LECTURES ON MIXED MOTIVES

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0. Introduction

0.1. What follows are notes on lectures I gave on mixed motives at the AMS summer conference in Santa Cruz. My experience with these lectures suggests that motives are like onions; they are complicated, multi-layered objects, and any attempt to cut too quickly to the heart of the matter can leave the audience in tears. A rigorous construction of **The category of motives** seems to founder on the rocks of Grothendieck's currently inaccessible "standard conjectures" about algebraic cycles [29]. (For surprising progress, however, see [28].) Instead of precision and realism, we have to settle for a sort of cubist approach which attempts to see the subject from many different viewpoints simultaneously. Here is a casual list of some properties one expects for the category of motives. For a more careful discussion, the reader should see [35], [30], and [21]:

0.1.1. Algebraic varieties give motives: $X \mapsto h(X)$. More precisely, one fixes a ground field F and one deals with motives associated to varieties over F. The category of motives is a $\mathbb{Q}$-linear abelian tensor category, and

$$h(X \times Y) = h(X) \otimes h(Y)$$

0.1.2. When X and Y are smooth and projective, a morphism $h(X) \to h(Y)$ is given by an algebraic cycle on $X \times Y$ of dimension $\dim Y$. Rationally equivalent cycles give the same morphism. The question of what the appropriate equivalence relation on cycles is to get a good category of motives is subtle, and we shall avoid it. Roughly speaking, one can take a fine equivalence relation like rational equivalence, leading to the theory of Chow motives, or one can take a coarser relation like homological or numerical equivalence. Whatever equivalence relation is chosen, one introduces for X smooth, projective, and p on $X \times X$ a projector (i.e. a cycle satisfying $p^2 = p$ in the correspondence ring),

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a formal image motive $h(X,p)$. In particular, one of the standard conjectures (known for varieties over a finite field, [29]) says the diagonal Δ_X for a smooth, projective variety X is homologically equivalent to a sum of projectors (Kunneth decomposition). The resulting motivic decomposition is written

$$h(X) = \oplus_{i=0}^{2\dim X} h^i(X)$$

0.1.3. The motive of a point $h^0(pt.)$ is written $\mathbb{Q}(0)$, and the Tate motive $\mathbb{Q}(-1)$ is defined to be $h^2(\mathbb{P}^1)$. In other words, $\mathbb{Q}(-1) = h(\mathbb{P}^1,\ \mathbb{P}^1 \times pt.)$ One introduces a formal tensor inverse $\mathbb{Q}(1)$ such that $\mathbb{Q}(-1) \otimes \mathbb{Q}(1) \cong \mathbb{Q}(0)$. One then has $\mathbb{Q}(n)$ defined for all $n \in \mathbb{Z}$ by tensoring. We write

$$h(X)(n) := h(X) \otimes \mathbb{Q}(n).$$

0.1.4. Motives have realizations like H_{Betti} (Betti realization), H_{DR} (de Rham realization), $H_{\text{ét}}$ (étale realization), and H_{crys} (H_{cris} in French) (Crystalline (French cristalline) realization). Realizations are functors from the category of motives to a category of vector spaces.

0.1.5. For X smooth, projective over a field F, $h^i(X)$ is a **pure motive** of **weight** i. The realization functors preserve weights. The category of pure motives is conjectured to be semi-simple (see [28]). More generally, one expects a larger abelian category of **mixed motives**. Mixed motives M have increasing weight filtrations $W_\bullet M$ which are preserved by the realization functors. The category of mixed motives itself should be the core (heart) of a larger triangulated category $\mathcal{D}(F)$, the **derived category of mixed motives** over F.

0.1.6. More generally, for any base scheme S, one should define the derived category $\mathcal{D}(S)$ of motivic sheaves over S. As in the case of sheaves, given $p : X \to S$, one expects various functors relating $\mathcal{D}(X)$ and $\mathcal{D}(S)$. For $p : X \to \mathrm{Spec}\,(F)$ smooth and projective, we should have (cf. [16])

$$\mathbf{R}p_*\mathbb{Q}(0)_X \cong \bigoplus h^i(X)[-i].$$

Examples of motivic sheaves are the polylogarithm sheaves in [4] and [5], and the elliptic polylogarithm sheaves in [6]. It would be very interesting to understand the relationship between the Chow polylogarithms of Goncharov, to be discussed below, and motivic sheaves on the Chow variety.

0.1.7. The category of **mixed Tate motives** or mixed Tate motivic sheaves is the full subcategory of motives over S generated by the $\mathbb{Q}(n)_S$. A very beautiful observation of Beilinson is that the functor $M \mapsto \oplus\mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{-2n}M)$ is a fibre functor from the category of mixed Tate motives to the category of $\mathbb{Z}$-graded vector spaces. This means that one may hope to write the category of mixed Tate motives as the category of finite dimensional graded representations of a graded Lie algebra. One candidate for such a Lie algebra will be discussed below in section 2. One may hope also that this point of view will lead to new insights about motives, just as galois groups give a powerful tool to understand number fields.

0.1.8. Let X be a smooth, projective variety over a field F, and let Z be a codimension r algebraic cycle on X. The pullback $Z\times(\mathbb{P}^1)^r$ on $X\times(\mathbb{P}^1)^r$ has dimension equal $\dim X$ and so induces a morphism of motives. It is straightforward to check this factors through the projector defining $h^2(\mathbb{P}^1)$, so Z defines a map $\mathbb{Q}(-r) \to h(X)$. Applying a Kunneth projector leads to a cycle class in $\mathrm{Hom}(\mathbb{Q}, h^{2r}(X)(r))$. With notation as in 0.1.6, let $CH^r(X)$ denote the Chow group of codimension r algebraic cycles modulo rational equivalence. One expects a map (isomorphism?)

$$CH^r(X)\otimes\mathbb{Q} \to \mathrm{Hom}_{\mathcal{D}(F)}(\mathbb{Q}, \mathbf{R}p_*\mathbb{Q}(r)_X[2r])$$

and hence a filtration $F^*CH^r(X)\otimes\mathbb{Q}$ with

$$\mathrm{gr}^i_F CH^r(X)\otimes\mathbb{Q} \overset{?}{\cong} \mathrm{Ext}^i_{\mathcal{D}(F)}(\mathbb{Q}, h^{2r-i}(X)(r))$$

For results on this filtration cf. [36], [28].

Algebraic cycles also give classes in other Ext groups. Namely, we may introduce "simplices" $\Delta^n := \mathrm{Spec}\left(F[t_0,\dots,t_n]/(\sum t_i - 1)\right)$ and consider cycles of codimension r on $X\times\Delta^n$ meeting all faces properly. One has a cosimplicial object $\Delta^{\bullet}: \ \Delta^0 \rightrightarrows \Delta^1 \cdots$ and hence by pullback of cycles a simplicial abelian group $\mathcal{Z}^r(X,\cdot)$ of codimension r cycles on $X\times\Delta^{\bullet}$. One defines the higher Chow groups $CH^r(X,\cdot) := H_{\bullet}(\mathcal{Z}^r(X,\cdot))$. Known candidates for the derived category of motives include cycle maps

$$CH^r(X,n) \to \mathrm{Ext}^{2r-n}_{\mathcal{D}(F)}(\mathbb{Q}(0), \mathbf{R}p_*\mathbb{Q}_X(r))$$

If we define the **motivic cohomology**

$$(0.1.8.1)\quad H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) := Ext^i_{\mathcal{D}(X)}(\mathbb{Q}(0)_X, \mathbb{Q}(j)_X) \cong Ext^i_{\mathcal{D}(F)}(\mathbb{Q}(0)_F, \mathbf{R}p_*(\mathbb{Q}(j)_X))$$

we obtain maps

$$CH^r(X,n)\otimes\mathbb{Q}\to H^{2r-n}_{\mathcal{M}}(X,\mathbb{Q}(r)) \tag{0.1.8.2}$$

which one may hope are isomorphisms.

For motives M and N over a number field F, one expects

$$\mathrm{Ext}^i_{\mathcal{D}(F)}(M,N)=(0),\ \ i\geq 2.$$

Since $\mathrm{Hom}(\mathbb{Q},h^{2r-n}(X)(r))=(0)$ for $n\geq 1$ because of the weights, we conclude in the number field case

$$H^{2r-n}_{\mathcal{M}}(X,\mathbb{Q}(r))\cong \mathrm{Ext}^1(\mathbb{Q},h^{2r-n-1}(X)(r)). \tag{0.1.8.3}$$

This suggests a natural interpretation of the map 0.1.8.2. Let Z be a codimension r cycle on $X\times\Delta^n$ representing a class in $CH^r(X,n)$. Let $\partial\subset\Delta^n$ be the union of the faces $t_i=0$. Mirroring the situation for étale or Betti realizations one may imagine a Gysin sequence of motives

(0.1.8.4)

$$\begin{aligned} 0\to h^{2r-1}(X\times\Delta^n,X\times\partial))(r)\to h^{2r-1}(X\times\Delta^n-|Z|,X\times\partial))(r)\\ \to h_{2\dim(Z)}(|Z|,|Z|\cap\partial)(-\dim(Z))\xrightarrow{a}h^{2r}(X\times\Delta^n,X\times\partial))(r)\end{aligned}$$

Because the cycle Z pulls back to zero on $X\times\Delta^{n-1}$, we should get a cycle map $[Z]:\mathbb{Q}(0)\to h_{2\dim(Z)}(|Z|,|Z|\cap\partial)(-\dim(Z))$. For weight reasons, the composition $a\circ[Z]=0$, so we may pull back 0.1.8.4 along $[Z]$. If we assume (again thinking of the familiar realizations) $h^{2r-1}(X\times\Delta^n,X\times\partial))(r)\cong h^{2r-n-1}(X)(r)$ we get an extension

$$0\to h^{2r-n-1}(X)(r)\to E_Z\to\mathbb{Q}(0)\to 0 \tag{0.1.8.5}$$

realizing the class 0.1.8.3 associated to Z.

0.1.9. Of course, motives are related to arithmetic in many ways. Classically, the motive of a variety over a finite field determines the characteristic polynomial of Frobenius acting on cohomology, which in turn gives the zeta function. From the point of view of mixed motives and motivic cohomology, which is central to these lectures, an important link is provided by the Beilinson conjectures. Let X be smooth and projective over a number field, and let $\mathfrak{X}$ be a regular model of X. Fix integers $r,n\geq 0$ with $n+1-2r\leq -2$. The Beilinson conjectures equate the rank of $H^{n+1}_{\mathcal{M}}(\mathfrak{X},\mathbb{Q}(r))$ with the order of vanishing of the Hasse-Weil L-function $L(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell),s)$ at the point $s=n+1-r$. (The actual conjectures are much more precise. They also determine the order of vanishing for weights $n+1-2r=-1,0$, and they relate the first non-vanishing term in the Taylor expansion of L to the determinant of a regulator map on motivic cohomology.)

0.1.10. Finally, although we have treated the categories of motives as $\mathbb{Q}$-linear categories, the fact that cycles and Chow groups are integrally defined suggests one may consider motives with $\mathbb{Z}/n\mathbb{Z}$-coefficients.A good portion of this paper will be devoted to an exposition of the Suslin-Voevodsky theorem relating cycles mod n to étale cohomology.

0.2. The content of these lectures is as follows (by section):

1. The derived category of motives over a field.
2. The Tannakian category of mixed Tate motives over a field.
3. Motives with $\mathbb{Z}/n\mathbb{Z}$-coefficients and étale cohomology.
4. Chow polylogarithms.

1. The derived category of mixed motives

1.1. My purpose in this section is to expose Voevodsky's work constructing a candidate for the derived category of mixed motives over a field. The references are [44],[45], and [46]. They, as well as a number of other relevant papers by Voevodsky are currently available from the K-theory web site maintained by Dan Grayson at UIUC. Competing theories by Hanamura and Levine were not as well documented in the literature at the time of the lectures, though now details of both Levine's and Hanamura's work are available ([32],[33], [23], [24]). It seems plausible that all three candidates are equivalent, but this has not been checked. Note that the construction of the derived category does not depend on any conjectures. Even in the case of mixed Tate motives, the existence of an abelian subcategory forming a core for the derived category seems to depend on a conjecture of Beilinson and Soulé [1],[39] which basically says there is no motivic cohomology in negative degrees. In terms of Chow groups, the conjecture says that for X smooth and complete over a field,

$$CH^p(X, q) \otimes \mathbb{Q} = (0), \ 2p - q \leq 0. \tag{1.1.0.1}$$

The relation between this and motivic cohomology is give by 0.1.8.2.

1.2. Voevodsky's constructions involve a considerable amount of abstract category theory. Although it will not be possible to give detailed justifications in this survey, I will try when I can to give references in the literature where the reader can find more detail on these points.

1.2.1. Let k be a perfect field. We first define an additive category $SmCor(k)$ by taking as objects smooth schemes of finite type over k and as morphisms $X \to Y$ the free abelian group on closed subvarieties $Z \subset X \times Y$ such that $pr_1 : Z \to X$ is finite and surjective onto a connected component of X. Roughly speaking, we may think of

$\mathrm{Hom}(X, Y)$ as the group associated to the monoid of morphisms of schemes

$$X \to \coprod_n \mathrm{Sym}^n(Y).$$

(When X is non-normal, this is not correct. For a precise statement, cf. [41].) Note for $U \to V$ a morphism of smooth schemes, the graph provides a morphism $U \to V$ in $SmCor(k)$.

Remark 1.2.1. *The alert reader will have remarked that the variance here is the opposit of that in the introduction; i.e. these are* **homological** *motives.*

1.2.2. We define $\mathcal{H}^b(SmCor(k))$ to be the homotopy category of bounded complexes of objects in $SmCor(k)$. Recall a homotopy of complexes $h : C \to D$ is a collection of maps $h^n : C^n \to D^{n-1}$. Maps

$$d_D \circ h^n - h^{n+1} \circ d_C : C \to D$$

for a homotopy h are said to be homotopic to zero. The homotopy category of complexes of objects in an additive category is obtained from the category of complexes by identifying morphisms whose difference is homotopic to zero, [27].

A triangulated category is a category $\mathcal{C}$ with an invertible shift operator $C \mapsto C[1]$ (inverse $C \mapsto C[-1]$) and a collection of diagrams of the form $A \to B \to C \to A[1]$ (distinguished triangles) satisfying certain axioms (op. cit.), the most important of which imply that for an object D of $\mathcal{C}$ and a distinguished triangle as above, one gets long exact sequences of abelian groups

$$\ldots \mathrm{Hom}(D, C[-1]) \to \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, A[1]) \to \ldots$$

$$\ldots \mathrm{Hom}(A[1], D) \to \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(C[-1], D) \ldots$$

The standard example is the homotopy category of complexes (possibly with some boundedness condition) over an additive category. $C[1]$ denotes the complex C shifted to the left one step, and distinguished triangles are diagrams isomorphic to a diagram of the form

$$C \xrightarrow{f} D \to \mathrm{Cone}(f) \to C[1].$$

Here $\mathrm{Cone}(f)$ denotes the complex

$$\mathrm{Cone}^n := D^n \oplus C^{n+1}$$

with differential $\partial : \mathrm{Cone}^{n-1} \to \mathrm{Cone}^n$ given by the matrix

$$\begin{pmatrix} -d_C^n & f^n \\ 0 & d_D^{n-1} \end{pmatrix}.$$

1.2.3. The next step is to impose on $\mathcal{H}^b(SmCor(k))$ some basic isomorphisms and long-exact sequences. Voevodsky does this by inverting certain morphisms in $\mathcal{H}^b(SmCor(k))$. For X over k we write $[X]$ for the corresponding object in $\mathcal{H}^b(SmCor(k))$. He first inverts all morphisms $[X \times \mathbb{A}^1] \to [X]$ induced by projections. Next, given an open covering $W = U \cup V$, one obtains in an obvious way a complex

$$[U \cap V] \xrightarrow{f} [U] \oplus [V] \xrightarrow{g} [W],$$

and hence a morphism $\gamma : \mathrm{Cone}(f) \to [W]$. Inverting γ forces distinguished triangles

$$[U \cap V] \xrightarrow{f} [U] \oplus [V] \xrightarrow{g} [W] \to [U \cap V][1].$$

The actual business of localizing categories with respect to sets of morphisms is quite technical. We refer the reader to [27] and [37] for more information. The category $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is defined to be the Karoubian envelope of this localized category, i.e. $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is obtained from the localized category by formally adjoining kernels and images of projectors (morphisms p such that $p^2 = p$).

1.2.4. The association $X \mapsto [X]$ defines a functor from the category of smooth varieties over k to $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. The latter category has a tensor structure, and $[X \times Y] = [X] \otimes [Y]$. One defines the Tate object $\mathbb{Z}(1)$ to be the complex $[\mathbb{P}_k^1] \to [\mathrm{Spec}\,(k)]$ placed in degrees 2 and 3. The category $DM_{\mathrm{gm}}(k)$ is defined by formally inverting the functor $? \otimes \mathbb{Z}(1)$ on $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Thus, $DM_{\mathrm{gm}}(k)$ contains objects $\mathbb{Z}(n)$ for all $n \in \mathbb{Z}$. To show we have not totally lost the way, one has for X and Y smooth projective, and assuming varieties over k admit resolutions of singularities, that

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}([X], [Y]) \cong CH_{\dim X}(X \times Y).$$

In fact, assuming resolution of singularities the link between $DM_{\mathrm{gm}}(k)$ and algebraic cycles is exactly what one would expect. For example, the higher Chow groups define the Borel-Moore homology of the theory. That is, for X a variety of finite type which is equidimensional of dimension n, one defines a compactified object $[X]^c$ in $DM_{\mathrm{gm}}(k)$, and

$$\mathrm{Hom}(\mathbb{Z}(i)[j], [X]^c) \cong CH^{n-i}(X, j - 2i).$$

There are similar links to Suslin homology [42], motivic cohomology [7], and bivariant cycle homology [17].

1.2.5. Sheaf-theoretic methods play an important role in understanding the structure of $DM_{\mathrm{gm}}(k)$. For X a smooth variety over k, let $L(X)$ be the functor of points in $SmCor(k)$; i.e. for Y irreducible, $L(X)(Y)$ is the free abelian group on closed subvarieties $Z \subset Y \times X$ finite surjective over Y. Recall the Nisnevich topology on a variety X has as coverings étale coverings $U \to X$ such that for every field K, $U(K) \twoheadrightarrow X(K)$. $L(X)$ is a sheaf for any reasonable topology, but a crucial property of the Nisnevich topology (also true for the étale topology, but false for the Zariski topology) is the following ([46], prop. 3.1.3):

Proposition 1.2.2. *Let $U \to X$ be a Nisnevich covering. Then the sequence of sheaves of abelian groups*

$$\ldots \to L(U \times_X U) \to L(U) \to L(X) \to 0$$

is exact.

A contravariant functor from $SmCor(k)$ to abelian groups is called a presheaf with transfers. The subcategory of preseaves with transfers whose underlying presheaf is a Nisnevich sheaf is denoted

$$\mathrm{Shv}_{\mathrm{Nis}}(SmCor)(k).$$

For example, $L(X)$ is an object in $\mathrm{Shv}_{\mathrm{Nis}}(SmCor)(k)$. Let

$$\mathcal{D} = \mathcal{D}^{-}(\mathrm{Shv}_{\mathrm{Nis}}(SmCor)(k))$$

denote the derived category of bounded above complexes. We define a functor $\underline{C}_n$ on presheaves with transfer by

$$\underline{C}_n(F)(X) := F(\Delta^n \times X),$$

where Δ^n is as in 0.1.8. This extends in an obvious way to a functor to complexes: $F \mapsto \underline{C}_{\bullet}(F)$ and then to a functor

$$\mathbf{R}\underline{C}_{\bullet} : \mathcal{D} \to \mathcal{D}.$$

A presheaf with transfers F is said to be homotopy invariant if the morphism $X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$F(X) \to F(X \times \mathbb{A}^1).$$

DM^{eff} is defined to be the full subcategory of $\mathcal{D}$ of objects whose homology sheaves are homotopy invariant. One shows that the functor $\mathbf{R}\underline{C}_{\bullet}$ factors through $DM^{\mathrm{eff}} \subset \mathcal{D}$, and one gets a diagram

$$\begin{array}{ccc} \mathcal{H}^b(SmCor(k)) & \xrightarrow{L} & \mathcal{D} \\ \downarrow & & \downarrow \mathbf{R}\underline{C} \\ DM^{\mathrm{eff}}_{\mathrm{gm}}(k) & \overset{i}{\dashrightarrow} & DM^{\mathrm{eff}}(k) \end{array}$$

The crucial point is the existence of the dashed arrow i. This embeds the motive category $DM^{\text{eff}}_{\text{gm}}(k)$ (which should be thought of as very mysterious) as a full subcategory of the derived category of bounded above Nisnevich sheaves with transfer and homotopy invariant cohomology. What's more, the image of i is big, in the sense that given any object $A \in DM^{\text{eff}}(k)$ there exists a distinguished triangle $A \to B \to C \to A[1]$ such that B and C are direct sums of objects in the image of i.

For X a smooth variety over k, the composition $\underline{C}^{\bullet}(X) := \mathbf{R}C \circ L(X)$ is the Suslin complex [42]. One has for X and Y smooth schemes over k,

$$\mathrm{Hom}_{DM^{\text{eff}}_{\text{gm}}(k)}([X], [Y][p]) \cong H^p_{\text{Nis}}(X, \underline{C}^{\bullet}(Y)).$$

2. The Tannakian Category of Mixed Tate Motives

2.1. In this section, I want to describe a program to construct the abelian category of mixed Tate motives over a field k. The work in question (in large measure joint with I. Kriz) is published in [9] and [11], but I should also acknowledge considerable inspiration from private conversations with A. Beilinson and P. Deligne. A general references for tannakian categories is [15].

2.1.1. What we are after is a category of what might be thought of as motivic sheaves over Spec (k). We consider only those sheaves which admit a weight filtration such that the successive quotients are (direct sums of) Tate objects $\mathbb{Q}(n)$. Our objects will have realizations, including ℓ-adic galois representations for ℓ prime to the characteristic of k, and mixed Tate Hodge structures if $k \hookrightarrow \mathbb{C}$. Presumably, when k is a function field over $\mathbb{C}$, one has a realization as a variation of mixed Tate Hodge structure, but that theory has not been worked out.

2.1.2. Our category will be the abelian category of corepresentations of a certain Hopf algebra $\mathcal{H} := H^0(\mathcal{B}(\mathfrak{N}^{\bullet}))$. Here $\mathfrak{N}^{\bullet}$ is a differential graded algebra built from algebraic cycles on $\Delta^{\bullet}_k$ and $\mathcal{B}$ denotes the bar (or cobar, depending on your point of view) construction. The fact that these corepresentations have realizations stems from the existence of algebraic cycle classes in ℓ-adic and Betti cohomology. It is very hard (we tried!) to envision a procedure for generating (open, relative,simplicial,...) varieties whose cohomology groups might give all mixed Tate motives. This seemingly chaotic problem is reinterpreted via the Tannakian formalism as a problem of constructing a pro-algebraic group over $\mathbb{Q}$. Because of the weight filtration, this pro-algebraic group will be unipotent, so the problem can be linearized.

Our Hopf algebra is essentially the coordinate ring of (our candidate for) this group.

2.1.3. This program for constructing motives can be criticized on several grounds. First, we only consider mixed Tate motives. This is not so serious. In fact, the weight grading functor from mixed motives to pure motives should have the properties of a fibre functor (basically, exactness, compatibility with $\otimes$ and faithfulness) so one may try to construct a Lie algebra or Hopf algebra in the category of pure motives whose representations give all mixed motives.

Second, and more seriously, the construction of an abelian category of motives seems to be deeper than the construction of a triangulated "derived category". We are force to assume a deep conjecture that our cycle complex $\mathfrak{N}^\cdot$ is a $K(\pi, 1)$ in the sense of Sullivan. This is necessary to insure that the motivic cohomology (which coincides with the cohomology of $\mathfrak{N}^\cdot$) agrees with the cohomology of the abelian category of corepresentations we construct.

Finally, and in some sense most serious of all, is the question whether our methods construct "all" mixed Tate motives. For example, in our category, $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q}(1)) = k^\times$, i.e. all extensions of $\mathbb{Q}$ by $\mathbb{Q}(1)$ are Kummer. However, in the real world, one has the following construction. Let V be a smooth, projective variety of dimension 3 over k. Let $A = \sum a_i A_i$ and $B = \sum b_j B_j$ be algebraic 1-cycles on V. Assume A and B are homologous to 0 and have disjoint support, $|A| \cap |B| = \emptyset$. It is an easy exercise in ℓ-adic or Betti cohomology to construct a subquotient $H^3(X - A, B)$ (depending on the coefficients a_i and b_j) of $H^3(X - |A|, |B|)$ whose weight graded pieces are $\mathbb{Q}(1)$, $H^3(V)(2)$, and $\mathbb{Q}$. It seems reasonable to interpret $H^3(X - A, B)$ as a motive. If $H^3(V) = (0)$, the middle piece disappears, and we get an extension of $\mathbb{Q}$ by $\mathbb{Q}(1)$. If either A or B is algebraically equivalent to 0, one can show this extension is Kummer in the sense that there exists $x \in k^\times$ such that the image of x in the ℓ-adic completion of $k^\times$ classifies the extension for all ℓ (resp. the image of $x \in \mathbb{C}^\times$ classifies the extension of Hodge structures if $k \subset \mathbb{C}$). However, if neither A nor B is algebraically equivalent to 0, I know of no way of proving this.

On the other hand, the Tannakian viewpoint is enormously powerful. Think of the central role of Galois groups in number theory or fundamental groups in topology!

2.2. We want to construct a category

$$\mathfrak{MTM} = \mathfrak{MTM}(k)$$

with the following properties

1. $\mathfrak{MTM}$ is an abelian category.
2. Objects in $\mathfrak{MTM}$ have finite length. The simple objects are objects $\mathbb{Q}(n)$, for $n \in \mathbb{Z}$.
3. $\mathrm{Hom}(\mathbb{Q}(m), \mathbb{Q}(n)) = \begin{cases} \mathbb{Q} & m = n \\ 0 & m \neq n \end{cases}$.
4. Objects M have functorial increasing weight filtrations $W_{\bullet}M$. Thus, $\mathbb{Q}(n)$ has weight $-2n$, and in general, $\mathrm{gr}^W_{2n}M \cong \oplus_{\mathrm{finite}}\mathbb{Q}(-n)$.

The beautiful remark of Beilinson is that, given such a category, one gets a fibre functor

$$\phi : \mathfrak{MTM} \to \{\mathbb{Z}\text{-Graded } \mathbb{Q}\text{- Vector Spaces}\}$$

by defining

$$\phi_{-n}(M) := \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{-2n}M).$$

(It would be better to double the index and call this ϕ_{-2n}, but we will follow established usage. The reader should bear in mind that this grading never involves signs.) For example if M has a two step filtration $0 \to \mathbb{Q}(1) \to M \to \mathbb{Q} \to 0$, $\phi(M) = \mathbb{Q}_{-1} \oplus \mathbb{Q}_0$.

One then deduces from Tannaka theory [2], [15] the existence of a graded pro-Lie algebra over $\mathbb{Q}$,

$$\mathcal{L} = \mathcal{L}_{-1} \oplus \mathcal{L}_{-2} \oplus \cdots$$

and an identification

$$\mathfrak{MTM} = \{\text{finite dim. graded } \mathbb{Q}\text{-representations of } \mathcal{L}\}.$$

The grading corresponds to the weights and will be referred to as the Adams grading to distinguish it from the cohomological grading.

2.3. **Examples.** We assume for a while we have the category $\mathfrak{MTM}$ and see what can be said about $\mathcal{L}$. We expect, by Kummer theory,

$$\mathrm{Ext}^1_{\mathfrak{MTM}}(\mathbb{Q}, \mathbb{Q}(1)) \cong k^\times \otimes \mathbb{Q}. \qquad (2.3.0.1)$$

Such an extension corresponds under the fibre functor as above to a representation

$$\mathcal{L} \to \mathrm{End}(\mathbb{Q}_{-1} \oplus \mathbb{Q}_0),\ \ell \mapsto \begin{pmatrix} 0 & \rho(\ell) \\ 0 & 0 \end{pmatrix}.$$

Here ρ is a homomorphism of graded Lie algebras

$$\mathcal{L} \to \mathbb{Q}_{-1} = \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}_{-1}),$$

i.e. an element of degree $+1$ in the linear dual space to $\mathcal{L}$. Note that $[\mathcal{L}, \mathcal{L}] \subset \oplus_{n \leq -2}\mathcal{L}_n$, so any element of degree $+1$ in the dual will be a homomorphism of Lie algebras with target $\mathbb{Q}$ with the trivial Lie structure. Write $\mathcal{M} := \mathcal{L}^\vee$ (dual as pro-object), so (2.3.0.1) implies

$$\mathcal{M}_1 \cong k^\times \otimes \mathbb{Q}.$$

Before moving on, it is perhaps worth saying a few words about realizations in this simple example. There is a simple dictionary between extensions of $\mathbb{Z}$ by a group object G in a category and G-torseurs. Given an extension, $0 \to G \to E \xrightarrow{\phi} \mathbb{Z} \to 0$, (the object represented by) $\phi^{-1}(1)$ is a G-torseur. Conversely, given a G-torseur T, one easily constructs an extension of $\mathbb{Z}$ such that T is identified with the inverse image of 1. Thus, to realize an extension of $\mathbb{Q}$ by $\mathbb{Q}(1)$ corresponding to an element $x \in k^\times$, it suffices to realize the corresponding torseur $T_x := \phi_x^{-1}(1)$. These realizations are the obvious ones:

$$\text{Hodge realization} \quad T_x := \{2\pi i n \log(x) \mid n \in \mathbb{Z}\}$$
$$\ell\text{-adic realization} \quad T_x := \{(x = x_1, x_2, \dots) \mid x_n^\ell = x_{n-1}\}.$$

Notice the Hodge realization is a torseur under $2\pi i\mathbb{Z}$ and the ℓ-adic realization is a torseur under $\mathbb{Z}_\ell(1)$. To obtain vector spaces, one simply tensors with $\mathbb{Q}$.

2.3.1. Consider next a 3-dimensional representation $V = \mathbb{Q}_0 \oplus \mathbb{Q}_{-1} \oplus \mathbb{Q}_{-2}$ of $\mathcal{L}$ given by a matrix

$$\rho := \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

with $x, y \in \mathcal{M}_1$ and $z \in M_2$. The dual of the Lie algebra bracket $[\,] : \mathcal{L} \wedge \mathcal{L} \to \mathcal{L}$ gives a graded map

$$d : \mathcal{M} \to \wedge^2 \mathcal{M}.$$

We extend d to a map $d : \bigwedge^\bullet \mathcal{M} \to \bigwedge^{\bullet+1} \mathcal{M}$ by $d(x_1 \wedge \dots \wedge x_n) = \sum (-1)^i x_1 \wedge \dots \widehat{x_i} \wedge \dots x_n$. The dual of the Jacobi identity is the statement $d^2 = 0$, so $\bigwedge^\bullet \mathcal{M}$ becomes a differential graded algebra.

For $a, b \in \mathcal{L}$ the condition that ρ be a representation of Lie algebras is

$$[\rho(a), \rho(b)] = \rho([a, b]),$$

where the bracket on the left is the commutator bracket in $\mathrm{End}(V)$. Both sides are zero unless a and b have degree -1. In this case, the identity to be checked is

$$z([a, b]) = x(a)y(b) - x(b)y(a)$$

i.e.

$$(x \wedge y)(a \wedge b) = dz(a \wedge b)$$

i.e.

$$dz = x \wedge y. \tag{2.3.1.1}$$

2.3.2. The cohomology of the Lie algebra $\mathcal{L}$ is given by the cohomology of the complex $(\wedge^{\bullet}\mathcal{M}, d)$ (with $\wedge^i$ in degree i) and is supposed to be the motivic cohomology $H^q_M(\mathrm{Spec}\,(k), \mathbb{Q}(p))$. The latter has been conjecturally identified by Beilinson with the γ-graded pieces of algebraic K-theory. In sum, we expect

$$H^q(\bigwedge^{\bullet}\mathcal{M}, d)_{\mathrm{deg.}\ p} = H^q_M(\mathrm{Spec}\,(k), \mathbb{Q}(p)) \cong \mathrm{gr}^p_\gamma K_{2p-q}(k) \otimes \mathbb{Q}.$$

In particular, since $\mathrm{gr}^2_\gamma K_2(k) \otimes \mathbb{Q} = K_2(k) \otimes \mathbb{Q}$, we look for an exact sequence

$$\mathcal{M}_2 \xrightarrow{d} \bigwedge^2 \mathcal{M}_1 \xrightarrow{s} K_2(k) \otimes \mathbb{Q} \to 0.$$

The map labeled s should be the symbol map. In particular, for $x \in k^\times - 1$ one has the Steinberg relation $s(x \wedge (1-x)) = 0$, so there should be an element

$$\theta_2(x) \in \mathcal{M}_2; \qquad d(\theta_2(x)) = x \wedge (1-x) \tag{2.3.2.1}$$

As a consequence of (2.3.1.1) and (2.3.2.1), the Steinberg relation in K_2 should correspond to a family of mixed Tate motives of dimension 3 parametrized by $\mathbb{P}^1 - \{0, 1, \infty\}$, with weight graded pieces $\mathbb{Q}, \mathbb{Q}(1), \mathbb{Q}(2)$ and underlying Kummer extensions of $\mathbb{Q}$ by $\mathbb{Q}(1)$ and $\mathbb{Q}(1)$ by $\mathbb{Q}(2)$ given by x and $1-x$ respectively. Here x is the standard parameter on $\mathbb{P}^1$. These are the dilogarithm motives. More generally, one looks for elements $\theta_n(x) \in \mathcal{M}_n$ satisfying

$$d\theta_n(x) = \theta_{n-1}(x) \otimes x. \tag{2.3.2.2}$$

The corresponding polylogarithm motives have been constructed [5],[9]. They are intimately related to the Zagier conjecture about values of zeta functions of number fields [47].

2.4. **Construction.** Here is an outline of the construction of $\mathcal{L}$ proposed in [11]. We construct an augmented Hopf algebra χ_{mot} over $\mathbb{Q}$, and $\mathcal{M} := I/I^2$ where $I \subset \chi$ is the augmentation ideal. Note that $\mathcal{M}$ and χ_{mot} should be graded (Adams grading, cf. 2.2). The construction is in several steps:

1. Using algebraic cycles we construct an augmented, graded, differential graded algebra over $\mathbb{Q}$

$$A^{\bullet} = \oplus_{n\geq 0} A^{\bullet}(n) \tag{2.4.0.3}$$

$$\ldots \to A^{-1} \to A^0 \to A^1 \to \ldots \tag{2.4.0.4}$$

$$0 \to A^+ \to A \to \mathbb{Q} \to 0 \tag{2.4.0.5}$$

$$A^p(m) \otimes A^q(n) \to A^{p+q}(m+n) \tag{2.4.0.6}$$

2. We define the bar construction $B(A^{\bullet})$.
3. We define $\chi_{\mathrm{mot}} := H^0(B(A))$.

2.4.1. *Construction of A.* Write $\square^1 = \mathbb{P}^1_k - \{1\}$, $\square^n = (\square^1)^n$. Each $\square^n$ has faces of all codimensions defined by setting various coordinates $= 0, \infty$. We define groups

$$G_n := \mathcal{S}_n \times \{\pm 1\}^n \qquad \text{semi-direct product}$$

where $\mathcal{S}_n$ is the symmetric group, acting by permutation on $\{\pm 1\}^n$. The group G_n acts on $\square^n$ with $\mathcal{S}_n$ acting by permutation of coordinates and the i-th $\{\pm 1\}$ acting by sending the i-th coodinate function to its inverse. Also, we have a homomorphism

$$\mathrm{Alt} : G_n \to \{\pm 1\}$$

which is the usual alternating map on $\mathcal{S}_n$ and the non-trivial character on each factor $\{\pm 1\}$. The group $A^i(r)$ is defined by applying the Alt-projector to the $\mathbb{Q}$-vector space spanned by all irreducible, codimension r closed subvarieties $V \subset \square^{2r-i}$ such that V meets properly (i.e. in the correct codimension) all faces of $\square^{2r-i}$. For example $A^1(2)$ is spanned by irreducible curves in $\square^3$ not meeting any of the loci where two coordinates are not invertible and not contained in the set where any single coordinate is not invertible.

One has face maps $\delta^k_j : \square^{2r-i} \to \square^{2r-(i-1)}$ for $1 \leq j \leq 2r - i + 1$ and $k = 0, \infty$. The alternating sum of the resulting pullback maps on cycles induces a map

$$\partial : A^{i-1}(r) \to A^i(r).$$

As usual, $\partial^2 = 0$, so $\oplus_{r\geq 0} A^{\bullet}(r)$ becomes a graded complex. We have

$$\square^{2r-i} \times \square^{2s-j} = \square^{2(r+s)-(i+j)}$$

The resulting exterior product on cycles $Z, W \mapsto Z \times W$ composed with the Alt projection gives $\oplus_{r\geq 0} A^{\bullet}(r)$ the structure of a graded differential graded algebra,

$$A^i(r) \otimes A^j(s) \mapsto A^{i+j}(r+s).$$

The augmentation $A^\bullet \to \mathbb{Q}$ is zero on $A^i(r)$ unless $i = r = 0$ (cod. 0 cycles on $\square^0 = \mathrm{Spec}\,(k)$), in which case it is the obvious isomorphism. The kernel of the augmentation is denoted $A^+ = \oplus A^{+i}$.

Concerning $A^\bullet$, the main result is [7]

$$H^q(A^\bullet(r)) \cong \mathrm{gr}^r_\gamma K_{2r-q}(k) \otimes \mathbb{Q} := H^q_M(\mathrm{Spec}\,(k), \mathbb{Q}(r)). \tag{2.4.1.1}$$

Note that the complex $A^\bullet$ is not connected, i.e. there are non-trivial terms in negative degrees.

Conjecture 2.4.1. *(Soulé-Beilinson) $H^q(A^{+\bullet}) = (0)$ for $q \leq 0$.*

The discussion below suggests a stronger conjecture:

Conjecture 2.4.2. *The differential graded algebra $A^\bullet$ is a $K(\pi, 1)$ in the sense of Sullivan* [40].

Recall Sullivan's theory associates to a differential graded algebra $A^\bullet$ another d.g.a. called the 1-minimal model

$$\mathfrak{M}^\bullet \xrightarrow{\phi} A^\bullet.$$

As an algebra,

$$\mathfrak{M}^\bullet \cong \wedge^\bullet \mathfrak{M}^1.$$

As mentioned in 2.3.1 above, the dual $\mathfrak{L}$ of $\mathfrak{M}^1$ inherits the structure of a (pro-)Lie algebra, with the bracket coming from the (pro-)dual of the differential $d : \mathfrak{M}^1 \to \wedge^2\mathfrak{M}^1$. When $A^\bullet$ is the differential graded algebra of forms on a manifold, one recovers from $\mathfrak{M}^\bullet$ the Lie algebra associated to the nilpotent quotient of the fundamental group. The algebra $A^\bullet$ is said to be a $K(\pi, 1)$ if it is quasi-isomorphic to its 1-minimal model. The idea is that in our case, the Lie algebra $\mathfrak{L}$ should be the Lie algebra of the Tannaka group of the category of mixed motives over k. For this to fit with established dogma (viz. the definition of motivic cohomology (2.4.1.1)) it should be the case that the Lie algebra cohomology of $\mathfrak{L}$ coincides with the cohomology of $A^\bullet$. This is the content of 2.4.2.

2.4.2. *Bar construction.* We write $A^\bullet$ for a (not necessarily reduced) augmented d.g.a. over $\mathbb{Q}$ as above. $A^{+\bullet} \subset A^\bullet$ is the kernel of the augmentation. The bar complex $\mathcal{B}(A)$ is a double complex as indicated below:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\partial\uparrow & \ddots & \partial\uparrow & & \partial\uparrow & & \\
\left(A^{+\otimes 3}\right)^2 & \xrightarrow{\delta} & \left(A^+\otimes A^+\right)^2 & \xrightarrow{\delta} & A^2 & \longrightarrow & 0 \\
\partial\uparrow & & \partial\uparrow & \ddots & \partial\uparrow & & \\
\left(A^{+\otimes 3}\right)^1 & \xrightarrow{\delta} & \left(A^+\otimes A^+\right)^1 & \xrightarrow{\delta} & A^1 & \longrightarrow & 0 \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow & \ddots & \\
\left(A^{+\otimes 3}\right)^0 & \xrightarrow{\delta} & \left(A^+\otimes A^+\right)^0 & \xrightarrow{\delta} & A^{+0} & \xrightarrow{0} & \mathbb{Q} \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

The complex is numbered so the diagonal indicated with dots represents total degree 0, and we define our Hopf algebra

$$\chi_{\mathrm{mot}} := H^0(\mathcal{B}(A)).$$

Technically, our bar complex is the two-sided bar complex $B(A) = B(\mathbb{Q}, A, \mathbb{Q})$. One writes $|$ rather than $\otimes$, and the differential δ is defined by $(a_i \in A^+)$

$$\delta[a_1|a_2|\ldots|a_r] = \sum_{i=1}^{r-1}(-1)^i\delta_i[a_1|a_2|\ldots|a_r]$$
$$\delta_i[a_1|a_2|\ldots|a_r] := [a_1|a_2|\ldots|a_i\cdot a_{i+1}|\ldots|a_r]$$

(The maps δ_0 and δ_r are both zero in this case.) The map ∂ is defined in the obvious way extending the differential on $A^{\bullet}$, and the total differential d is the sum with appropriate signs of δ and ∂. For more details see [11], §2.

Theorem 2.4.3. *Let $\mathcal{C}$ be the category of finite dimensional graded co-represntations of χ_{mot}.*
(i) Objects of $\mathcal{C}$ admit mixed Tate Hodge and ℓ-adic realizations. These arise from homomorphisms of Hopf algebras

$$\chi_{mot} \to \chi_{\acute{e}t} \ (resp.\ \chi_{betti})$$

where χ on the right is the Hopf algebra associated to the Tannakian category of mixed Tate ℓ-adic (resp. Hodge) realizations. (In the ℓ-adic case, one should assume that k contains only finitely many ℓ-th power roots of 1 in order to control $\mathrm{Hom}(\mathbb{Q}_\ell(m), \mathbb{Q}_\ell(n))$*.)*
(ii) $Ext^1_{\mathcal{C}}(\mathbb{Q}, \mathbb{Q}(1)) \cong k^\times \otimes \mathbb{Q}$.

(iii) $\mathcal{C}$ contains polylogarithm corepresentations corresponding to (2.3.2.2). *The Hodge realizations of these corepresentations are the Hodge polylogarithms as given in* [5].

Example 2.4.4. *The dilogarithm corepresentation associated to $a \in k - \{0,1\}$ may be described as follows. Define*

$$\theta_2(a) = Alt(\{(t, 1-t, 1-at^{-1}) \mid t \in \mathbb{P}^1\} \cap \square^3) \in A^1(2).$$

Then $(\theta_2(a), (a) \otimes (1-a)) \in A^1(2) \oplus (A^1(1) \otimes A^1(1))$ is a cocycle representing a class $Li_2(a)$ in $\chi_{mot} = H^0(\mathcal{B}(A))$. The comultiplication map $\psi : \chi_{mot} \to \chi_{mot} \otimes \chi_{mot}$ (cf. [11], *(2.1.4)) maps $Li_2(a)$ to $(a) \cup (1-a)$. This is the condition to get a rank 3 corepresentation as desired.*

3. Motives with $\mathbb{Z}/n\mathbb{Z}$-coefficients and étale cohomology

3.1. A beautiful link between motivic cohomology and étale cohomology has been established by Suslin and Voevodsky [42], [43]. There are two main theorems. The first, Theorem 3.1.1 below, links étale cohomology to the derived category of mixed motives as constructed in §1. The second, Theorem 3.1.2, relates étale cohomology and higher Chow groups.

Let k be an algebraically closed field. Results in this section depend on resolution of singularities, so we will eventually need to assume $\mathrm{Char}(k) = 0$. Recall we have defined 0.1.8 a cosimplicial variety $\Delta^\bullet$. For $F : (\mathrm{Sch}/k)^0 \to (\mathrm{ab})$ we have a complex 1.2.5

$$C_\bullet(F), \qquad C_n(F) := F(\Delta^n).$$

In particular, for $F = L(X)$ as in 1.2.5, we have

$$C_n(X) := C_n(L(X)) = \{\text{free abel. grp. on } Z \subset X \times \Delta^n \mid Z \to \Delta^n \text{ finite surjective}\}.$$

For X/S there is also a relative version $C_\bullet(X/S)$ generated by cycles Z such that $Z \to S \times \Delta^n$ is finite surjective. The *algebraic singular homology* of a variety X/k with $\mathbb{Z}/n\mathbb{Z}$-coefficients is defined by

$$H_*^{\mathrm{sing}}(X, \mathbb{Z}/n\mathbb{Z}) := H_*(C_\bullet(X) \otimes \mathbb{Z}/n\mathbb{Z}). \tag{3.1.0.1}$$

Theorem 3.1.1. *Let X be a variety over an algebraically closed field k of characteristic 0. Then*

$$H_*^{sing}(X, \mathbb{Z}/n\mathbb{Z}) \cong H_*^{\acute{e}t}(X, \mathbb{Z}/n\mathbb{Z}).$$

One way to calculate the étale homology on the right is to embed $k \hookrightarrow \mathbb{C}$ and compute the singular homology of $X(\mathbb{C})$. From this perspective, the isomorphism arises from a map (+ indicates group completion of the indicated algebraic or topological monoid)

$$C_{\bullet}(X) \cong \mathrm{Hom}_{\mathrm{algebraic}}(\Delta^{\bullet}, \coprod_{d=0}^{\infty} \mathrm{Sym}^d(X))^+ \to$$
$$\mathrm{Hom}_{\mathrm{topological}}(\Delta^{\bullet}_{\mathrm{top.}}, \coprod_{d=0}^{\infty} \mathrm{Sym}^d(X)_{\mathrm{top.}})^+$$

The higher chow groups with finite coefficients are given by

$$CH^r(X, n; \mathbb{Z}/m\mathbb{Z}) := H_n(\mathcal{Z}^r(X, \cdot) \otimes \mathbb{Z}/m\mathbb{Z}) \tag{3.1.0.2}$$

where $\mathcal{Z}^r(X, \cdot)$ is defined in 0.1.8.

Theorem 3.1.2. *Let X be an equi-dimensional quasiprojective variety over an algebraically closed field of characteristic 0. Let $i \geq d = \dim X$. Then*

$$CH^i(X, n; \mathbb{Z}/m\mathbb{Z}) \cong H^{2(d-i)+n}_{\acute{e}t,c}(X, \mathbb{Z}/m\mathbb{Z}(d-i))^*.$$

(The group on the right is the dual of the cohomology with compact supports.) When X is smooth, this can be rewritten

$$CH^i(X, n; \mathbb{Z}/m\mathbb{Z}) \cong H^{2i-n}_{\acute{e}t}(X, \mathbb{Z}/m\mathbb{Z}(i)).$$

Remark 3.1.3. (i) *Note the essential stability hypothesis $i \geq \dim X$. Without this, the theorem is false, even when $n = 0$.*

(ii) *Although the proofs of these results require some delicate work with exotic Grothendieck topologies, the results themselves are totally "motivic". The singular homology (with finite coefficients) is calculated wholly in terms of algebraic cycles.*

The remainder of this section is devoted to a sketch of the proof of theorem 3.1.1.

3.2. **Rigidity.** We begin with a simple algebraic singular homology calculation.

Proposition 3.2.1. *Consider a diagram of schemes*

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Assume X is a dense affine open subscheme of $\overline{X}$ smooth over S with fibre dimension 1. Let $Y = \overline{X} - X$. Assume S is affine and normal, $\overline{X}$ is normal and proper over S, and that there exists an affine open

$U \subset \overline{X}$ with $Y \subset U$. Define $Pic(\overline{X}, Y)$ to be the group of line bundles on $\overline{X}$ trivialized on Y. Then

$$H_0^{sing}(X/S) \cong Pic(\overline{X}, Y)$$
$$H_i^{sing}(X/S) = (0); \quad i > 0.$$

sketch of proof. One has by normality

$$\mathrm{Pic}(\overline{X}, Y) \cong \mathrm{Pic}(\overline{X} \times \Delta^n, Y \times \Delta^n).$$

Consider the sheaf $(1+I_n)^\times \subset \mathcal{O}_{\overline{X}\times\Delta^n}$ of invertible functions $\equiv 1$ along $Y \times \Delta^n$. Let $A_\bullet$ be the simplicial abelian group of global sections of $(1+I_\bullet)^\times$. Let $G_\bullet$ be the corresponding simplicial group of sections defined on some neighborhood of $Y \times \Delta^\bullet$. One gets an exact sequence

$$0 \to A_\bullet \to G_\bullet \to C_\bullet(X/S) \to \mathrm{Pic}(\overline{X} \times \Delta^\bullet, Y \times \Delta^\bullet) \to 0.$$

The proposition follows from an argument (omitted) showing $G_\bullet$ is acyclic. □

Recall we have the notion (1.2.5) of a homotopy-invariant presheaf with transfers $\Phi : (\mathrm{Sch}/k)^0 \to (\mathrm{ab})$.

Theorem 3.2.2 (Rigidity). *Let Φ be a homotopy-invariant presheaf with transfers. Let n be an integer prime to the characteristic of k such that $n \cdot \Phi = (0)$. Let $S = Spec(\Lambda)$, where Λ is the henselization of $\mathbb{A}_k^\ell$ at the origin. Then $\Phi(S) \cong \Phi(k)$. (Here S is interpreted as a pro-scheme, $S = \varprojlim_\alpha S_\alpha$, and $\Phi(S) := \varinjlim \Phi(S_\alpha)$.)*

Sketch of proof. The proof is by induction on ℓ, the case $\ell = 0$ being trivial. Let T be the henselization at 0 of $\mathbb{A}^{\ell-1}$. S can be thought of as a limit over smooth T-pointed affine curves X over T. We consider the resulting fibre diagram:

$$\begin{array}{ccc} X_S & \rightarrow & X \\ \downarrow\uparrow & \tau\nearrow & \downarrow\uparrow\sigma \\ S & \stackrel{\nu}{\rightarrow} & T. \end{array}$$

It will suffice to show the two resulting pullback maps

(3.2.0.3) $$\tau^*,\ \nu^* \circ \sigma^* : \Phi(X) \to \Phi(S)$$

coincide.

Because Φ has transfers, we can define a map

$$C_0(X/T) \otimes \Phi(X) \to \Phi(T)$$

associating to a generator $Z \subset X$ of $C_0(X/T)$ the composition

$$\Phi(X) \to \Phi(Z) \stackrel{\mathrm{Tr}}{\to} \Phi(T)$$

Because Φ is homotopy invariant, this factors through a pairing

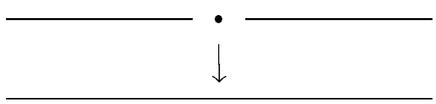
$$\mathrm{Pic}(\overline{X}, \overline{X} - X) \otimes \Phi(X) \cong H_0^{\mathrm{sing}}(X/T) \otimes \Phi(X) \to \Phi(T).$$

Let a be a section of X/T. Then a represents a class $[a] \in \mathrm{Pic}(\overline{X}, \overline{X} - X)$, and the map induced by $[a]$ from the above equals the pullback $a^* : \Phi(X) \to \Phi(T)$. If a and b are sections of X/T which agree at the closed point of T, standard properties of Pic show that $[a] - [b]$ is divisible by n, so we conclude

$$a^* = b^* : \Phi(X) \to \Phi(T).$$

We apply this to X_S/S and the two sections $S \to X_S$ induced by τ and $\sigma \circ \nu$ to get the desired equality (3.2.0.3). $\square$

3.3. **Topologies.** One of the really new ideas in the proof of theorem 3.1.1 involves the systematic use of very fine topologies. (So fine, in fact, that functors of points of representable objects are not sheaves.) A morphism $f : Y \to X$ is a topological epimorphism if $U \subset X$ open $\iff$ $f^{-1}(U) \subset Y$ open.(Note this implies f is surjective.) f is a universal topological epimorphism if this condition continues to hold under pullback. The h-topology has as coverings all universal topological epimorphisms. Coverings for the qfh-topology are maps ϕ which are quasi-finite and h. A crucial property of qfh-covers $Y \to X$ of connected normal schemes X is that taking $Y' \subset Y$ to be the union of the irreducible components of Y which dominate X, $Y' \to X$ is a qfh-cover. Thus the following is not a qfh-cover.



This means that a qfh-cover of a strictly henselian local ring admits as a refinement another strictly henselian local ring. It follows without much difficulty that writing

$$(\mathrm{Sch}/k)_{\mathrm{qfh}} \xrightarrow{\beta} (\mathrm{Sch}/k)_{\text{ét}}$$

for the morphism of sites, one has

$$(\mathbb{Z}/n\mathbb{Z})_{\text{ét}} \cong \beta_*(\mathbb{Z}/n\mathbb{Z})_{\mathrm{qfh}} \cong \mathbf{R}\beta_*(\mathbb{Z}/n\mathbb{Z})_{\mathrm{qfh}} \tag{3.3.0.4}$$

Example 3.3.1. *Recall in section 1.2.5 we have defined $L(X)$ to be the functor of points of a scheme X in the category $SmCor(k)$. Let $\mathbb{Z}' := \mathbb{Z}[1/p]$ with $p = Char(k)$. Define $\mathbb{Z}'[T]$ for a set T to be the free*

$\mathbb{Z}'$-module on T. Let $\mathbb{Z}'[X]_{qfh}$ denote the qfh-sheaf associated to the presheaf

$$S \mapsto \mathbb{Z}'[Map_{Sch}(S, X)].$$

Then, for S normal and connected

$$\Gamma(S, \mathbb{Z}'[X]_{qfh}) \cong L(X)(S).$$

3.4. **Homological algebra.** In some sense, the crucial point of the proof is the following:

Theorem 3.4.1. *Let $k = \bar{k}$ be an algebraically closed field of characteristic 0. Let Φ be a homotopy-invariant presheaf with transfers on the category of schemes over k. Write $\Phi_{\acute{e}t}$, Φ_{qfh}, Φ_h for the corresponding sheaves on the big sites for the indicated topologies on schemes over k. Then*

$$Ext^*(\Phi_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) \cong Ext^*(\Phi_{qfh}, \mathbb{Z}/n\mathbb{Z}) \cong Ext^*(\Phi_h, \mathbb{Z}/n\mathbb{Z}) \cong Ext_{Ab}(\Phi(\mathrm{Spec}\,(k)), \mathbb{Z}/n\mathbb{Z}). \tag{3.4.0.5}$$

Sketch of proof. The first three isomorphisms follow from (3.3.0.4) and related ideas. For the last isomorphism, we work with h-sheaves throughout and so we surpress the subscript h. Let Φ_0 be the h-sheaf associated to the constant presheaf (on the big site) $X \mapsto \Phi(\mathrm{Spec}\,(k))$. Define Φ' by the exact sequence of h-sheaves

$$0 \to \Phi_0 \to \Phi \to \Phi' \to 0.$$

Let ${}_n\Phi_0 \subset \Phi_0$ and ${}_n\Phi \subset \Phi$ denote the kernels of multiplication by n. The rigidity theorem 3.2.2 implies for S smooth and strictly henselian

$${}_n\Phi_0(S) \cong {}_n\Phi(S);\ \big(\Phi_0/n\Phi_0\big)(S) \cong \big(\Phi/n\Phi\big)(S)$$

However, resolution of singularities is an h-covering, so we conclude that multiplication by n is an isomorphism on the h-sheaf Φ', and

$$\mathrm{Ext}^*_h(\Phi, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}^*_h(\Phi_0, \mathbb{Z}/n\mathbb{Z}).$$

Finally, we need to understand the Ext group on the right. We have

$$H^*(X_h, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}^*(\mathbb{Z}[X]_h, \mathbb{Z}/n\mathbb{Z}), \tag{3.4.0.6}$$

since both sides calculate the derived functors of global sections. Let $A = \Phi(\mathrm{Spec}\,(k))$. Also $\Phi_0 \cong A[\mathrm{Spec}\,(k)]_h$, since both sides are associated to the same presheaf. Using $H^*(\mathrm{Spec}\,(k)_h, \mathbb{Z}/n\mathbb{Z}) = (0)$, it is now easy to see

$$\mathrm{Ext}^*_h(\Phi_0, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Ext}^*_{\mathrm{Ab}}(\Phi(\mathrm{Spec}\,(k), \mathbb{Z}/n\mathbb{Z}).$$

□

3.5. **Spectral Sequences.** We continue to work in the category of algebraic k-schemes, where k is an algebraically closed field of characteristic 0. Let $F : (\mathrm{Sch}/k)^0 \to (\mathrm{Ab})$ be a sheaf (h, qfh, or ét). We have a surjection

$$\bigoplus_{(S,f)} \mathbb{Z}[S]_t \twoheadrightarrow F$$

where the subscript t denotes sheafification for the appropriate topology, and the sum runs over pairs S, f with S a k-scheme and $f \in F(S)$. This construction may be iterated to get a presentation of F in terms of direct sums of $\mathbb{Z}[S]_t$.

Proposition 3.5.1. *With notation as above (all functors are sheafified for the t topology, so we drop the t), the natural map $\mathbb{Z}[\mathbb{A}^1] \to \mathbb{Z}$ induces isomorphisms*

$$Ext^*(F, \mathbb{Z}/n\mathbb{Z}) \cong Ext^*(F \otimes \mathbb{Z}[\mathbb{A}^1], \mathbb{Z}/n\mathbb{Z}). \tag{3.5.0.7}$$

Proof. Suppose first $F = \mathbb{Z}[X]$. Then $F \otimes \mathbb{Z}[\mathbb{A}^1]$ is the t-sheaf associated to the presheaf

$$S \mapsto \mathbb{Z}[\mathrm{Map}(S, X)] \otimes \mathbb{Z}[\mathrm{Map}(S, \mathbb{A}^1)] \cong \mathbb{Z}[\mathrm{Map}(S, X \times \mathbb{A}^1)]$$

from which it follows that $\mathbb{Z}[X] \otimes \mathbb{Z}[\mathbb{A}^1] \cong \mathbb{Z}[X \times \mathbb{A}^1]$. In this case, by (3.4.0.6), we must show

$$H^*(X_t, \mathbb{Z}/n\mathbb{Z}) \cong H^*((X \times \mathbb{A}^1)_t, \mathbb{Z}/n\mathbb{Z}).$$

This is well known for $t =$ ét. In the other cases it follows from theorem 3.4.1. For general F, we use the above construction to write an exact sequence of sheaves

$$0 \to D \to E \to F \to 0,$$

where the assertion of the proposition is true for E. Note in general the map in (3.5.0.7) is injective (it has a section). The long-exact Ext-sequence and induction on i gives

$$\begin{array}{ccccccc}
\mathrm{Ext}^{i-1}(D, \mathbb{Z}/n\mathbb{Z}) & \to & \mathrm{Ext}^i(F, \mathbb{Z}/n\mathbb{Z}) & \to & \cdot & \to & \cdot \\
\| & & \cap & & \| & & \cap \\
\mathrm{Ext}^{i-1}(D \otimes \mathbb{Z}[\mathbb{A}^1], \mathbb{Z}/n\mathbb{Z}) & \to & \mathrm{Ext}^i(F \otimes \mathbb{Z}[\mathbb{A}^1], \mathbb{Z}/n\mathbb{Z}) & \to & \cdot & \to & \cdot
\end{array}$$

The assertion of the proposition follows from a diagram chase. □

3.5.1. *Contractible sheaves.* A sheaf F is said to be contractible if there exists a map $F \otimes \mathbb{Z}[\mathbb{A}^1] \xrightarrow{\phi} F$ such that $\phi \circ i_0 = 0$ and $\phi \circ i_1 = \mathrm{Id}$. Here $i_j : \mathbb{Z} \to \mathbb{Z}[\mathbb{A}^1]$ correspond to the points $j = 0, 1$. As a consequence of (3.5.0.7), F contractible implies $\mathrm{Ext}^*(F, \mathbb{Z}/n\mathbb{Z}) = (0)$.

Given a sheaf F and an integer q, we write F_q for the sheaf

$$X \mapsto F(X \times \mathbb{A}^q).$$

Let $\tilde{F}_q := \mathrm{Coker}(F \to F_q)$.

Proposition 3.5.2. *The sheaf $\tilde{F}_q$ is contractible. In particular,*

$$Ext^*(F, \mathbb{Z}/n\mathbb{Z}) \cong Ext^*(F_q, \mathbb{Z}/n\mathbb{Z}) \tag{3.5.1.1}$$

Proof. Let $\mu : \mathbb{A}^1 \times \mathbb{A}^q \to \mathbb{A}^q$ be the map $(t, u) \mapsto tu$. We define a map $\psi : F_q \otimes \mathbb{Z}[\mathbb{A}^1] \to F_q$ by associating to a section $s \in F(X \times \mathbb{A}^q)$ and a map $f : X \to \mathbb{A}^1$ the section $g^*(s)$ where $g : X \times \mathbb{A}^q \to X \times \mathbb{A}^q$ is the map

$$X \times \mathbb{A}^q \xrightarrow{f \times \mathrm{Id}} \mathbb{A}^1 \times X \times \mathbb{A}^q \xrightarrow{\mathrm{Id}_X \times \mu} X \times \mathbb{A}^q.$$

It is straightforward to check that ψ induces a contraction map

$$\tilde{F}_q \otimes \mathbb{Z}[\mathbb{A}^1] \to \tilde{F}_q$$

with the required properties. □

3.5.2. Let F be a sheaf as above. Recall we have defined in 1.2.5 a homological complex of sheaves

$$\underline{C}_\bullet(F) := \cdots \to F_2 \to F_1 \to F.$$

Define the hyperext groups

$$\mathbf{E}^* := \mathbf{Ext}^*(\underline{C}_\bullet(F), \mathbb{Z}/n\mathbb{Z}).$$

There are three spectral sequences

$$I_1^{p,q} := \mathrm{Ext}^q(F_p, \mathbb{Z}/n\mathbb{Z}) \Rightarrow \mathbf{E}^{p+q} \tag{3.5.2.1}$$

$$II_2^{p,q} := \mathrm{Ext}^p(H^q(\underline{C}_\bullet(F)), \mathbb{Z}/n\mathbb{Z}) \Rightarrow \mathbf{E}^{p+q} \tag{3.5.2.2}$$

$$\mathrm{Ext}^p_{\mathrm{Ab}}(H^q(C_\bullet(F)), \mathbb{Z}/n\mathbb{Z}) \Rightarrow \tag{3.5.2.3}$$

$$\mathbf{Ext}^{p+q}_{\mathrm{Ab}}(C_\bullet(F), \mathbb{Z}/n\mathbb{Z}) := H^*_{\mathrm{sing}}(F, \mathbb{Z}/n\mathbb{Z})$$

The first spectral sequence degenerates by virtue of (3.5.1.1), so

$$\mathbf{E}^p \cong \mathrm{Ext}^p(F, \mathbb{Z}/n\mathbb{Z}). \tag{3.5.2.4}$$

Now we assume the sheaf F is endowed with transfer maps as in 1.2.5. The homology groups $H_q(\underline{C}_\bullet)$ are then shown to be homotopy invariant sheaves with transfer, so by theorem 3.4.1 the spectral sequences (3.5.2.2) and (3.5.2) coincide at E_2. Since there are maps from the one to the other, we conclude with notation as in (3.5.2)

$$\mathrm{Ext}^p(F, \mathbb{Z}/n\mathbb{Z}) \cong \mathbf{E}^* \cong H^*_{\mathrm{sing}}(F, \mathbb{Z}/n\mathbb{Z}) \tag{3.5.2.5}$$

It suffices finally to take $F = \mathbb{Z}[X]_{\text{ét}}$ and substitute in the above, getting

$$H^p_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) \cong H^*_{\mathrm{sing}}(X, \mathbb{Z}/n\mathbb{Z}).$$

This completes the proof of theorem 3.1.1. (We leave to the reader the minor business of translating the cohomological formula above into the homological statement of the theorem.)

4. Polylogarithms and Chow Regulators

In this final section I will explain the work of Beilinson and Deligne on polylogarithms [5], and some recent generalizations, due to Goncharov [18], involving Chow varieties.

4.1. Let k be a field. As has already been mentioned briefly (2.3.2.2) the polylogarithm Lie algebra is the graded pro-Lie algebra

$$\mathcal{L}^{pl} = \mathcal{L}^{pl}_{-1} \oplus \mathcal{L}^{pl}_{-2} \oplus \dots$$

with dual $\mathcal{M}^{pl}$ given by

$$\mathcal{M}^{pl}_1 = k^\times \otimes \mathbb{Q}$$
$$\mathcal{M}^{pl}_n = \text{some quotient of } k - \{0,1\}; \quad z \mapsto \theta_n(z)$$

The dual of the bracket is a differential $d : \mathcal{M}^{pl} \to \wedge^2 \mathcal{M}^{pl}$ defined by

$$d\theta_n(z) = (z) \wedge \theta_{n-1}(z).$$

(Here $\theta_1(z) := 1 - z$.) For $N \geq 2$ given, the N-polylogarithm representation $\mathfrak{M}^N_z$ associated to $z \in k - \{0,1\}$ has the following shape. For $N = 1$, it is the Kummer extension

$$0 \to \mathbb{Q}(1) \to \mathfrak{M}^1_z \to \mathbb{Q} \to 0.$$

For $N \geq 2$ one gets an extension

$$0 \to (\mathrm{Sym}^{N-1}(\mathfrak{M}^1_z))(1) \to \mathfrak{M}^N_z \to \mathbb{Q} \to 0.$$

4.1.1. The Betti and deRham realizations of this family of mixed motives give rise to a variation of mixed Tate Hodge structure over $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. To describe this, recall the polylogarithm functions

$$\mathrm{Li}_1(z) := -\log(1 - z); \quad \mathrm{Li}_n(z) := \int_0^z \mathrm{Li}_{n-1}(t)\frac{dt}{t}.$$

These are multivalued functions which are analytic away from $\{0, 1, \infty\}$. The principal branch of $\mathrm{Li}_n(z)$ is given by the power series

$$\mathrm{Li}_n(z) = \sum_{p=1}^{\infty} \frac{z^p}{p^n}.$$

Denote by $L(z)$ (resp. $L^{(N)}(z)$) the infinite (resp. $(N+1)\times(N+1)$) matrix (resp. top left hand corner matrix)

$$\begin{pmatrix} 1 & 0 & 0 & \dots\dots \\ -\mathrm{Li}_1(z) & 1 & 0 & \dots\dots \\ -\mathrm{Li}_2(z) & \log(z) & 1 & \dots\dots \\ -\mathrm{Li}_3(z) & \frac{(\log(z))^2}{2!} & \log(z) & 1 \quad \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Define

$$A(z) := L(z)\tau(2\pi i); \quad \text{resp. } A^{(N)}(z) := L^{(N)}(z)\tau^{(N)}(2\pi i)$$

where $\tau(\lambda)$ (resp. $\tau^{(N)}(\lambda)$) denotes the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \lambda & 0 & \dots \\ 0 & 0 & \lambda^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(resp. the evident $(N+1)\times(N+1)$-truncation).

For $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ write $R^{[0,N]}$ for the space of all $N+1$-column vectors

$$\begin{pmatrix} r_0 \\ \vdots \\ r_N \end{pmatrix}$$

with $r_i \in R$. For $0 \le a \le b \le N$, let $R^{[a,b]} \subset R^{[0,N]}$ be the subspace with $r_i = 0$ for $r \notin [a,b]$. Write $\mathfrak{M}^{(N)}(R)$ for the R-structure in the fibre of the variation of mixed Hodge structure associated to $\mathfrak{M}^{(N)}$ over the point $z \in \mathbb{P}^1 - \{0,1,\infty\}$. We have

$$\mathfrak{M}_z^{(N)}(\mathbb{Q}) = A^{(N)}(z)\cdot \mathbb{Q}^{[0,N]}$$

$$W_{-2k}\mathfrak{M}_z^{(N)}(\mathbb{Q}) = A^{(N)}(z)\cdot \mathbb{Q}^{[k,N]} \quad \text{(weight filtration)}$$

$$F^{-k}\mathfrak{M}_z^{(N)}(\mathbb{C}) = \mathbb{C}^{[0,k]} \quad \text{(Hodge filtration)}$$

4.1.2. Notice that even though the matrix $A^{(N)}(z)$ is multiple-valued, the effect of analytic continuation around a loop is to replace it by $A^{(N)}(z)\cdot B$ where B is lower triangular with 1's on the diagonal and entries in $\mathbb{Q}$. In particular, the $\mathbb{Q}$-structure and weight filtration above are independent of the branch. This remark leads [5] to a very beautiful construction of single-valued $\mathbb{R}$-valued variants of the polylogarithm. As above, the image of $A^{(N)}(z)$ in $GL_{N+1}(\mathbb{C})/GL_{N+1}(\mathbb{R})$ is independent of the branch. Consider the matrix

$$B := \tau(i)^{-1}A\overline{A}^{-1}\tau(i)$$

Clearly, B is independent of the branch, and

$$\overline{B}^{-1} = \tau(-1)B\tau(-1).$$

Since B is triangular with 1's on the diagonal, we can take the logarithm, and

$$\log(B) = -\mathrm{ad}(\tau(-1))(\log(\overline{B})),$$

i.e.

$$\log(B)_{r,s} \in \begin{cases} \sqrt{-1}\,\mathbb{R} & r \equiv s \mod 2 \\ \mathbb{R} & r \not\equiv s \mod 2. \end{cases}$$

An elegant matrix calculation leads to the expression

$$-\log(B^{1/2}) = \begin{pmatrix} 0 & \dots \\ D_1(z) & \dots \\ D_2(z) & \dots \\ D_3(z) & \dots \\ \vdots & \vdots \end{pmatrix}$$

where the $D_k(z)$ are given by

(4.1.2.1)

$$D_k(z) = \begin{cases} \sqrt{-1}\,\sum_{\ell=0}^{k-1} \frac{b_\ell}{\ell!}(\log(|z|^2))^\ell \mathrm{Im}(\mathrm{Li}_{k-\ell}(z)) & k \equiv 0 \mod 2 \\ \sum_{\ell=0}^{k-1} \frac{b_\ell}{\ell!}(\log(|z|^2))^\ell \mathrm{Re}(\mathrm{Li}_{k-\ell}(z)) & k \equiv 1 \mod 2 \end{cases}$$

(Here the b_ℓ are Bernoulli numbers.) For example

$$D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log(|z|)\arg(1-z).$$

To my knowledge, these particular single-valued variants of the polylogarithms were first written down in [47]. Related functions were defined in [13] and [38].

4.1.3. *Exercise.* Recall in 2.4.1 we constructed a graded differential graded algebra based on cycles on affine spaces $\square^r := (\mathbb{P}^1 - \{1\})^r$ over a field k. The polylogarithm map

$$\theta_n : k - \{0,1\} \to A^1(n) \subset \left\{\text{codim. } n \text{ alg. cycles on } \square^{2n-1}\right\}$$

is defined by taking $\theta_n(a)$ to be $(-1)^n(n-1)/2$ times the skew-symmetrization in the sense of 2.4.1 of the locus parametrized in non-homogeneous coordinates by

$$\Big(x_1 \dots, x_{n-1}, 1-x_1, 1-\frac{x_2}{x_1}, \dots, 1-\frac{x_{n-1}}{x_{n-2}}, 1-\frac{a}{x_{n-1}}\Big).$$

We define B_n to be the subgroup of $A^1(n)/d(A^0(n))$ generated by the $\theta_n(a)$ for $a \in k,\ a \neq 0,1$. Try to use the differential d on $A^\bullet$ to define a map $B_n \to B_{n-1} \otimes k^\times$. Note there is a problem. Try to write down a polylogarithm representation using the bar complex [11].

4.2. **Chow regulators.** This section is an introduction to the paper of Goncharov, [18].

4.2.1. Fix non-negative integers $n = p + q$ with $n \geq 1$. Let $L = \{L_0, \dots, L_n\}$ be a set of hyperplanes in general position in $\mathbb{P}^n_{\mathbb{C}}$. By abuse of notation, we will sometimes write $L = \bigcup L_i$. If $L_i : z_i = 0$, then $\mathbb{P}^n - L \cong \mathbb{C}^{\times n}$ with coordinates z_i/z_0. The differential form

$$\Omega_L := (2\pi i)^{-n} d\log(z_1/z_0) \wedge \dots \wedge d\log(z_n/z_0)$$

has integer periods, so $\mathrm{Im}\Omega_L$ is exact. There is no canonical primitive for this form, but if we choose an auxillary hyperplane H in general position with respect to L we may scale the z_i so $H : \sum z_i = 0$. This fixes the coordinate functions, and we may fix a primitive as follows. For any meromorphic functions $f_1, \dots, f_n$, we define (alt_n is the alternating projection for the action of the symmetric group on the f_i):

$$r_n(f_1, \dots, f_n) := \mathrm{alt}_n\Big(\sum_{j\geq 0} \frac{1}{(2j+1)!(n-2j-1)!} \log|f_1| d\log|f_2| \wedge \dots \wedge d\log|f_{2j+1}| \wedge d(i \arg f_{2j+2}) \wedge \dots \wedge d(i \arg f_n)\Big). \tag{4.2.1.1}$$

Then

$$dr_n = \mathrm{Im}\big((2\pi i)^{-n} d\log f_1 \wedge \dots \wedge d\log f_n\big) \tag{4.2.1.2}$$

In particular, writing $r_n(L, H) := r_n(z_1/z_0, \dots, z_n/z_0)$, we have

$$dr_n(L, H) = \mathrm{Im}\Omega_L.$$

4.2.2. Let $\mathcal{Z}^q_p(L)$ be the open set in the Chow variety of codimension q algebraic cycles on $\mathbb{P}^n$ consisting of cycles meeting all faces $L_I = \cap_{i\in I} L_i$ properly, i.e. in the correct dimension. For example $\mathcal{Z}^n_0(L) = \mathbb{P}^n - L = \mathbb{C}^\times n$. Let

$$a_i : \mathcal{Z}^q_p(L) \to \mathcal{Z}^q_{p-1}(L_i) \tag{4.2.2.1}$$

$$b_j : \mathcal{Z}^q_p(L) \to \mathcal{Z}^{q-1}_p(L_j) \tag{4.2.2.2}$$

be defined by intersection (resp. projection) of cycles.

By definition, the space of p-currents on a manifold is the dual space to the space of $\mathcal{C}_\infty$ p-forms with compact support. A current on a singular reduced analytic set is a current on the open set of smooth points. Goncharov defines a $(q-p-1)$-current $\omega_p^q(L,H)$ on $\mathcal{Z}_p^q(L)$ with the following properties:

$$d\omega_0^q(L,H) = \mathrm{Im}\Omega_L \tag{4.2.2.3}$$

$$\omega_p^q(L,H) = \sum_{i=0}^{p+q}(-1)^i a_i^*\omega_{p-1}^q(L^{(i)},H^{(i)}) \tag{4.2.2.4}$$

$$\sum_{j=0}^{p+q+1}(-1)^j b_j^*\omega_p^q(L,H) = 0 \tag{4.2.2.5}$$

Here $L^{(i)}$ and $H^{(i)}$ are the evident restrictions to $\mathbb{P}^{n-1} = L_i$.

4.2.3. Let Γ_p^q be the incidence correspondence

$$\begin{tikzcd} & \Gamma_p^q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & \mathcal{Z}_p^q(L), \end{tikzcd}$$

and define

$$\omega_p^q(L,H) := \pi_{2*}\pi_1^* r_n(L,H).$$

Note π_2 is proper, so pushforward of currents is defined. Actually, one has to be a bit careful, because the pullback $\psi = \pi_2^*(\phi)$ of a $\mathcal{C}^\infty$ function on $\mathcal{Z}_p^q(L)$ with compact support contained in the smooth locus does not necessarily have support contained in the smooth locus on Γ_p^q. However, ψ is continuous, and, viewing $\pi_1^* r_n(L,H)$ as being defined on an open set, the integral

$$\int_{\Gamma_p^q} \psi\pi_1^* r_n(L,H)$$

is defined.

4.2.4. One verifies that $\omega_p^q(L,H)$ satisfies (4.2.2.3)-(4.2.2.5).

Theorem 4.2.1. (i) *ω_p^q is a $(q-p-1)$-current. It is a $\mathcal{C}^\infty$-form on the open set*

$$\widehat{\mathcal{Z}}_p^q \subset \mathcal{Z}_p^q(L)$$

parametrizing smooth cycles in transverse position relative to L. In particular, ω_p^{p+1} is an L^1-function on $\mathcal{Z}_p^{p+1}(L)$.

(ii) *Restriction of cycles to faces makes $\widehat{\mathcal{Z}}_\cdot^q$ a simplicial variety:*

$$\widehat{\mathcal{Z}}_\cdot^q : \cdots \widehat{\mathcal{Z}}_{p+1}^q \rightrightarrows \widehat{\mathcal{Z}}_p^q \cdots$$

Equations (4.2.2.3) and (4.2.2.4) imply that ω_p^q is a $2q$-cocycle representing a class in the Deligne cohomology

$$[w_p^q] \in H_{\mathcal{D}}^{2q}(\widehat{\mathcal{Z}}_\cdot^q, \mathbb{R}(q)).$$

(iii) *For $a \in \mathbb{C} - \{0,1\}$, let h_a denote the point in the Chow variety of codimension n cycles in $\mathbb{P}^{2n-1}$ corresponding to the closure in $\mathbb{P}^{2n-1}$ of the affine $n-1$-plane defined in affine coordinates by*

$$z_n = 1 - z_1;\ z_{n+i} = z_i - z_{i+1};\ 1 \le i \le n-2$$
$$z_{2n-1} = z_{n-1} - a$$

This cycle is not in good position, but the function ω^n_{n-1} is defined and continuous at h_a, and

$$\omega^n_{n-1}(h_a) = (2\pi i)^{-n} D_n(a)$$

where $D_n(a)$ is defined in (4.1.2.1).

Remark 4.2.2. (i) *One would like to better understand the functions ω^n_{n-1}. Is there a family of motives over $\widehat{\mathcal{Z}}^n_{n-1}$ giving rise to a mixed Tate variation of mixed Hodge structure?*

(ii) *Do the functions ω^n_{n-1}, or more generally the currents ω^q_p, admit some interpretation in terms of symplectic geometry for the group acting on $\mathbb{P}^{p+q}$?*

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