that $\mu^{\mathbb{N}} \times \nu_{\mathbb{P}}$ is a T-invariant probability measure on $G^{\mathbb{N}} \times \mathbb{P}(V)$.

By the proof of the Atkinson's lemma ([At], [Ke], see e.g. [BQ1, Lem. 3.18]), for $\mu^{\mathbb{N}} \times \nu_{\mathbb{P}}$-almost every $(b, v) \in G^{\mathbb{N}} \times \mathbb{P}(V)$, there is an infinite sequence $\{n_k\}_k$ such that

$$\left|\log \frac{\|b_{n_k} \ldots b_1 v\|}{\|v\|}\right| \leq 1. \tag{1}$$

Fix a nonzero $v \in V$ such that (1) holds for $\mu^{\mathbb{N}}$-almost every $b \in G^{\mathbb{N}}$. For each such word $b \in G^{\mathbb{N}}$, for each $n \geq 1$, take k large enough so that $n_k > n$. Then

$$\log \frac{\|b_n \ldots b_1 v\|}{\|v\|} = \log \frac{\|b_{n_k} \ldots b_1 v\|}{\|v\|} - \log \frac{\|b_{n_k} \ldots b_1 v\|}{\|b_n \ldots b_1 v\|}.$$

Now on the right hand side, the first term is at least -1 by (1), and the second term is at least $-\log C(\sigma^n(b))$ by definition of C. Therefore

$$\log \frac{\|b_n \ldots b_1 v\|}{\|v\|} \geq -1 - \log C(\sigma^n(b)).$$

However note that $C(\sigma^n(b))$ does not depend on $b_1, b_2, \ldots, b_n$. Therefore we can replace b by one of the words that starts with $b_1, b_2, \ldots, b_n$ and satisfies $\sigma^n(b) \in \mathcal{B}$ so that $C(\sigma^n(b)) < C'$ for the uniform constant C' chosen above. Thus for $\mu^{\mathbb{N}}$-almost every word b, for all $n \geq 1$,

$$\log \frac{\|b_n \ldots b_1 v\|}{\|v\|} \geq -1 - \log C'.$$

Now consider the sequence of measures on V

$$\frac{1}{N}\sum_{n=0}^{N-1}\delta_{b_n b_{n-1}\dots b_1 v}.$$

Then any weak-$*$ limit ν is a μ-stationary measure on V by Breiman's Law of Large Numbers ([Br], also see e.g. [BQ1, Cor. 3.4]), and is a probability measure since there is a uniform bound from above on the sequence $\{b_n b_{n-1}\dots b_1 \mid n \geq 1\}$ by assumption. Since

$$\frac{\|b_n \dots b_1 v\|}{\|v\|} \geq C'' \qquad \text{for all } n \geq 1$$

for some uniform C'', ν is not δ_0, so it is a nontrivial μ-stationary probability measure on V. By Proposition 3.1, Γ_μ is compact in $GL(V)$. ☐

The same is true if the order of the matrix product $b_1 b_2 \dots b_n$ is reversed.

Corollary 3.3. Let μ be a Borel probability measure on $G = GL(V)$ with finite first moment. Assume that Γ_μ is irreducible. If for $\mu^{\mathbb{N}}$-almost every $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$, the sequence

$$\{b_1 b_2 \dots b_n \mid n \geq 1\}$$

is bounded from above (with respect to the operator norm on $GL(V)$), then either μ has negative top exponent on V, or Γ_μ is compact in $GL(V)$.

Proof. Apply Proposition 3.2 to the pushforward μ^T of μ via the adjoint map $GL(V) \to GL(V^*)$ defined by $g \mapsto g^T$ (i.e. the matrix transpose). Note that $\|g\|_{GL(V)} = \|g^T\|_{GL(V^*)}$, so the first moments of μ and μ^T are the same. Similarly the top exponents of μ and μ^T are the same. Finally Γ_μ is irreducible if and only if Γ_{μ^T} is, and Γ_μ is compact if and only if Γ_{μ^T} is. ☐

4 Properties of a contracting-by-compact action

In this section, we list a few properties of subspaces with a contracting-by-compact action by μ, i.e. there is a proper subspace (possibly zero) with negative top exponent with respect to μ and Γ_μ acts compactly on the quotient.

The following proposition shows that for such action, almost every word is bounded from above with respect to the operator norm (though this bound may depend on the word).

Proposition 4.1. Let μ be a Borel probability measure on $GL(W)$ with finite first moment. Moreover there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Then there exists a measurable map $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \dots)$,

$$\|b_1 b_2 \dots b_n\| < C(b) \qquad \text{for all } n \geq 1.$$

Proof. By choosing suitable basis, we can write each $b_i \in \operatorname{supp}\mu$ as

$$\begin{bmatrix} x_i & y_i \\ 0 & z_i \end{bmatrix},$$

where $x_i \in GL(W')$, $z_i \in GL(W/W')$ and $y_i \in \operatorname{Hom}(W/W', W')$.

Now we expand $b_1 b_2 \dots b_n$ in terms of x_i, y_i, z_i,

$$b_1 b_2 \dots b_n = \begin{bmatrix} X_n & Y_n \\ 0 & Z_n \end{bmatrix},$$

where

$$X_n = x_1 x_2 \dots x_n, \qquad Y_n = \sum_{k=1}^{n} x_1 \dots x_{k-1} y_k z_{k+1} \dots z_n, \qquad Z_n = z_1 z_2 \dots z_n.$$

Since μ has negative top exponent on W', $x_1 x_2 \dots x_n \to 0$ for $\mu^{\mathbb{N}}$-almost every word b by Theorem 2.2. Since Γ_μ acts compactly on W/W', Z_n is uniformly bounded by some constant C'. Hence it remains to find a bound on Y_n that is independent of n (but may depend on the word b).

If $W' = 0$, we are done. If $W' \neq 0$, let $\lambda_{1,W'} < 0$ be the top exponent of μ on W'. Then for $\mu^{\mathbb{N}}$-almost every word b,

$$\lim_{k\to\infty} \frac{1}{k} \log \|x_1 x_2 \dots x_k\| = \lambda_{1,W'} < 0.$$

Since μ has finite first moment in $GL(W)$, in particular, we have

$$\int_G \log^+(\|g\|) \, d\mu < \infty,$$

where $\log^+(x) := \max(\log(x), 0)$. This implies that (since $\|b_k\| \geq \|y_k\|$)

$$\sum_{k=1}^{\infty} \mu\left(\log^+(\|y_k\|) > -\frac{k\lambda_{1,W'}}{2}\right) \leq \sum_{k=1}^{\infty} \mu\left(\log^+(\|b_k\|) > -\frac{k\lambda_{1,W'}}{2}\right) < \infty.$$

By Borel-Cantelli Lemma, for $\mu^{\mathbb{N}}$-almost every word b,

$$\limsup_k \frac{1}{k} \log^+ \|y_k\| \leq -\frac{\lambda_{1,W'}}{2}.$$

This implies that

$$\limsup_k \frac{1}{k} \log \|x_1 \dots x_{k-1} y_k\| \leq \limsup_k \frac{1}{k} \log(\|x_1 \dots x_{k-1}\| \|y_k\|) < \frac{\lambda_{1,W'}}{2}.$$

Since $\lambda_{1,W'} < 0$, and z_i is in a compact subgroup of $GL(W/W')$ with a uniform upper bound C', there exist $n_0 = n_0(b)$ and $C'' = C''(b)$ such that for all large enough n,

$$\|Y_n\| \leq \sum_{k=1}^{n} \|x_1 \dots x_{k-1} y_k z_{k+1} \dots z_n\| \leq C'' + C' \sum_{k=n_0}^{n} e^{k\lambda_{1,W'}/2} \leq C'' + \frac{C'}{1 - e^{\lambda_{1,W'}/2}} < \infty,$$

as desired. ◻

The following proposition shows that there is at least one nontrivial stationary measure in the subspace W.

Proposition 4.2. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Suppose there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Then for all $x \in W \setminus W'$, any weak-$*$ limit point of the sequence of probability measures

$$\nu_{x,n} := \frac{1}{n} \sum_{i=0}^{n-1} \mu^{(i)} * \delta_x$$

is a nontrivial μ-stationary probability measure on W.

Proof. Let $\widehat{W} := W \cup \{\infty\}$ be the one-point compactification of W. Then the space of probability measures $\mathcal{M}(\widehat{W})$ is compact, hence there exists a subsequence $\{n_k\}$ such that ν_{x,n_k} converges to a probability measure $\nu \in \mathcal{M}(\widehat{W})$. Moreover,

$$\mu * \nu_{x,n_k} - \nu_{x,n_k} = \frac{1}{n_k}(\mu^{(n_k)} * \delta_x - \delta_x) \to 0.$$

Hence ν is μ-stationary. Since ∞ is a fixed point, we may consider ν as a μ-stationary measure on W (*a priori* may not be a probability measure). It remains to show that $\nu(W \setminus \{0\}) = 1$. Let $\pi : W \to W/W'$ be the quotient map.

First of all since Γ_μ acts compactly on W/W' and $x \in W \setminus W'$, $\Gamma_\mu \pi(x) \subset W/W'$ is compact and does not contain the origin in W/W'. Therefore there exists a compact subset $\mathcal{C}_x \subset W \setminus W'$ such that $\Gamma_\mu x \subset \mathcal{C}_x + W'$. Note that $0 \notin \mathcal{C}_x + W'$. Now clearly the support of $\nu_{x,n}$ is contained in $\Gamma_\mu x \subset \mathcal{C}_x + W'$ for all n and hence the support of ν is also contained in the closed set $\mathcal{C}_x + W'$. In particular $\nu(\{0\}) = 0$.

It remains to show that for all $\varepsilon > 0$, there exists $C'' = C''(\varepsilon, x) > 0$ such that

$$\nu(\{w \in W \mid \|w\| < C''\}) > 1 - \varepsilon.$$

Since $\nu_{x,n_k} \to \nu$, applying this convergence to the indicator function $\mathbf{1}_{\{w \in W \mid \|w\| < C''\}}$, we have

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mu^i(\{(b_1, b_2, \ldots, b_i) \in G^i \mid \|b_1 b_2 \ldots b_i x\| < C''\}) = \nu(\{w \in W \mid \|w\| < C''\}).$$

But the left hand side can be bounded from below using Fatou's lemma:

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mu^i(\{(b_1, b_2, \ldots, b_i) \in G^i \mid \|b_1 b_2 \ldots b_i x\| < C''\}) \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b)\, d\mu^{\mathbb{N}}(b) \\
&\geq \int \liminf_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b)\, d\mu^{\mathbb{N}}(b)
\end{aligned}$$

Moreover, by Proposition 4.1, there exists a measurable function $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that, for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots)$,

$$\|b_1 b_2 \ldots b_n\| < C(b).$$

Now take a subset $\mathcal{B}_\varepsilon \subset G^{\mathbb{N}}$ and large enough $C'_\varepsilon > 0$ such that $\mu^{\mathbb{N}}(\mathcal{B}_\varepsilon) > 1 - \varepsilon$ and $C(b) < C'_\varepsilon$ for all $b \in \mathcal{B}_\varepsilon$. Let $C'' = C''(\varepsilon, x) := C'_\varepsilon \|x\|$. Then for all $b \in \mathcal{B}_\varepsilon$,

$$\liminf_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b) = 1.$$

Thus

$$\nu(\{w \in W \mid \|w\| < C''\}) \geq \int \liminf_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b)\, d\mu^{\mathbb{N}}(b) \geq \mu^{\mathbb{N}}(\mathcal{B}_\varepsilon) > 1 - \varepsilon.$$

□

The following proposition shows that any stationary measure in such subspace W is uniquely determined by its pushforward on the quotient W/W'.

Proposition 4.3. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Let $W' \subsetneq W$ be a Γ_μ-invariant flag. Suppose

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Suppose that we have two μ-stationary measures ν and ν' on W that satisfy $\pi_* \nu = \pi_* \nu'$ for the quotient map $\pi : W \to W/W'$, then $\nu = \nu'$.

Proof. By Proposition 4.1, there exists a measurable map $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$, we have

$$\|b_1 b_2 \dots b_n\|_{GL(W)} < C(b).$$

Also for almost every word b, we have the limit measure

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \dots b_n)_* \nu.$$

Therefore we can take a limit point b_∞ of the sequence $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ in $\mathrm{End}(W)$, and

$$\nu_b = (b_\infty)_* \nu.$$

Similarly, we have, for almost every word b,

$$\nu'_b := \lim_{n\to\infty} (b_1 b_2 \dots b_n)_* \nu' = (b_\infty)_* \nu'.$$

Now since μ has negative top exponent on W', for almost every word b,

$$\lim_{n\to\infty} b_1 b_2 \dots b_n v = 0 \qquad \text{for every vector } v \in W'.$$

Therefore $W' \subset \ker b_\infty$, hence the map $b_\infty : W \to W$ factors through W/W', i.e. there exists a linear map $b'_\infty : W/W' \to W$ such that $b_\infty = b'_\infty \circ \pi$, where $\pi : W \to W/W'$ is the quotient map. Since $\pi_* \nu = \pi_* \nu'$, for $\mu^{\mathbb{N}}$-almost every word b, we have

$$\nu_b = (b_\infty)_* \nu = (b'_\infty)_* \pi_* \nu = (b'_\infty)_* \pi_* \nu' = (b_\infty)_* \nu' = \nu'_b.$$

Thus by Theorem 1.5,

$$\nu = \int_{G^{\mathbb{N}}} \nu_b d\mu^{\mathbb{N}}(b) = \int_{G^{\mathbb{N}}} \nu'_b d\mu^{\mathbb{N}}(b) = \nu'.$$

□

In particular the above proof shows that each limit measure ν_b is supported on a compact subset of W. We record this in the following proposition (which proves Theorem 1.6).

Proposition 4.4. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Let $W' \subsetneq W$ be a Γ_μ-invariant flag. Suppose

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Given an ergodic μ-stationary measure ν on W, for $\mu^{\mathbb{N}}$-almost every word b, the limit measure

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \dots b_n)_* \nu$$

is supported on the pushforward of a single compact Γ_μ-orbit on W/W' via a linear injection $p_b : W/W' \to W$. In particular, ν_b is compactly supported on W.

Proof. Take p_b to be the linear map b'_∞ defined in the proof of Proposition 4.3. Since $\pi_* \nu$ is an ergodic μ-stationary measure on W/W' and μ acts compactly on W/W', $\pi_* \nu$ is an ergodic Γ_μ-invariant measure and is supported on a single compact Γ_μ-orbit in W/W' by Proposition 2.1. Thus $\nu_b = (b'_\infty)_* \pi_* \nu$ is also compactly supported on W. □

Using Proposition 4.3, one can refine Proposition 4.2.

Proposition 4.5. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Suppose there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

For all $x \in W \setminus W'$, let

$$\nu_{x,n} := \frac{1}{n} \sum_{i=0}^{n-1} \mu^{(i)} * \delta_x.$$

Then the weak-$*$ limit

$$\nu_x := \lim_{n\to\infty} \nu_{x,n}$$

exists and is a nontrivial μ-stationary probability measure on W.

Proof. By Proposition 4.2, we know that any limit point of the sequence $\{\nu_{x,n}\}_n$ is a nontrivial μ-stationary measure on W. Moreover, since the projection of $\nu_{x,n}$ on W/W' lies in the compact Γ_μ-orbit of $x+W' \in W/W'$, any weak-$*$ limit point projects to a μ-stationary measure supported on the single compact orbit $\Gamma_\mu x + W' \subset W/W'$, hence is in fact the unique invariant measure supported on the compact set $\Gamma_\mu x + W'$. In particular, any limit point of $\{\nu_{x,n}\}_n$ is a μ-stationary probability measure that projects to the same measure on W/W'. By Proposition 4.3, all such limit points agree, so the sequence $\{\nu_{x,n}\}_n$ converges. □

In fact, if we start with any initial measure that projects to the Haar measure supported on a compact Γ_μ-orbit in W/W', then the convolution powers are not just Cesáro summable, but themselves converge.

Proposition 4.6. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Suppose there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

For all $x \in W \setminus W'$, let $\mathcal{C}_x$ be the Γ_μ-orbit of the image x in W/W', and m_x be the Haar (probability) measure on W/W' supported on $\mathcal{C}_x$. Then for any linear section $s : W/W' \to W$, i.e. a linear map such that $\pi \circ s = \mathrm{id}$, we have the following weak-$*$ convergence

$$\nu_x := \lim_{n\to\infty} \mu^{(n)} * (s_* m_x).$$

Moreover, ν_x is a nontrivial μ-stationary probability measure on W that does not depend on the choice of the linear section s. The map $x \mapsto \nu_x$ is constant along the orbit $\mathcal{C}_x$.

Proof. By Proposition 4.2, for all $x \in W \setminus W'$, there exists a nontrivial μ-stationary measure ν_x on W that projects to m_x on W/W'.

Similar to the proof of Proposition 4.3, there exists a measurable function $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots)$, we have

$$\|b_1 b_2 \ldots b_n\|_{GL(W)} < C(b), \qquad \text{and} \qquad \nu_b = \lim_{n\to\infty} (b_1 \ldots b_n)_* \nu_x$$

exists. Moreover, for any limit point b_∞ of $\{b_1 b_2 \ldots b_n \mid n \geq 1\}$ in $\mathrm{End}(W)$, there exists a linear map $b'_\infty : W/W' \to W$ such that $b_\infty = b'_\infty \circ \pi$. Let $\{n_k\}_k$ be the indices of the subsequence such that

$$\lim_{k\to\infty} b_1 b_2 \ldots b_{n_k} = b_\infty = b'_\infty \circ \pi.$$

Now for any linear section $s : W/W' \to W$, we have

$$\lim_{k\to\infty} (b_1 \ldots b_{n_k})_* (s_* m_x) = (b'_\infty)_* \pi_* s_* m_x = (b'_\infty)_* m_x$$

since $\pi \circ s = \mathrm{id}$. On the other hand since the stationary measure ν_x projects to m_x on W/W', we also have

$$\nu_b = \lim_{n\to\infty} (b_1 \ldots b_n)_* \nu_x = (b_\infty)_* \nu_x = (b'_\infty)_* \pi_* \nu_x = (b'_\infty)_* m_x.$$

Thus

$$\nu_b = (b'_\infty)_* m_x = \lim_{k\to\infty} (b_1 \ldots b_{n_k})_* (s_* m_x)$$

for any convergent subsequence $\{b_1 \dots b_{n_k} \mid k \geq 1\}$. Since the left hand side does not depend on the subsequence, we have the convergence

$$\nu_b = \lim_{n\to\infty} (b_1 \dots b_n)_*(s_* m_x).$$

Since this holds for $\mu^{\mathbb{N}}$-almost every b, we have

$$\nu_x = \int \nu_b d\mu^{\mathbb{N}}(b) = \int \lim_{n\to\infty} (b_1 \dots b_n)_*(s_* m_x) d\mu^{\mathbb{N}}(b) = \lim_{n\to\infty} \mu^{(n)} * (s_* m_x).$$

□

5 Properties of the span of the support of a stationary measure

In this section, we prove a few properties of the action on the span of the support of a given stationary measure. The main statement is that the span of the support of a given stationary measure must have a contracting-by-compact action by μ (Proposition 5.5). An important auxiliary proposition leading towards this fact is Proposition 5.2.

Lemma 5.1. Let μ be a Borel probability measure on $GL(V)$, ν be a μ-stationary probability measure on V. Let W be the linear span of the support of ν. Then

(i) W is Γ_μ-invariant.

(ii) For $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$, the sequence $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ is bounded from above in $GL(W)$.

Proof. (i) is clear since supp ν is Γ_μ-invariant. The proof of (ii) is similar to the proof of [BP, Lem. 3.3], using ideas of [F, Thm. 1.2]. By considering the restriction of the action to W we may assume that $V = W$ and thus $G = GL(W)$ without loss of generality. For $b \in G^{\mathbb{N}}$ for which the limit measure ν_b exists, assume the contrary that the sequence $\{b_1 b_2 \dots b_n \mid n \geq 0\}$ is not bounded from above in $GL(W)$. Then we can find a subsequence $\{n_k \mid k \in \mathbb{N}\}$ and $b_\infty \in \operatorname{End}(W)$ with $\|b_\infty\|_{\operatorname{End}(W)} = 1$ such that

$$\lim_{n\to\infty} \|b_1 b_2 \dots b_{n_k}\|_{GL(W)} = \infty, \qquad \text{and} \qquad \lim_{k\to\infty} \frac{b_1 b_2 \dots b_{n_k}}{\|b_1 b_2 \dots b_{n_k}\|_{GL(W)}} = b_\infty.$$

Let $W_b := \ker b_\infty \subset W$. For $v \in W \setminus W_b$, we have

$$\lim_{k\to\infty} \|b_1 b_2 \dots b_{n_k} v\|_W = \infty.$$

Thus for any continuous function $\phi : W \to \mathbb{R}$ with compact support, for all $v \in W \setminus W_b$,

$$\phi(b_1 b_2 \dots b_{n_k} v) \to 0 \qquad \text{as} \qquad k \to \infty.$$

Therefore

$$\begin{aligned}
\int \phi(v) d\nu_b(v) &= \lim_{k\to\infty} \int \phi(v) d(b_1 b_2 \dots b_{n_k})_* \nu(v) \\
&= \lim_{k\to\infty} \int \phi(b_1 b_2 \dots b_{n_k} v) d\nu(v) \\
&= \lim_{k\to\infty} \int \mathbf{1}_{W_b}(v) \phi(b_1 b_2 \dots b_{n_k} v) d\nu(v) \\
&\leq \nu(W_b) \sup_{v\in W} |\phi(v)|.
\end{aligned}$$

Since ϕ is an arbitrary continuous function on W with compact support, by taking a sequence of such ϕ supported on balls of radius $n \to \infty$ and takes value 1 within a slightly smaller open ball, we can conclude that $\nu(W_b) = 1$. Since W_b is closed, we have supp $\nu \subset W_b$.

Since W_b is a subspace of W and supp ν spans W, we have $\ker b_\infty = W_b = W$, i.e. b_∞ is the zero map. But this is a contradiction since $\|b_\infty\|_{\operatorname{End}(W)} = 1$. □

We shall show the following important auxiliary proposition.

Proposition 5.2. Let μ be a Borel probability measure μ on $G = GL(V)$ with finite first moment. Suppose there exists a μ-stationary measure ν on V such that V is the span of supp ν. Suppose there exist Γ_μ-invariant subspaces $0 \subset W' \subset W \subsetneq V$ such that

(i) Γ_μ acts compactly on W';

(ii) if $W' \neq W$, μ has negative top exponent on W/W';

(iii) Γ_μ acts compactly on V/W.

Then there is a Γ_μ-invariant splitting of V:

$$V = W' \oplus W''$$

for some Γ_μ-invariant subspace $W'' \subset V$.

We first prove a lemma which allows us to reduce the proposition to the case when the acting group Γ_μ is uniformly bounded from above in $GL(V)$.

Lemma 5.3. Under the assumptions of Proposition 5.2, if in addition, Γ_μ is unbounded from above with respect to the operator norm on $GL(V)$, i.e. there exists a sequence $\{g_k\} \subset \Gamma_\mu$ such that $\|g_k\|_{GL(V)} \to \infty$, then there is a nonzero Γ_μ-invariant subspace $W_0 \subset W$ such that

$$W' \cap W_0 = 0.$$

Proof. The proof is similar to that of Lemma 5.1(ii). By Lemma 5.1(ii), for $\mu^{\mathbb{N}}$-almost every word $b \in G^{\mathbb{N}}$, the sequence

$$\{b_1 b_2 \dots b_n \mid n \geq 1\}$$

is bounded from above in $GL(V)$. Let b_∞ be a limit point of this sequence in $\mathrm{End}(V)$. Moreover, by Lemma 1.5, for all $g \in \Gamma_\mu$ and each positive integer k, we have

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \dots b_n g g_k)_* \nu = (b_\infty g g_k)_* \nu.$$

Let g_∞ be a limit point of the sequence $\{g_k/\|g_k\|\}_k$ in $\mathrm{End}(V)$. Then by the same argument as the proof of Lemma 5.1(ii), using the fact that $\|g_k\| \to \infty$, one can conclude that

$$b_\infty g g_\infty \equiv 0,$$

the zero map on V. Hence for all $g \in \Gamma_\mu$,

$$g g_\infty V \subset \ker b_\infty.$$

Let W_0 be the span of $\{g g_\infty V \mid g \in \Gamma_\mu\}$. Then $W_0 \subset \ker b_\infty$. Since $\|g_\infty\| = 1$, $g_\infty V$ is nonzero, so W_0 is a nonzero Γ_μ-invariant subspace of V. Moreover, since Γ_μ acts compactly on W' and b_∞ is in the closure of Γ_μ in $\mathrm{End}(V)$, $W' \cap \ker b_\infty = 0$.

On the other hand, we claim that $\ker b_\infty \subset W$. In fact, for $v \notin W$, since $b_\infty \in \Gamma_\mu$ acts compactly on V/W, we have $b_\infty v \notin W$, in particular $b_\infty v \neq 0$, so $v \notin \ker b_\infty$.

Now since $W_0 \subset \ker b_\infty$, we have that $W_0 \subset W$ and $W' \cap W_0 = 0$, as desired. $\square$

We also need an algebraic fact about compact subsemigroups of $\mathrm{End}(V)$.

Lemma 5.4. [HM, A.1.22] Let $\mathcal{S} \subset \mathrm{End}(V)$ be a nonempty compact subsemigroup. Then there exists $h \in \mathcal{S}$ such that

(a) h is idempotent, i.e. $h^2 = h$,

(b) $h\mathcal{S}h := \{hgh \mid g \in \mathcal{S}\}$ has the structure of a compact group with identity element h,

(c) there is a group action by $h\mathcal{S}h$ on hV.

For completeness we include a sketch of the proof here.

Sketch of Proof. Let r be the smallest rank among elements in $\mathcal{S}$, and let $\mathcal{S}_0 := \{g \in S \mid \operatorname{rank}(g) = r\}$. Then $\mathcal{S}_0$ is itself a compact subsemigroup of $\operatorname{End}(V)$ since the rank cannot increase when taking products and limits. By Ellis-Numakura lemma ([HM, A.1.16]), any nonempty compact semigroup has an idempotent element, so there exists $h \in \mathcal{S}_0$ with $h^2 = h$. Then $h\mathcal{S}h$ is a compact semigroup with h acting as the identity element.

We claim that h is the only idempotent element in $h\mathcal{S}h$. In fact let h' be another idempotent element in $h\mathcal{S}h$. Then the image of h' is contained in the image of h. But h has minimal rank in $\mathcal{S}$ and $h\mathcal{S}h$ is contained in $\mathcal{S}$, so the images of h and h' are the same. Moreover, since h and h' are idempotents in $\operatorname{End}(V)$, we have the decompositions

$$V = \operatorname{im} h \oplus \ker h = \operatorname{im} h' \oplus \ker h'.$$

Since $h' \in h\mathcal{S}h$, $\ker h \subset \ker h'$. But since $\operatorname{im} h = \operatorname{im} h'$, the dimensions of $\ker h$ and $\ker h'$ agree, so $\ker h = \ker h'$. Any idempotent in $\operatorname{End}(V)$ is determined by its image and kernel, so $h = h'$.

On the other hand, one can check that if a compact semigroup K with identity has no other idempotent, then it is a compact group. In fact, for any $t \in K$, tK and Kt are nonempty compact subsemigroups of K, so they also have idempotent elements. But by assumption, this idempotent element must be the identity, so t has left and right inverses for all $t \in K$, as desired.

Thus we have shown that $K = h\mathcal{S}h$ is a compact group with identity h. $h\mathcal{S}h$ acts on hV since the identity element h acts trivially on hV. □

Now we are ready to prove Proposition 5.2.

Proof of Proposition 5.2. We prove the statement by induction on $\dim V$.

Base case: $\dim V = 1$**.**
Since W is a proper subspace of V, we have $W' = W = 0$. Therefore we can take $W'' = V$.

Induction step.
If Γ_μ is unbounded from above in $GL(V)$, by Lemma 5.3, there exists a nonzero Γ_μ-invariant subspace $W_0 \subset W$ with $W' \cap W_0 = 0$. Now consider the Γ_μ-invariant flag

$$0 \subsetneqq W' \subset W/W_0 \subsetneqq V/W_0.$$

Since W_0 is nonzero, $\dim V/W_0 < \dim V$, so by the induction hypothesis, there exists a Γ_μ-invariant subspace $W_2 \subset V$ with $W_0 \subset W_2$ such that there is the Γ_μ-invariant splitting

$$V/W_0 = W' \oplus W_2/W_0.$$

Thus we can take $W'' = W_2$.

Hence in the remaining part of the proof we assume also that there exists $C > 0$ such that $\|g\| \leq C$ for all $g \in \Gamma_\mu$. Let $\overline{\Gamma_\mu}$ be the (topological) closure of Γ_μ in $\operatorname{End}(V)$, then $\overline{\Gamma_\mu}$ is a compact semigroup in $\operatorname{End}(V)$. By Lemma 5.4, there exists an idempotent $h \in \overline{\Gamma_\mu}$ (i.e. $h^2 = h$) such that

$$K := h\overline{\Gamma_\mu}h$$

is a compact group with identity h. Moreover K acts on hV, and preserves W' (note that $hW' = W'$ since Γ_μ acts compactly on W'). Since K is compact, there exists a K-invariant complementary subspace $W_1 \subset hV$ of W', i.e.

$$hV = W' \oplus W_1.$$

Note that $hW_1 = W_1$ since $h \in K$. Now let W'' be the span of $\{ghW_1 \mid g \in \overline{\Gamma_\mu}\}$. Then W'' is Γ_μ-invariant.

Let $v \in W'' \cap W'$. On one hand, $hv \in hW' = W'$, on the other hand,

$$hv \in \operatorname{span}(\{hghW_1 \mid g \in \overline{\Gamma_\mu}\}) = W_1$$

since $hgh \in K$ for $g \in \overline{\Gamma_\mu}$ and W_1 is K-invariant. Thus $hv \in W' \cap W_1 = 0$, i.e. $v \in \ker h$. Now since Γ_μ acts compactly on W', $\ker h \cap W' = 0$. But $v \in \ker h \cap W'$, so $v = 0$. Therefore $W'' \cap W' = 0$.

Hence we have found a Γ_μ-invariant subspace W'' with trivial intersection with W'. It remains to show that $W'' + W' = V$.

We first observe that $\ker h \subset W$. In fact, consider $v \notin W$. Since h acts compactly on V/W, $hv \neq 0$ in V/W, so $hv \neq 0$ in V, thus $v \notin \ker h$.

Since h is idempotent, we have that

$$V = \mathrm{im}\, h \oplus \ker h = W' \oplus W_1 \oplus \ker h.$$

Since $W_1 \subset W''$ and $W' \oplus \ker h \subset W$, we have

$$V = W'' + W.$$

If $W' = W$, we are done. If $W' \neq W$, by assumption, μ has negative top exponent on W/W'. Now

$$V/W'' = (W'' + W)/W'' = W/(W'' \cap W).$$

Since W' is Γ_μ-invariant, $W' \subset W$ and $W' \cap W'' = 0$, we have the following Γ_μ-equivariant identification

$$V/(W'' \oplus W') = W/((W'' \cap W) \oplus W') = (W/W')/(W'' \cap (W/W')).$$

Since μ has negative top exponent on W/W', it also has negative top exponent on $(W/W')/(W'' \cap (W/W'))$, thus on $V/(W'' \oplus W')$. Therefore the only μ-stationary measure on $V/(W'' \oplus W')$ is δ_0. On the other hand, since ν is a μ-stationary measure on V with $\mathrm{span}(\mathrm{supp}\ \nu) = V$, the pushforward of ν on $V/(W'' \oplus W')$ also spans. But this pushforward is μ-stationary on $V/(W'' \oplus W')$, so it equals δ_0. Therefore $V = W'' \oplus W'$, as desired. $\square$

Now we are ready to prove that the μ-action on the span of the support of a stationary measure is contracting-by-compact.

Proposition 5.5. [Bou, Thm. 5.1 necessity direction] Let μ be a Borel probability measure μ on $G = GL(V)$ with finite first moment, and ν be a nontrivial μ-stationary measure on V. Let W be the linear span of $\mathrm{supp}\ \nu$. Then there exists a Γ_μ-invariant proper subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Proof. We prove this by induction on $\dim W$.

Base case: $\dim W = 1$.
In this case, Γ_μ acts irreducibly on W. By Proposition 3.1, Γ_μ acts compactly on W and we can take $W' = 0$.

Induction step.
If Γ_μ acts irreducibly on W, then again by Proposition 3.1, Γ_μ acts compactly on W and we can take $W' = 0$.

If Γ_μ does not act irreducibly on W, take a minimal nonzero Γ_μ-invariant proper subspace $0 \subsetneq W_0 \subsetneq W$. The pushforward of ν under the map $W \to W/W_0$ is a stationary measure on W/W_0 whose support spans W/W_0. Since $\dim W/W_0 < \dim W$, by the induction hypothesis, we know that there exists a Γ_μ-invariant proper subspace $W_1 \subsetneq W$ such that

(i) $0 \subsetneq W_0 \subset W_1 \subsetneq W$,

(ii) Γ_μ acts compactly on W/W_1, and

(iii) either $W_1 = W_0$ or μ has negative top exponent on W_1/W_0.

By minimality of W_0, we know that Γ_μ acts irreducibly on W_0. Since W is the linear span of $\mathrm{supp}\ \nu$, by Lemma 5.1, for $\mu^{\mathbb{N}}$-almost every word $b \in G^{\mathbb{N}}$, the sequence $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ is bounded from above in $GL(W)$. By Lemma 2.4, $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ is also bounded from above in $GL(W_0)$. Thus by Corollary 3.3, either μ has negative top exponent on W_0 or Γ_μ acts compactly on W_0.

Case 1: μ has negative top exponent on W_0.
We claim that in this case, μ has negative top exponent on W_1. The claim is clear if $W_1 = W_0$. If $W_0 \subsetneq W_1$,

since μ has negative top exponent on W_1/W_0, by Lemma 2.3, μ also has negative top exponent on W_1. Thus we can take $W' = W_1$.

Case 2: μ acts compactly on W_0.
In this case, by Proposition 5.2, there exists a proper Γ_μ-invariant subspace $W_2 \subsetneq W$ such that

$$W = W_0 \oplus W_2.$$

Let $W_2' := W_1 \cap W_2$. Then we can Γ_μ-equivariantly identify W_2' and W_1/W_0. Thus either $W_2' = 0$ or μ has negative top exponent on W_2', and Γ_μ acts compactly W_2/W_2'. Moreover, since

$$W/W_2' = W_0 \oplus W_2/W_2',$$

and Γ_μ acts compactly on W_0 and W_2/W_2', we have that Γ_μ acts compactly on W/W_2'. Therefore we can take $W' = W_2'$.

□

6 Proofs of the main theorems

Using properties proved in the previous two sections, we can now prove the main theorems.

Proof of Theorem 1.1. Let $W \subset V$ be the Γ_μ-invariant subspace of maximal dimension such that $W = \mathrm{span}(\mathrm{supp}\ \nu_0)$ for some μ-stationary measure ν_0 on V.

We now claim that every μ-stationary measure ν satisfies $\mathrm{supp}\ \nu \subset W$. In fact, assume that there is some stationary measure ν' such that $\mathrm{supp}\ \nu' \not\subset W$. Let $U = \mathrm{span}(\mathrm{supp}\ \nu')$. Now let $\nu'' = \frac{1}{2}\nu + \frac{1}{2}\nu'$. Then $W + U = \mathrm{span}(\mathrm{supp}\ \nu'')$. Since $W + U$ has strictly larger dimension than W, this contradicts the maximality of $\dim W$, hence condition (i) in the theorem holds.

By Proposition 5.5, there exists a Γ_μ-invariant proper subspace $W' \subsetneq W$ such that Γ_μ acts compactly on W/W', and if $W' \neq 0$, μ has negative top exponent on W'. Thus (ii) and (iii) in the theorem hold. □

Proof of Theorem 1.2. Let $\pi : W \to W/W'$ be the quotient map. By Theorem 1.1 and Proposition 2.1, the map

$$\Phi : \{\text{ergodic } \mu\text{-stationary measure on } V\} \quad \to \quad \{\text{compact } \Gamma_\mu\text{-orbit in } W/W'\},$$

defined by $\Phi(\nu) := \mathrm{supp}\ \pi_*\nu$ is well-defined.

- **Φ is injective.**
  This follows from Proposition 4.3 and the uniqueness of the Γ_μ-invariant measure supported on a single compact Γ_μ-orbit.
- **Φ is surjective and determine Φ^{-1}**
  The origin 0 of W/W' is a compact invariant subset of W/W', and is the image of the invariant measure δ_0 on V. Now given a compact Γ_μ-invariant subset $\mathcal{C} \neq \{0\}$ in W/W', let $x \in \pi^{-1}(\mathcal{C}) \subset W \setminus W'$. By Proposition 4.6, $\nu_x = \lim_{n\to\infty} \mu^{(n)} * (s_* m_x)$ is a μ-stationary probability measure on V such that $\mathrm{supp}\ \pi_*\nu_x$ is $\mathcal{C}$, where as we recall, $s : W/W' \to W$ is any linear section and m_x is the unique Γ_μ-invariant measure supported on $\mathcal{C}$. Thus $\mathcal{C} \mapsto \nu_x$ is the inverse of Φ.

□

Proof of Theorem 1.3. The first claim was proved in Proposition 4.6. The second claim was shown in the proof of Theorem 1.2. □

Proof of Theorem 1.4. The convergence of the limit in the first claim was shown in Proposition 4.5. That the limiting measure is $\nu_{\mathcal{C}}$ follows from the injectivity of Φ in Theorem 1.2. The second claim is true since by Breiman's law of large number [Br], for $\mu^{\mathbb{N}}$-almost every word $b \in G^{\mathbb{N}}$, every weak-* limit point of the empirical measures is a μ-stationary probability measure. Now the rest follows from the same argument as Proposition 4.5 and the injectivity of Φ in Theorem 1.2. □

Proof of Theorem 1.6. This follows from Proposition 4.4. □

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