GAUSS-MANIN DETERMINANT CONNECTIONS AND PERIODS FOR IRREGULAR CONNECTIONS

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Abstract. Gauß-Manin determinant connections associated to irregular connections on a curve are studied. The determinant of the Fourier transform of an irregular connection is calculated. The determinant of cohomology of the standard rank 2 Kloosterman sheaf is computed modulo 2 torsion. Periods associated to irregular connections are studied in the very basic $\exp(f)$ case, and analogies with the Gauß-Manin determinant are discussed.

Everything's so awful reg'lar a body can't stand it.

The Adventures of Tom Sawyer
Mark Twain

1. Introduction

A very classical area of mathematics, at the borderline between applied mathematics, algebraic geometry, analysis, mathematical physics and number theory is the theory of systems of linear differential equations (connections). There is a vast literature focusing on regular singular points, Picard-Fuchs differential equations, Deligne's Riemann-Hilbert correspondence and its extension to $\mathcal{D}$-modules, and more recently various index theorems in geometry.

The classification of irregular singular points in terms of Stokes structures is explained in [8], chap. IV. It involves the Riemann-Hilbert correspondence for holonomic D-modules with regular singular points, together with a filtration related to the rate of growth of local solutions.

Nontheless, there are some very modern themes in the subject which remain virtually untouched. On the arithmetic side, one may ask, for example, how irregular connections can be incorporated in the modern theory of motives? How deep are the apparent analogies between wild ramification in characteristic p and irregularity? Can one define periods for irregular connections? If so, do the resulting period matrices have

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anything to do with ϵ-factors for ℓ-adic representations? On the geometric side, one can ask for a theory of characteristic classes $c_i(E, \nabla)$ for irregular connections such that c_0 is the rank of the connection and c_1 is the isomorphism class of the determinant. With these, one can try to attack the Riemann-Roch problem.

In this note, we describe some conjectures and examples concerning what might be called a families index theorem for irregular connections. Let

$$f : X \to S \tag{1.1}$$

be a smooth, projective family of curves over a smooth base S. Let $\mathcal{D}$ be an effective relative divisor on X. Let E be a vector bundle on X, $\nabla : E \to E \otimes \Omega^1_X(\mathcal{D})$ an integrable, absolute connection with poles along $\mathcal{D}$. The relative de Rham cohomology $\mathbb{R}f_*(\Omega^*_{X-\mathcal{D}} \otimes E)$ inherits a connection (Gauß-Manin connection), and the families index or Riemann-Roch problem is to describe the isomorphism class of the line bundle with connection

$$\det \mathbb{R}f_*(\Omega^*_{X-\mathcal{D}} \otimes E). \tag{1.2}$$

The Gauß-Manin construction is fairly standard and we do not recall it in detail. By way of example, we cite two classical formulas (Gauß hypergeometric and Bessel functions, respectively) :

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; z) = \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} du$$
$$J_n(z) = \frac{1}{2\pi i} \int_{S_0} u^{-n} \exp\ \frac{z}{2}\Big(u + \frac{1}{u}\Big)\frac{du}{u} \qquad (S_0 = \text{circle about } 0).$$

In both cases, the integrand is a product of a solution of a rather simple degree 1 differential equation in u, the solution being either

$$u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a} \quad \text{or} \quad u^{-n} \exp\ \frac{z}{2}\Big(u + \frac{1}{u}\Big),$$

with an algebraic 1-form (du or $\frac{du}{u}$). The integral is taken over a chain in the u-plane. The resulting functions $F(a, b; c; z)$ and $J_n(z)$ satisfy Gauß-Manin equations, which are much more interesting degree 2 equations in z.

It is not our purpose to go further into the classical theory, but, to understand the role of the determinant, we remark that in each of the above cases, there is a second path and a second algebraic 1-form such that the two integrals, say $f_1(z)$ and $f_2(z)$, satisfy the same second order equation. The Wronskian determinant

$$\begin{vmatrix} f_1 & f_2 \\ \frac{df_1}{dz} & \frac{df_2}{dz} \end{vmatrix}$$

satisfies the degree 1 equation given by the determinant of Gauß-Manin.

We are working here with algebraic de Rham cohomology. Analytically (i.e. permitting coordinate and basis transformations with essential singularities on $\mathcal{D}$), the bundle $E^{\mathrm{an}}|_{X-\mathcal{D}}$ can be transformed (locally on S) to have regular singular points along $\mathcal{D}$, but the algebraic problem we pose is more subtle.

If $\mathcal{D} = \emptyset$, or, more generally, if ∇ has regular singular points, the answer is known:

$$\det \mathbb{R}f_*(\Omega^*_{X-\mathcal{D}} \otimes E) = f_*((\det E^\vee, -\det\nabla)\cdot c_1(\omega_{X/S})) \tag{1.3}$$

(In the case of regular singular points, the c_1 has to be taken as a relative class [9], [1].)

In a recent article [2], we proved an analogous formula in the case when E was irregular and rank 1. For a suitable $\mathcal{D}$, the relative connection induces an isomorphism

$$\nabla_{X/S,\mathcal{D}} : E|_{\mathcal{D}} \cong E|_{\mathcal{D}} \otimes (\omega(\mathcal{D})/\omega),$$

i.e. a trivialization of $\omega(\mathcal{D})|_{\mathcal{D}}$. The connection pulls back from a rank 1 connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ on the relative Picard scheme $\mathrm{Pic}(X, \mathcal{D})$, and the Gauß-Manin determinant connection is obtained by evaluating $(\mathcal{E}, \nabla_{\mathcal{E}})$ at the priviledged point $(\omega(\mathcal{D}), \nabla_{X/S,\mathcal{D}}) \in \mathrm{Pic}(X, \mathcal{D})$.

We want now to consider two sorts of generalizations. First, we formulate an analogous conjecture for higher rank connections which are admissible in a suitable sense. We prove two special cases of this conjecture, computing the determinant of the Fourier transform of an arbitrary connection and, up to 2-torsion, the determinant of cohomology of the basic rank 2 Kloosterman sheaf.

Second, we initiate in the very simplest of cases $E = \mathcal{O}$, $\nabla(1) = df$, the study of periods for irregular connections. Let $m = \deg f + 1$. We are led to a stationary phase integral calculation over the subvariety of $\mathrm{Pic}(\mathbb{P}^1, m\cdot\infty)$ corresponding to trivializations of $\omega(m\cdot\infty)$ at ∞. The subtle part of the integral is concentrated at the same point

$$(\omega(m\cdot\infty), \nabla_{X/S,m\cdot\infty}) \in \mathrm{Pic}(\mathbb{P}^1, m\cdot\infty)$$

mentioned above.

Our hope is that the geometric methods discussed here will carry over to the sort of arithmetic questions mentioned above. We expect that the distinguished point given by the trivialization of $\omega(\mathcal{D})$ plays some role in the calculation of ϵ-factors for rank 1 ℓ-adic sheaves and that the higher rank conjectures have some ℓ-adic interpretation as well.

We would like to thank Pierre Deligne for sharing his unpublished letters with us. The basic idea of using the relative Jacobian to study ϵ-factors for rank 1 sheaves we learned from him. We also have gotten considerable inspiration from the works of T. Saito and T. Terasoma cited in the bibliography. The monograph [5] is an excellent reference for Kloosterman sheaves, and the subject of periods for exponential integrals is discussed briefly at the end of [6].

2. The Conjecture

Let S be a smooth scheme over a field k of characteristic 0. We consider a smooth family of curves $f: X \to S$ and a vector bundle E with an absolute, integrable connection $\nabla : E \to E \otimes \Omega^1_X(*D)$. Here $D \subset X$ is a divisor which is smooth over S. We are interested in the determinant of the Gauß-Manin connection

$$\begin{aligned}(2.1)\quad \det \mathbb{R}f_*(\Omega^*_{(X-D)/S} \otimes E) \in \mathrm{Pic}^\nabla(S) &:= \mathbb{H}^1(S, \mathcal{O}_S^\times \xrightarrow{d\log} \Omega^1_S) \\ &\cong \Gamma(S, \Omega^1_S / d\log \mathcal{O}_S^\times).\end{aligned}$$

Proposition 2.1. *Let $K = k(S)$ be the function field of S. Then the restriction map $\mathrm{Pic}^\nabla(S) \to \mathrm{Pic}^\nabla(\mathrm{Spec}\,(K))$ is an injection.*

Proof. In view of the interpretation of Pic^∇ as $\Gamma(\Omega^1/d\log \mathcal{O}^\times)$, the proposition follows from the fact that for a meromorphic function g on S, we have g regular at a point $s \in S$ if and only if $\frac{dg}{g}$ is regular at s. □

Thus, we do not lose information by taking the base S to be the spectrum of a function field, $S = \mathrm{Spec}\,(K)$. We shall restrict ourselves to that case.

Let

$$(2.2)\qquad \mathcal{D} = \sum_{x \in D(\overline{K})} m_x x$$

be an effective divisor supported on D. Suppose that the relative connection has poles of order bounded by $\mathcal{D}$, i.e.

$$(2.3)\qquad \nabla_{X/S} : E \to E \otimes \omega(\mathcal{D}),$$

and that $\mathcal{D}$ is minimal with this property. (We frequently write ω in place of $\Omega_{X/S}$.) We view $\mathcal{D}$ as an artinian scheme with sheaf of functions $\mathcal{O}_\mathcal{D}$ and relative dualizing sheaf $\omega_\mathcal{D} := \omega(\mathcal{D})/\omega$. (To simplify, we do not use the notation $\omega_{\mathcal{D}/K}$).

Proposition 2.2. *Define* $E_{\mathcal{D}} := E \otimes \mathcal{O}_{\mathcal{D}}$. *then* $\nabla_{X/S}$ *induces a function-linear map*

$$\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$$

Proof. Straightforward. □

Let $j : X - D \hookrightarrow X$. There are now two relative de Rham complexes we might wish to study

$$\begin{gathered} j_* j^* E \to j_* j^* (E \otimes \omega) \\ E \to E \otimes \omega(\mathcal{D}). \end{gathered}$$

The first is clearly the correct one. For example, its relative cohomology carries the Gauß-Manin connection. On the other hand, the second complex is sometimes easier to study, as the relative cohomology of the sheaves involved has finite dimension over K. Since we are primarily interested in the irregular case, that is when points of $\mathcal{D}$ have multiplicity ≥ 2, the following proposition clarifies the situation.

Proposition 2.3. *Let notation be as above, and assume every point of* $\mathcal{D}$ *has multiplicity* ≥ 2. *Then the natural inclusion of complexes*

$$\iota : \{E \to E \otimes \omega(\mathcal{D})\} \hookrightarrow \{j_* j^* E \to j_* j^* (E \otimes \omega)\}$$

is a quasiisomorphism if and only if $\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$ *in proposition 2.2 is an isomorphism.*

Proof. Let $\mathcal{D} : h = 0$ be a local defining equation for $\mathcal{D}$. Write $\mathcal{D} = \mathcal{D}' + D$ where D is the reduced divisor with support $= \mathrm{supp}(\mathcal{D})$. We claim first that the map ι is a quasiisomorphism if and only if for all $n \geq 1$ the map G defined by the commutative diagram

$$\begin{tikzcd} E/E(-\mathcal{D}) \arrow[r, "\nabla_{X/S,\mathcal{D}} - n \cdot id \otimes \frac{dh}{h}"] \arrow[d, "\cong"', "\text{“}\cdot h^{-n}\text{”}"] & (E(\mathcal{D}')/E(-D)) \otimes \omega(D) \arrow[d, "\cong"', "\text{“}\cdot h^{-n}\text{”}"] \\ E(n\mathcal{D})/E((n-1)\mathcal{D}) \arrow[r, "G"] & \Big(E(n\mathcal{D} + \mathcal{D}')/E((n-1)\mathcal{D} + \mathcal{D}')\Big) \otimes \omega(D) \end{tikzcd}$$

given by $h^{-n} e \mapsto h^{-n} \nabla_{X/S,\mathcal{D}}(e) - n h^{-n-1} e \otimes dh$ is a quasi-isomorphism. This follows by considering the cokernel of ι

$$j_* j^* E/E \to j_* j^* (E \otimes \omega)/E \otimes \omega(\mathcal{D})$$

and filtering by order of pole. The assertion of the proposition follows because $n h^{-n-1} e \otimes dh$ has a pole of order strictly smaller than the multiplicity of $(n+1)\mathcal{D}$ at every point of $\mathcal{D}$. □

We will consider only the case

$$\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \cong E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}. \tag{2.4}$$

We now consider the sheaf $j_*j^*\Omega^1_X$ of absolute (i.e. relative to k) 1-forms on X with poles on D. Let $\mathcal{D} = \sum m_x x$ be an effective divisor supported on D as above, and write $\mathcal{D}' = \mathcal{D} - D$.

Definition 2.4. *The sheaf $\Omega^p_X\{\mathcal{D}\} \subset j_*j^*\Omega^p_X$ is defined locally around a point $x \in D$ with local coordinate z by*

$$\Omega^p_X\{\mathcal{D}\}_x = \Omega^p_X(\mathcal{D}')_x + \Omega^{p-1}_{X,x} \wedge \frac{dz}{z^{m_x}}.$$

The graded sheaf $\bigoplus_p \Omega^p_X\{\mathcal{D}\}$ is stable under the exterior derivative and independent of the choice of local coordinates at the points of D. One has exact sequences

$$0 \to f^*\Omega^p_S(\mathcal{D}') \to \Omega^p_X\{\mathcal{D}\} \to \Omega^{p-1}_S(\mathcal{D}') \otimes \omega(D) \to 0. \tag{2.5}$$

Definition 2.5. *An integrable absolute connection on E will be called admissible if there exists a divisor $\mathcal{D}$ such that $\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\}$ and such that $\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \cong E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$.*

Remark 2.6. *When E has rank 1, there always exists a $\mathcal{D}$ such that ∇ is admissible for $\mathcal{D}$ ([2], lemma 3.1). In higher rank, this need not be true, even if $\nabla_{X/S,\mathcal{D}}$ is an isomorphism for some $\mathcal{D}$. For example, let $\eta \in \Omega^1_K$ be a closed 1-form. Let $n \geq 1$ be an integer and let $c \in k,\ c \neq 0$. The connection matrix*

$$A = \begin{pmatrix} \frac{cdz}{z^m} & \frac{\eta}{z^n} \\ 0 & \frac{cdz}{z^m} - \frac{ndz}{z} \end{pmatrix}$$

satisfies $dA + A \wedge A = 0$ for all $m, n \in \mathbb{N}$, but the resulting integrable connection is not admissible for $n > m$, although $\nabla_{X/S,(0)}$ is an isomorphism for $c \neq n$ if $m = 1$. Note in this case it is possible to change basis to get an admissible connection. We don't know what to expect in general. There do exist connections for which $\nabla_{X/S,\mathcal{D}}$ is not an isomorphism for any $\mathcal{D}$, for example, if one takes a sum of rank 1 connections with different m_x as above (see notations (2.2)) and a local basis adapted to this direct sum decomposition.

Henceforth, $S = \mathrm{Spec}\,(K)$ is the spectrum of a function field, and we consider only integrable, admissible connections $\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\}$ with $\mathcal{D} \neq \emptyset$. In sections 3 and 4 we will see many importnat examples (Fourier transforms, Kloosterman sheaves) of admissible connections. By abuse of notation, we write

$$H^*_{DR/S}(E) := \mathbb{H}^*(X, E \to E \otimes \omega(\mathcal{D})). \tag{2.6}$$

We assume (proposition 2.3) this group coincides with $H^*_{DR}(E|_{X-D})$. The isomorphism class of the Gauß-Manin connection on the K-line $\det H^*_{DR/S}(E)$ is determined by an element

$$\det H^*_{DR/S}(E) \in \Omega^1_{K/k}/d\log(K^\times)$$

which we would like to calculate.

Suppose first that E is a line bundle. Twisting E by $\mathcal{O}(\delta)$ for some divisor δ supported on the irregular part of the divisor D, we may assume $\deg E = 0$. In this case, the result (the main theorem in [2]) is the following. Since E has rank 1, $\nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \cong E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}$ can be interpreted as a section of $\omega_{\mathcal{D}}$ which generates this sheaf as an $\mathcal{O}_{\mathcal{D}}$-module. The exact sequence

$$0 \to \omega \to \omega(\mathcal{D}) \to \omega_{\mathcal{D}} \to 0 \tag{2.7}$$

yields an element $\partial\nabla_{X/S,\mathcal{D}} \in H^1(X, \omega) \cong K$ which is known to equal $\deg E = 0$. Thus, we can find some $s \in H^0(X, \omega(\mathcal{D}))$ lifting $\nabla_{X/S,\mathcal{D}}$. We write (s) for the divisor of s as a section of $\omega(\mathcal{D})$ (so (s) is disjoint from D). Then the result is

$$\det H^*_{DR/S}(E) \cong -f_*((s) \cdot E). \tag{2.8}$$

(When (s) is a disjoint union of K-points, the notation on the right simply means to restrict E with its absolute connection to each of the points and then tensor the resulting K-lines with connection together.) Notice that unlike the classical Riemann-Roch situation (e.g. (1.3)) the divisor (s) depends on $(E, \nabla_{X/S})$.

Another way of thinking about (2.8) will be important when we consider periods. It turns out that the connection (E, ∇) pulls back from a rank 1 connection $(\mathcal{E}, \nabla_{\mathcal{E}})$ on the relative Picard scheme $\mathrm{Pic}(X, \mathcal{D})$ whose points are isomorphism classes of line bundles on X with trivializations along $\mathcal{D}$. The pair $(\omega(\mathcal{D}), \nabla_{X/D,\mathcal{D}})$ determine a point $t \in \mathrm{Pic}(X, \mathcal{D})(K)$, and (2.8) is equivalent to

$$\det H^*_{DR/S}(E) \cong -(\mathcal{E}, \nabla_{\mathcal{E}})|_t. \tag{2.9}$$

Let $\omega^\times_{\mathcal{D}} \subset \omega_{\mathcal{D}}$ be the subset of elements generating $\omega_{\mathcal{D}}$ as an $\mathcal{O}_{\mathcal{D}}$-module. Let $\partial : \omega_{\mathcal{D}} \to H^1(X, \omega) = K$. Define $\tilde{B} = \omega^\times_{\mathcal{D}} \cap \partial^{-1}(0)$. One has a natural action of $K^\times$, and the quotient $\omega^\times_{\mathcal{D}}/K^\times$ is identified with isomorphism classes of trivializations of $\omega(\mathcal{D})|_{\mathcal{D}}$, and hence with a subvariety of $\mathrm{Pic}(X, \mathcal{D})$. One has

$$t \in B := \tilde{B}/K^\times \subset \omega^\times_{\mathcal{D}}/K^\times \subset \mathrm{Pic}(X, \mathcal{D}). \tag{2.10}$$

The relation between $t, B, \mathcal{E}$ is the following. $\mathcal{E}|_B \cong \mathcal{O}_B$, so the connection $\nabla_{\mathcal{D}}|_B$ is determined by a global 1-form Ξ. Then

$$\Xi(t) = 0 \in \Omega^1_{B/K} \otimes K(t). \tag{2.11}$$

Indeed, t is the unique point on B where the relative 1-form Ξ/K vanishes (cf [2], Lemma 3.10).

Now suppose the rank of E is > 1. We will see when we consider examples in the next section that $\det H^*_{DR/S}(E)$ depends on more than just the connection on $\det E$ (remark 3.3). Thus, it is hard to imagine a simple formula like (2.8). Indeed, there is no obvious way other than by taking the determinant to get rank 1 connections on X from E. The truly surprising thing is that if we rewrite (2.8) algebraically we find a formula which does admit a plausible generalization. We summarize the results, omitting proofs (which are given in detail in [2]). For each $x_i \in D$, choose a local section s_i of $\Omega^1_X\{\mathcal{D}\}$ whose image in $\omega(\mathcal{D})$ generates at x_i. Write the local connection matrix in the form

$$A_i = g_i s_i + \frac{\eta_i}{z_i^{m_i-1}} \tag{2.12}$$

with z_i a local coordinate and $\eta_i \in f^*\Omega^1_S$. Let s be a meromorphic section of $\omega(\mathcal{D})$ which is congruent to the $\{s_i\}$ modulo $\mathcal{D}$. Define

$$c_1(\omega(\mathcal{D}), \{s_i\}) := (s) \in \mathrm{Pic}(X, \mathcal{D}). \tag{2.13}$$

As in (2.8), we can define

$$f_*(c_1(\omega(\mathcal{D}), \{s_i\}) \cdot \det(E, \nabla)) \in \Omega^1_K / d\log K^\times. \tag{2.14}$$

We further define

(2.15)

$$\{c_1(\omega(\mathcal{D})), \nabla\} := f_*\Big(c_1(\omega(\mathcal{D}), \{s_i\}) \cdot \det(E, \nabla)\Big) - \sum_i \mathrm{res}\ \mathrm{Tr}(dg_i g_i^{-1} A_i).$$

Here res refers to the map

$$\Omega^2_X\{\mathcal{D}\} \to \Omega^1_K \otimes \omega(\mathcal{D}) \to \Omega^1_K \otimes \omega_{\mathcal{D}} \xrightarrow{\text{transfer}} \Omega^1_K. \tag{2.16}$$

Conjecture 2.7. *Let $\nabla : E \to E \otimes \Omega^1_X\{\mathcal{D}\}$ be an admissible connection as in definition 2.5. Then*

$$\det H^*_{DR/S}(X - D, E) = -\{c_1(\omega(\mathcal{D})), \nabla\} \in \Omega^1_K / d\log(K^\times) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Our main objective here is to provide evidence for this conjecture. Of course, one surprising fact is that the right hand side is independent of choice of gauge, etc. Again, the proof is given in detail in [2] but we reproduce two basic lemmas. There are function linear maps

(2.17)
$$\nabla_{X,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \left(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\right); \quad \nabla_{X/S,\mathcal{D}} : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \omega_{\mathcal{D}},$$

and it makes sense to consider the commutator

$$[\nabla_{X,\mathcal{D}}, \nabla_{X/S,\mathcal{D}}] : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \left(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\right) \otimes \omega_{\mathcal{D}}.$$

Lemma 2.8. $[\nabla_{X,\mathcal{D}}, \nabla_{X/S,\mathcal{D}}] = 0.$

Proof. With notation as in (2.12), we take $s_i = \frac{dz_i}{z_i^{m_i}}$, where z_i is a local coordinate. Integrability implies

$$dA_i = dg_i \wedge \frac{dz_i}{z_i^{m_i}} + d\Big(\frac{\eta_i}{z_i^{m_i-1}}\Big) = A_i^2 = [\eta_i, g_i]\frac{dz_i}{z_i^{2m_i-1}} + \epsilon \tag{2.18}$$

with $\epsilon \in \Omega^2_K \otimes K(X)$. Multiplying through by $z_i^{m_i}$, we conclude that $[\frac{\eta_i}{z_i^{m_i-1}}, g_i]$ is regular on D, which is equivalent to the assertion of the lemma. □

The other lemma which will be useful in evaluating the right hand term in (2.15) is

Lemma 2.9. *We consider the situation from* (2.12) *and* (2.15) *at a fixed $x_i \in D$. For simplicity, we drop the i from the notation. Assume $ds = 0$. Then*

$$\text{res Tr}(dgg^{-1}A) = \text{res Tr}(dgg^{-1}\frac{\eta}{z^{m-1}}).$$

Proof. We must show res $\text{Tr}(dgs) = 0$. Using (2.18) and $\text{Tr}[g,\eta] = 0$ we reduce to showing $0 = \text{res Tr}(d(\eta z^{1-m})) \in \Omega^1_K$. Since $\eta \in f^*\Omega^1_K$, we may do the computation formally locally and replace d by d_z. The desired vanishing follows because an exact form has no residues. □

With lemma 2.8, we can formulate the conjecture in a more invariant way in terms of an AD-cocycle on X. Recall [4]

$$AD^2(X) := \mathbb{H}^2(X, \mathcal{K}_2 \xrightarrow{d\log} \Omega^2_X) \cong H^1(X, \Omega^2_X/d\log\mathcal{K}_2) \tag{2.19}$$

The AD-groups are the cones of cycle maps from Chow groups to Hodge cohomology, and as such they carry classes for bundles with connections. There is a general trace formalism for the AD-groups, but in this simple case the reader can easily deduce from the right hand isomorphism in (2.19) a trace map

$$f_* : AD^2(X) \to AD^1(S) = \Omega^1_K/d\log K^\times. \tag{2.20}$$

When the connection ∇ has no poles (or more generally, when it has regular singular points) it is possible to define a class

$$\epsilon = c_1(\omega)\cdot c_1(E,\nabla) \in AD^2(X) \tag{2.21}$$

with $f_*(\epsilon) = [\det H^1_{DR/S}(X,E)]$. Remarkably, though it no longer has the product description (2.21), one can associate such a class to any admissible connection.

Fix the divisor $\mathcal{D}$ and consider tuples $\{E,\nabla,\mathcal{L},\mu\}$ where (E,∇) is an admissible, absolute connection, $\mathcal{L}$ is a line bundle on X, and $\mu : E_{\mathcal{D}} \cong E_{\mathcal{D}}\otimes\mathcal{L}_{\mathcal{D}}$. We require

$$0 = [\mu, \nabla_{X,\mathcal{D}}] : E_{\mathcal{D}} \to E_{\mathcal{D}}\otimes\mathcal{L}_{\mathcal{D}}\otimes\Big(\Omega^1_X\{\mathcal{D}\}/\Omega^1_X\Big). \tag{2.22}$$

Of course, the example we have in mind, using lemma 2.8, is

$$\{E,\nabla\} := \{E,\nabla,\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}}\}. \tag{2.23}$$

To such a tuple satisfying (2.22), we associate a class $\epsilon(E,\nabla,\mathcal{L},\mu)\in AD^2(X)$ as follows. Choose cochains $c_{ij}\in GL(r,\mathcal{O}_X)$ for E, $\lambda_{ij}\in\mathcal{O}_X^\times$ for $\mathcal{L}$, $\mu_i\in GL(r,\mathcal{O}_{\mathcal{D}})$ for μ, and $\omega_i\in M(r\times r,\Omega^1_X\{\mathcal{D}\})$ for ∇. Choose local liftings $\tilde{\mu}_i\in GL(r,\mathcal{O}_X)$ for the μ_i.

Proposition 2.10. *The Cech hypercochain*

$$\Big(\{\lambda_{ij},\det(c_{jk})\}, d\log\lambda_{ij}\wedge\mathrm{Tr}(\omega_j), \mathrm{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i{}^{-1}\wedge\omega_i)\Big)$$

represents a class

$$\epsilon(E,\nabla,\mathcal{L},\mu)\in\mathbb{H}^2(X,\mathcal{K}_2\to\Omega^2_X\{\mathcal{D}\}\to\Omega^2_X\{\mathcal{D}\}/\Omega^2_X)\cong AD^2(X).$$

This class is well defined independent of the various choices. Writing $\epsilon(E,\nabla)=\epsilon(E,\nabla,\omega(\mathcal{D}),\nabla_{X/S,\mathcal{D}})$, *we have*

$$f_*\epsilon(E,\nabla) = \{c_1(\omega(\mathcal{D})),\nabla\}$$

where the right hand side is defined in (2.15).

Proof. Again the proof is given in detail in [2] and we omit it. ☐

As a consequence, we can restate the main conjecture:

Conjecture 2.11. *Let* ∇ *be an integrable, admissible, absolute connection as above. Then*

$$\det H^*_{DR/S}(E,\nabla) = -f_*\epsilon(E,\nabla).$$

To finish this section, we would like to show that behind the quite technical cocyle written in proposition 2.10, there is an algebraic group playing a role similar to $\mathrm{Pic}(X,\mathcal{D})$ in the rank 1 case. Let G be the algebraic group whose K-points are isomorphism classes $(\mathcal{L},\mu)$, where $\mathcal{L}$ is an invertible sheaf, and $\mu : E_{\mathcal{D}} \to E_{\mathcal{D}} \otimes \mathcal{L}_{\mathcal{D}}$ is an isomorphism commuting with $\nabla_{X,\mathcal{D}}$. It is endowed with a surjective map $q : G \to \mathrm{Pic}(X)$. As noted, G contains the special point $(\omega(\mathcal{D}), \nabla_{X/S,\mathcal{D}})$. The cocyle of proposition 2.10 defines a class in $\mathbb{H}^2(X \times_K G, \mathcal{K}_2 \to \Omega^2_{X\times G}\{\mathcal{D} \times G\} \to \Omega^2_{X\times G}\{\mathcal{D} \times G\}/\Omega^2_{X\times G})$. Taking its trace (2.20), one obtains a class in $(\mathcal{L}(E), \nabla(E)) \in AD^1(G)$, that is a rank one connection on G. Then $f_*\epsilon(E,\nabla)$ is simply the restriction of $(\mathcal{L}(E), \nabla(E))$ to the special point $(\omega(\mathcal{D}), \nabla_{X/S,\mathcal{D}})$.

Now we want to show that this special point, as in the rank 1 case, has a very special meaning. By analogy with (2.10), we define

(2.24)
$$\tilde{B} = \Big(\mathrm{Ker}(\mathrm{Hom}(E_{\mathcal{D}}, E_{\mathcal{D}} \otimes \omega_{\mathcal{D}}) \xrightarrow{\text{res Tr}} K)\Big) \cap \mathrm{Isom}(E_{\mathcal{D}}, E_{\mathcal{D}} \otimes \omega_{\mathcal{D}})$$

(2.25)
$$B = \tilde{B}/K^\times \subset G.$$

We observe that lemma 2.9 shows that $g \in B$. Choosing a local trivialization $\omega_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}\frac{dz}{z^m}$ and a local trivialization of $E_{\mathcal{D}}$, we write $\nabla_{X/S}|_{\mathcal{D}}$ as a matrix $g\frac{dz}{z^m}$, with $g \in GL_r(\mathcal{O}_{\mathcal{D}})$. Θ is then identified with a translation invariant form on the restriction of scalars $\mathrm{Res}_{\mathcal{D}/K}GL_r$

(2.26)
$$\Theta := (\mathcal{L}(E), \nabla_{G/K}(E)) = \text{res Tr}(d\mu\mu^{-1}g\frac{dz}{z^m}).$$

The assumption that $g \in B$ implies that Θ descends to an invariant form on $\mathrm{Res}_{\mathcal{D}/K}GL_r/\mathbb{G}_m \supset G_0$, where $G_0 := \{(\mathcal{O}, \mu) \in G\}$. By invariance, it gives rise to a form on the G_0-torsor $G_{\omega(\mathcal{D})} := \{(\omega(\mathcal{D}), \mu) \in G\}$. We have

$$B \subset G_{\omega(\mathcal{D})} \subset G.$$

Let $S \subset G_0$ be the subgroup of points stabilizing B.

Proposition 2.12. *$\Theta|_B$ vanishes at a point $t \in B$ if and only if t lies in the orbit $g \cdot S$.*

Proof. Write $th = g$. Write the universal element in $\mathrm{Res}_{\mathcal{D}/K}GL_r$ as a matrix $X = \sum_{k=0}^{m-1}(X^{(k)}_{ij})_{ij}z^k$. The assertion that $\Theta|_B$ vanishes in the fibre at t means

$$\text{res Tr}(\sum_k (d(X^{(k)}_{ij})_{ij}z^k)h\frac{dz}{z^m})(t) = a(\sum_i dX^{(m-1)}_{ii})(t)$$

for some $a \in K$. Note this is an identity of the form

(2.27)
$$0 = \sum c^{(k)}_{ij}dX^{(k)}_{ij} \in \Omega^1_{G/K} \otimes K(t); \quad c^{(k)}_{ij} \in K.$$

We first claim that in fact this identity holds already in $\Omega^1_{G/K}$. To see this, write $\mathcal{G} = \mathrm{Res}_{\mathcal{D}/K} GL_r$. Note $\Omega^1_{\mathcal{G}}$ is a free module on generators $dX^{(k)}_{ij}$. Also, $G \subset \mathcal{G}$ is defined by the equations

$$[\sum_{k=0}^{m-1}(X^{(k)}_{ij})_{ij}z^k, \sum_{k=0}^{m-1}(g^{(k)}_{ij})_{ij}z^k] = 0$$
$$[\sum_{k=0}^{m-1}(X^{(k)}_{ij})_{ij}z^k, \sum_{k=0}^{m-2}(\eta^{(k)}_{ij})_{ij}z^k] = 0,$$

which are of the form

$$\sum_{i,j,k} b^{(k)}_{ijp}X^{(k)}_{ij} = 0, \quad p = 1, 2, \ldots, M; \quad b^{(k)}_{ijp} \in K,$$

that is are linear equations in the $X^{(k)}_{ij}$ with K-coefficients. Thus, we have an exact sequence

$$0 \to N^\vee \to \Omega^1_{\mathcal{G}/K} \otimes \mathcal{O}_G \to \Omega^1_{G/K} \to 0, \tag{2.28}$$

where $N^\vee$ is generated by K-linear combinations of the $dX^{(k)}_{ij}$. We have, therefore, a reduction of structure of the sequence (2.28) from $\mathcal{O}_G$ to K, and therefore $\Omega^1_{G/K} \cong \Omega^1_0 \otimes_K \mathcal{O}_G$, where $\Omega^1_0 \subset \Omega^1_{G/K}$ is the K-span of the $dX^{(k)}_{ij}$. Hence, any K-linear identity among the $dX^{(k)}_{ij}$ which holds at a point on G holds everywhere on G. As a consequence, we can integrate to an identity

$$\mathrm{res}\ \mathrm{Tr}(Xh\frac{dz}{z^m}) = a(\sum_i X^{(m-1)}_{ii}) + \kappa \tag{2.29}$$

with $\kappa \in K$. If we specialize $X \to t$ we find

$$0 = \mathrm{res}\ \mathrm{Tr}(g\frac{dz}{z^m}) = a \cdot \mathrm{res}\ \mathrm{Tr}(t\frac{dz}{z^m}) + \kappa = \kappa \tag{2.30}$$

We conclude from (2.29) and (2.30) that $h \in S$. □

3. The Fourier Transform

In this section we calculate the Gauß-Manin determinant line for the Fourier transform of a connection on $\mathbb{P}^1 - D$ and show that it satisfies the conjecture 2.7. Let $\mathcal{D} = \sum m_\alpha \alpha$ be an effective k-divisor on $\mathbb{P}^1_k$. Let $\mathcal{E} = \oplus_r \mathcal{O}$ be a rank r free bundle on $\mathbb{P}^1_k$, and let $\Psi : \mathcal{E} \to \mathcal{E} \otimes \omega(\mathcal{D})$ be a k-connection on $\mathcal{E}$. Let $(\mathcal{L}, \Xi)$ denote the rank 1 connection on $\mathbb{P}^1 \times \mathbb{P}^1$ with poles on $\{0, \infty\} \times \{0, \infty\}$ given by $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ and $\Xi(1) = d(\frac{z}{t})$.

Here z, t are the coordinates on the two copies of the projective line. Let $K = k(t)$. We have a diagram

$$\begin{array}{ccc} (\mathbb{P}^1_z - \mathcal{D}) \times \mathbb{P}^1_t & \hookleftarrow & (\mathbb{P}^1_z - \mathcal{D}) \times \mathrm{Spec}\,(K) \\ \downarrow p_1 & & \downarrow p_2 \\ \mathbb{P}^1_z - \mathcal{D} & & \mathrm{Spec}\,(K) \end{array} \tag{3.1}$$

The Gauß-Manin determinant of the Fourier transform is given at the generic point by

$$\det H^*_{DR/K}\Big((\mathbb{P}^1_z - \mathcal{D})_K,\ p_1^*(\mathcal{E}, \Psi) \otimes (\mathcal{L}, \Xi)\Big) = \det H^*_{DR/K}\Big((\mathbb{P}^1_z - \mathcal{D})_K, (E, \nabla)\Big) \tag{3.2}$$

with $E := p_1^*\mathcal{E} \otimes \mathcal{L}|_{(\mathbb{P}^1_z - \mathcal{D})_K}$ and $\nabla = \Psi \otimes 1 + 1 \otimes \Xi$. We have the following easy

Remark 3.1. *Write*

$$\Psi = \sum_\alpha \sum_{i=1}^{m_\alpha} \frac{g_i^\alpha dz}{(z-\alpha)^i} + d(g_1^\infty z + \ldots + g_{m_\infty - 1}^\infty z^{m_\infty - 1}) \tag{3.3}$$

where $g_i^\alpha \in M(r \times r, k)$. Then

$$\nabla = \Psi + \frac{dz}{t} - \frac{z dt}{t^2} \tag{3.4}$$

is admissible if and only if either

(i) *$m_\infty \leq 2$ and $g_{m_\alpha}^\alpha$ is invertible for all $\alpha \neq \infty$, or*
(ii) *$m_\infty \geq 3$, $g_{m_\alpha}^\alpha$ is invertible for all $\alpha \neq \infty$, and $g_{m_\infty - 1}^\infty$ is invertible.*

Theorem 3.2. *The connection (E, ∇) satisfies conjecture 2.7.*

Proof. We first consider the case when Ψ has a pole of order ≤ 1 at infinity, so the $g_i^\infty = 0$ in (3.3). A basis for

$$H^0(\mathbb{P}^1_K, E \otimes \omega(\sum m_\alpha \alpha + 2\infty))$$

is given by

$$e_j \otimes dz; \quad e_j \otimes \frac{dz}{(z-\alpha)^i},\ 1 \leq i \leq m_\alpha,\ 1 \leq j \leq r. \tag{3.5}$$

$H^0_{DR/K} = (0)$ and $H^1_{DR/K} = \mathrm{coker}(H^0(E) \to H^0(E \otimes \omega(\sum m_\alpha \alpha + 2\infty)))$ has basis

$$e_j \otimes \frac{dz}{(z-\alpha)^i}; \quad 1 \leq i \leq m_\alpha,\ 1 \leq j \leq r. \tag{3.6}$$

To compute the Gauß-Manin connection, we consider the diagram (here $\mathcal{D} = \sum m_\alpha \alpha + 2\infty$ and $\mathcal{D}' = \mathcal{D} - D = \sum (m_\alpha - 1)\alpha + \infty$)

$$
(3.7)\quad \begin{array}{ccccccc}
 & & & & H^0(E) & = & H^0(E) \\
 & & & & \downarrow \nabla_X & & \downarrow \nabla_{X/S} \\
0 \longrightarrow & & H^0(E(\mathcal{D}')) \otimes \Omega^1_K & \longrightarrow & H^0(E \otimes \Omega^1_{\mathbb{P}^1}\{\mathcal{D}\}) & \xrightarrow{a} & H^0(E \otimes \omega(\mathcal{D})) \longrightarrow 0 \\
 & & \downarrow \nabla_{X/S} \otimes 1 & & \downarrow \nabla_X & & \\
 & & H^0(E(\mathcal{D}') \otimes \omega(\mathcal{D})) \otimes \Omega^1_K & \xrightarrow{\cong} & H^0(E \otimes \Omega^2_{\mathbb{P}^1}\{\mathcal{D} + \mathcal{D}'\}) & &
\end{array}
$$

One deduces from this diagram the Gauß-Manin connection

(3.8)

$$H^1_{DR/K}(E) \cong \mathrm{coker}(\nabla_{X/S}) \xrightarrow{\nabla_{GM}} H^1_{DR/K}(E) \otimes \Omega^1_K; \quad w \mapsto \nabla_X(a^{-1}(w)).$$

We may choose $a^{-1}(e_j \otimes \frac{dz}{(z-\alpha)^i}) = e_j \otimes \frac{dz}{(z-\alpha)^i}$, so by (3.4)

$$(3.9)\quad \nabla_{GM}\Big(e_j \otimes \frac{dz}{(z-\alpha)^i}\Big) = \nabla_X\Big(e_j \otimes \frac{dz}{(z-\alpha)^i}\Big) = e_j \otimes \frac{zdz \wedge dt}{(z-\alpha)^i t^2}$$

In $H^1_{DR/K} \cong \mathrm{coker}(\nabla_{X/S})$ we have the identity

$$(3.10)\quad e_j \otimes dz = -t\Psi e_j.$$

We conclude

$$(3.11)\quad \nabla_{GM}\Big(e_j \otimes \frac{dz}{(z-\alpha)^i}\Big)$$
$$= \begin{cases} \Big(e_j \otimes \frac{dz}{(z-\alpha)^{i-1}} + \alpha e_j \otimes \frac{dz}{(z-\alpha)^i}\Big) \wedge \frac{dt}{t^2} & 2 \le i \le m_\alpha \\ -t\Psi e_j + \alpha e_j \otimes \frac{dz}{z-\alpha}\Big) \wedge \frac{dt}{t^2} & i = 1. \end{cases}$$

In particular, the determinant connection, which is given by $\mathrm{Tr}\nabla_{GM}$, can now be calculated:

$$(3.12)\quad \mathrm{Tr}\nabla_{GM} = \sum_\alpha \frac{rm_\alpha \alpha dt}{t^2} - \mathrm{Tr}\sum_\alpha \frac{g_1^\alpha dt}{t}.$$

We compare this with the conjectured value which is the negative of (2.15). Define

$$(3.13)\quad F(z) := \sum_\alpha \frac{1}{(z-\alpha)^{m_\alpha}} - 1 = \frac{G(z)}{(z-\alpha)^{m_\alpha}}; \quad s := F(z)dz.$$

One has

$$(3.14)\quad c_1(\omega(\mathcal{D}), \Big\{\frac{dz}{(z-\alpha)^{m_\alpha}}, dz\Big\}) = (G),$$

the divisor of zeroes of G. We need to compute $(\det E, \det\nabla)|_{(G)}$. We have

$$(3.15)\quad G(z) = \sum_\alpha \prod_{\beta\neq\alpha}(z-\beta)^{m_\beta} - \prod_\alpha (z-\alpha)^{m_\alpha}$$
$$= -z^{\sum m_\alpha} + \Big(\sum m_\alpha\alpha + \#\{\alpha\,|\,m_\alpha = 1\}\Big) z^{(\sum m_\alpha)-1} + \dots$$

Note that the coefficients of G do not involve t, so the dz part of the connection dies on (G) and we get

$$(3.16)\quad \mathrm{Tr}\nabla|_{(G)} = \frac{-rzdt}{t^2}|_{(G)} = -\frac{rdt}{t^2}\sum_{\substack{\beta \\ G(\beta)=0}} \beta$$
$$= -\frac{rdt}{t^2}\Big(\sum m_\alpha\alpha + \#\{\alpha\,|\,m_\alpha = 1\}\Big)$$

It remains to evaluate the correction terms res $\mathrm{Tr}(dgg^{-1}A)$ occurring in (2.15). In the notation of (2.12), $\frac{\eta}{z^{m-1}} = \frac{-zdt}{t^2}$, and by lemma 2.9 we have res $\mathrm{Tr}(dgg^{-1}A) = -$res $\mathrm{Tr}(dgg^{-1}\frac{zdt}{t^2})$. Clearly, the only contribution comes at $z=\infty$. Take $u = z^{-1}$. At ∞ the connection is

$$(3.17)\qquad A = -\Big(\sum_\alpha\sum_i \frac{g_i^\alpha u^i}{(1-u\alpha)^i} + \frac{1}{t}\Big)\frac{du}{u^2} - \frac{dt}{ut^2}.$$

We rewrite this in the form $A = gs + \frac{\eta}{u}$ as in (2.12) with s as in (3.13) and $\eta = -\frac{dt}{t^2}$. We find

$$(3.18)\qquad g = \frac{\sum_{i,\alpha}\frac{g_i^\alpha u^i}{(1-u\alpha)^i} + t^{-1}}{\sum_\alpha \frac{u^{m_\alpha}}{(1-u\alpha)^{m_\alpha}} - 1} = \frac{\kappa}{v},$$

(defining κ and v to be the numerator and denominator, respectively.) Then

$$(3.19)\quad \text{res } \mathrm{Tr}(dgg^{-1}A) = -\text{res } \mathrm{Tr}(dgg^{-1}\frac{dt}{ut^2})$$
$$= \Big(-\text{res } \mathrm{Tr}(d\kappa\kappa^{-1}u^{-1}) + r\cdot\text{res } \mathrm{Tr}(dvv^{-1}u^{-1})\Big)\frac{dt}{t^2}$$
$$= \Big(-t\sum \mathrm{Tr}(g_1^\alpha) - r\#\{\alpha\,|\,m_\alpha = 1\}\Big)\frac{dt}{t^2}$$

Combining (3.19), (3.16), and (3.12) we conclude

$$(3.20)\qquad \mathrm{Tr}\nabla_{GM} = -\Big(\mathrm{Tr}\nabla|_{(G)} - \text{res } \mathrm{Tr}(dgg^{-1}A)\Big),$$

which is the desired formula.

We turn now to the case where Ψ has a pole of order ≥ 2 at infinity. We write

$$\Psi = \sum_{\alpha} \sum_{i=1}^{m_\alpha} \frac{g_i^\alpha dz}{(z-\alpha)^i} + g^\infty dz; \quad g^\infty = g_2^\infty + \ldots + g_{m_\infty}^\infty z^{m_\infty - 2} \tag{3.21}$$

$$\nabla = \Psi + \frac{dz}{t} - \frac{z dt}{t^2}. \tag{3.22}$$

A basis for $\Gamma(\mathbb{P}^1, \omega(\sum m_\alpha \alpha + m_\infty \infty))$ is given by

$$e_j \otimes \frac{dz}{(z-\alpha)^i}; \;\; 1 \leq i \leq m_\alpha; \quad e_j \otimes z^i dz; \; 0 \leq i \leq m_\infty - 2. \tag{3.23}$$

A basis for the Gauß-Manin bundle is given by omitting $e_j \otimes z^{m_\infty - 2} dz$. As in (3.7)-(3.9), the Gauß-Manin connection is

$$w \mapsto zw \wedge \frac{dt}{t^2}. \tag{3.24}$$

To compute the trace, note that in $H^1_{DR/K}$, we have if $m_\infty \geq 3$

$$\begin{aligned} &e_j \otimes z^{m_\infty - 2} dz \\ &\quad = -(g_{m_\infty}^\infty)^{-1} \Big(\sum_{i,\alpha} g_i^\alpha(e_j) \otimes \frac{dz}{(z-\alpha)^i} + \\ &\qquad (g_2^\infty(e_j) + \ldots + g_{m_\infty - 1}^\infty(e_j) z^{m_\infty - 3} + t^{-1} e_j) \otimes dz \Big). \end{aligned} \tag{3.25}$$

If $m_\infty = 2$,

$$e_j \otimes dz = -(g_2^\infty + t^{-1})^{-1} \Big(\sum_{i,\alpha} g_i^\alpha(e_j) \otimes \frac{dz}{(z-\alpha)^i} \Big). \tag{3.26}$$

It follows that

(3.27)

$$\mathrm{Tr}\nabla_{GM} = \begin{cases} \Big(\sum_\alpha r m_\alpha \alpha - \mathrm{Tr}\big((g_{m_\infty}^\infty)^{-1} g_{m_\infty - 1}^\infty\big) \Big) \frac{dt}{t^2} & m_\infty \geq 3 \\ \Big(\sum_\alpha r m_\alpha \alpha - \sum_\alpha \mathrm{Tr}\big((g_2^\infty + t^{-1})^{-1} g_1^\alpha\big) \Big) \frac{dt}{t^2} & m_\infty = 2 \end{cases}$$

To compute the right hand side in conjecture 2.7, we take as trivializing section

$$s = \sum_\alpha \frac{dz}{(z-\alpha)^{m_\alpha}} - z^{m_\infty - 2} dz = \frac{G(z) dz}{\prod_\alpha (z-\alpha)^{m_\alpha}} \tag{3.28}$$

where

$$G(z) = \sum_\alpha \prod_{\beta \neq \alpha} (z-\beta)^{m_\beta} - z^{m_\infty - 2} \prod_\alpha (z-\alpha)^{m_\alpha} \tag{3.29}$$

We have $(s) = (G)$, the divisor of zeroes of the polynomial G. Again, G does not involve t, so if $G = -\prod(z - a_k)$, we have

$$\mathrm{Tr}\nabla|_{z=a_p} = -\frac{ra_p dt}{t^2} \tag{3.30}$$

Thus

$$\mathrm{Tr}\nabla|_{(s)} = \begin{cases} -\frac{(\sum rm_\alpha\alpha + r\cdot\#\{\alpha\,|\,m_\alpha=1\})dt}{t^2}, & m_\infty = 2 \\ -\frac{\sum rm_\alpha\alpha dt}{t^2}, & m_\infty \geq 3. \end{cases} \tag{3.31}$$

Finally we have to deal with the correction term res $\mathrm{Tr}(dgg^{-1}(-\frac{zdt}{t^2}))$. There is no contribution except at ∞. We put $u = z^{-1}$ as before, and $g = \frac{\kappa}{v}$ with

$$\kappa = \sum_\alpha \sum_{i=1}^{m_\alpha} \frac{u^{m_\infty-2+i}g_i^\alpha}{(1-\alpha u)^i} + (g_2^\infty + t^{-1})u^{m_\infty-2} + \ldots + g_{m_\infty}^\infty \tag{3.32}$$

$$v = \sum_\alpha \frac{u^{m_\infty-2+m_\alpha}}{(1-\alpha u)^{m_\alpha}} - 1 \tag{3.33}$$

Assume first $m_\infty = 2$. Then

$$\begin{aligned} \text{res } \mathrm{Tr}(dgg^{-1}A) &= -\text{res } \mathrm{Tr}(dgg^{-1}\frac{dt}{ut^2}) \\ &= -\text{res } \mathrm{Tr}(d_z\kappa\kappa^{-1}u^{-1})\frac{dt}{t^2} + r\cdot\text{res } \mathrm{Tr}(d_z vv^{-1}u^{-1})\frac{dt}{t^2} \\ &= -\mathrm{Tr}\Big((g_2^\infty + t^{-1})^{-1}\sum g_1^\alpha\Big)\frac{dt}{t^2} - r\cdot\#\{\alpha\,|\,m_\alpha = 1\}\frac{dt}{t^2} \end{aligned} \tag{3.34}$$

In the case $m_\infty \geq 3$ we find

$$\text{res } \mathrm{Tr}(dgg^{-1}A) = -\mathrm{Tr}\Big((g_{m_\infty}^\infty)^{-1}g_{m_\infty-1}^\infty\Big)\frac{dt}{t^2} \tag{3.35}$$

The theorem follows by comparing (3.27), (3.31), (3.34), and (3.35). □

Remark 3.3. *The presence of nonlinear terms in the g_i^α in (3.27) means that the connection on* $\det H^*_{DR/K}(E)$ *is not determined by* $\det E$ *alone.*

4. Kloosterman Sheaves

In this section we show that the main conjecture holds at least up to 2-torsion for the basic rank 2 Kloosterman sheaf [5]. The base field k is $\mathbb{C}$. Fix $a, b \in K^\times$ (in fact one can work over $K = \mathbb{C}[a, b, a^{-1}, b^{-1}]$). Let $\alpha, \beta \in \mathbb{C} - \mathbb{Z}$ and assume also $\alpha - \beta \in \mathbb{C} - \mathbb{Z}$. Consider two connections $(\mathcal{L}_i, \nabla_i)$ on the trivial bundle on $\mathbb{G}_m$ given by $1 \mapsto \alpha d\log(t) + d(at)$

and $1 \mapsto \beta d\log(u) + d(bu)$ where t, u are the standard parameters on two copies of $\mathbb{G}_m$. Let $X := \mathbb{G}_m \times \mathbb{G}_m$ and consider the exterior tensor product connection on X

(4.1)
$$\mathcal{L} := \mathcal{L}_1 \boxtimes \mathcal{L}_2 = (\mathcal{O}_X, \nabla); \quad \nabla(1) = \alpha d\log(t) + \beta d\log(u) + d(at + bu).$$

Note all the above are integrable, absolute connections.

Proposition 4.1. 1.

$$H^i(DR(\mathcal{L}_i/K) \cong \begin{cases} K & i = 1 \\ 0 & i \neq 1 \end{cases}$$

2.

$$H^p_{DR}(\mathcal{L}/K) \cong \begin{cases} H^1_{DR}(\mathcal{L}_1/K) \otimes H^1_{DR}(\mathcal{L}_2/K) & p = 2 \\ 0 & p \neq 2 \end{cases}$$

Proof. The de Rham complex (of global sections on X) for $(\mathcal{L}, \nabla)$ is the tensor product of the corresponding complexes for the ∇_i, so (2) follows from (1). For (1) we have e.g. the complex of global sections $\mathcal{O} \xrightarrow{\nabla_{1,K}} \omega$, $1 \mapsto \alpha d\log(t) + adt$. In H^1_{DR} this gives for all $n \in \mathbb{Z}$

$$at^n dt \equiv -(\alpha + n)t^{n-1} dt.$$

Assertion (1) follows easily. $\square$

We now compute Gauß-Manin. We will (abusively) use sheaf notation when we mean global sections over $\mathbb{G}_m$ or X. Also we write ∇ for either one of the ∇_i or the exterior tensor connection. ∇_K is the corresponding relative connection. One has the diagram

$$\begin{tikzcd}
 & \mathcal{O} \arrow[r, equal] \arrow[d, "\nabla"] & \mathcal{O} \arrow[d, "\nabla_K"] \\
\mathcal{O} \otimes \Omega^1_K \arrow[r] \arrow[d, "\nabla_K \otimes 1"] & \Omega^1 \arrow[r] \arrow[d, "\nabla"] & \omega \arrow[l, shift right, "\sigma"'] \\
\omega \otimes \Omega^1_K \arrow[r, "\iota", "\cong"'] & \Omega^2/F^2. &
\end{tikzcd}$$

Here σ is the obvious function linear map (e.g. for $\nabla = \nabla_1$, $\sigma(t^n dt) = t^n dt$), and $F^2 \subset \Omega^2$ is the subgroup of 2 forms coming from the base.

This leads to the Gauß-Manin diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\nabla-\sigma\nabla_K} & \mathcal{O}\otimes\Omega^1_K \\ \Big\downarrow{\scriptstyle \nabla_K} & & \Big\downarrow{\scriptstyle \nabla_K\otimes 1} \\ \omega & \xrightarrow{-\iota^{-1}\nabla\sigma} & \omega\otimes\Omega^1_K. \end{array}$$

For example when $\nabla=\nabla_1$ we get on H^1_{DR}

$$\nabla_{GM}(t^n dt) = -\iota^{-1}(\nabla(1)\wedge t^n dt) = -\iota^{-1}(t da\wedge t^n dt) = t^{n+1}dt\otimes da.$$

Since $tdt \equiv \frac{-(\alpha+1)}{a}dt$, we get

$$\nabla_{GM}(dt) = -(\alpha+1)dt\otimes d\log(a) \equiv -\alpha dt\otimes d\log(a) \mod d\log(K^\times).$$

On the (rank 1) tensor product connection $H^2_{DR}(\mathcal{L}) = H^1_{DR}(\mathcal{L}_1)\otimes H^1_{DR}(\mathcal{L}_2)$, the Gauß-Manin determinant connection is therefore

$$-\alpha d\log(a) - \beta d\log(b). \tag{4.2}$$

Note that we computed this determinant here by hand, but we could have as well applied directly theorem 4.6 of [2]: the determinant of $H^1_{DR}(\mathcal{L}_1)$ is just the restriction of ∇_1 to the divisor of $\mathbb{P}^1$ defined by the trivializing section $\alpha d\log(t)+adt$ of $\omega(0+2\infty)$, that is by $\frac{a}{t}+\alpha=0$. Thus the determinant is $-\alpha d\log(a)\in\Omega^1_K/d\log K^\times$, and similarly for $H^1_{DR}(\mathcal{L}_2)$, the determinant is $-\beta d\log(b)\in\Omega^1_K/d\log K^\times$.

The idea now is to recalculate that determinant connection using the Leray spectral sequence for the map $\pi: X\to\mathbb{G}_m$, $\pi(t,u)=tu$. Write v for the coordinate on the base, so $\pi^*(v)=tu$.

Proposition 4.2. *We have*

$$R^i\pi_{*,DR}(\mathcal{L}) = \begin{cases} 0 & i\neq 1 \\ \textit{rank 2 bundle on } \mathbb{G}_m & i=1 \end{cases}$$

Proof. Let ∇_π be the relative connection on $\mathcal{L}$ with respect to the map π, and take t to be the fibre coordinate for π. Write $u=\frac{v}{t}$. Then

$$\nabla_\pi(1) = \alpha d\log(t)+\beta d\log(u)+adt+bdu = (\alpha-\beta)d\log(t)+adt-bv\frac{dt}{t^2}$$

so in $R^1\pi_{*,DR}$ we have

$$0\equiv\nabla_\pi(t^n) = (\alpha-\beta+n)t^{n-1}dt+at^n dt-bvt^{n-2}dt.$$

It follows that $R^1\pi_{*,DR}(\mathcal{L})$ has rank 2 (generated e.g. by dt and $\frac{dt}{t}$), and the other $R^i=(0)$ as claimed. ☐

Define

$$\mathcal{E} := R^1\pi_{*,DR}(\mathcal{L}),\ \nabla=\nabla_{GM}:\mathcal{E}\to\mathcal{E}\otimes\Omega^1_{\mathbb{G}_m}.$$

Theorem 4.3. *The Gauß-Manin connection on $H^*_{DR/K}(\mathbb{G}_m, \mathcal{E})$ satisfies conjecture 2.7 up to 2-torsion.*

Remark 4.4. *In fact, we will see that ∇ is not admissible in the sense of definition 2.5, but its inverse image via a degree 2 covering is. Since the new determinant of de Rham cohomology obtained in this way is twice the old one, we lose control of the 2-torsion. We do not know whether the conjecture holds exactly in this case or not.*

Proof. We can now calculate the connection $\nabla := \nabla_{GM}$ on $\mathcal{E}$ just as before. We have the Gausß-Manin diagram ($\sigma(t^n dt) = t^n dt$).

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\nabla - \sigma\nabla_\pi} & \mathcal{O}_X \otimes \Omega^1_{\mathbb{G}_m} \\ \Big\downarrow \nabla_\pi & & \Big\downarrow \nabla_\pi \otimes 1 \\ \omega_\pi & \xrightarrow{-\iota^{-1}\nabla\sigma} & \omega_\pi \otimes \Omega^1_{\mathbb{G}_m}. \end{array}$$

Here

$$\nabla(1) = (\alpha - \beta)\frac{dt}{t} + \beta\frac{dv}{v} + adt + tda + t^{-1}d(bv) + bvd(t^{-1}).$$

We get

$$\begin{aligned} \nabla_{GM}(\frac{dt}{t}) = -\iota(\nabla(1) \wedge \frac{dt}{t}) &= \frac{dt}{t} \otimes \beta\frac{dv}{v} + dt \otimes da + \frac{dt}{t^2} \otimes d(bv) \\ \nabla_{GM}(dt) = -\iota^{-1}((\beta\frac{dv}{v} &+ tda + t^{-1}d(bv)) \wedge dt) = \\ dt \otimes \beta\frac{dv}{v} &+ tdt \otimes da + \frac{dt}{t} \otimes d(bv) \end{aligned}$$

We can now substitute

$$\begin{aligned} \frac{dt}{t^2} &\equiv (bv)^{-1}((\alpha - \beta)\frac{dt}{t} + adt) \\ tdt &\equiv \frac{bv}{a}\frac{dt}{t} - \frac{\alpha - \beta + 1}{a}dt \end{aligned}$$

getting finally

$$\begin{aligned} \nabla_{GM}(\frac{dt}{t}) &= \frac{dt}{t} \otimes ((\alpha - \beta)\frac{db}{b} + \alpha\frac{dv}{v}) + dt \otimes (da + a\frac{db}{b} + a\frac{dv}{v}) \\ \nabla_{GM}(dt) &= dt \otimes (\beta\frac{dv}{v} - (\alpha - \beta + 1)\frac{da}{a}) + \frac{dt}{t} \otimes (bv\frac{da}{a} + d(bv)). \end{aligned}$$

For convenience define

$$\theta = \frac{da}{a} + \frac{db}{b} + \frac{dv}{v}.$$

Representing an element in our rank two bundle as a column vector

$$\begin{pmatrix} r \\ s \end{pmatrix} = rdt + s\frac{dt}{t}$$

the matrix for the connection on $\mathcal{E}$ becomes

$$A := \begin{pmatrix} \beta\theta - (\alpha+1)\frac{da}{a} - \beta\frac{db}{b} & a\theta \\ bv\theta & \alpha\theta - \alpha\frac{da}{a} - \beta\frac{db}{b} \end{pmatrix}.$$

The corresponding connection has a regular singular point at $v = 0$ and an irregular one at $v = \infty$. Extending $\mathcal{E}$ to $\mathcal{O}^2$ on $\mathbb{P}^1_K$, we can take $\mathcal{D} = (0) + 2(\infty)$ but the matrix g is not invertible at ∞. In order to remedy this, make the base change $z^{-2} = v$, and adjoin to K the element $\sqrt{ab}$. Notice the base change modifies the Gauß-Manin determinant computation. Let us ignore this for a while and continue with the determinant calculation.

Define $\gamma := \frac{\sqrt{ab}}{z}$. Make the change of basis

$$A_{\text{new}} = \begin{pmatrix} 1 & \frac{-z^2}{b}(\beta-\gamma) \\ 0 & \frac{z^2}{2b} \end{pmatrix} \begin{pmatrix} \beta\theta - (\alpha+1)\frac{da}{a} - \beta\frac{db}{b} & a\theta \\ \frac{b}{z^2}\theta & \alpha\theta - \alpha\frac{da}{a} - \beta\frac{db}{b} \end{pmatrix} \times \begin{pmatrix} 1 & 2(\beta-\gamma) \\ 0 & \frac{2b}{z^2} \end{pmatrix} + \begin{pmatrix} 1 & \frac{-z^2}{b}(\beta-\gamma) \\ 0 & \frac{z^2}{2b} \end{pmatrix} \begin{pmatrix} 0 & -2d\gamma \\ 0 & d(\frac{2b}{z^2}) \end{pmatrix}.$$

This works out to

$$A_{\text{new}} = \begin{pmatrix} \gamma\theta - (\alpha+1)\frac{da}{a} - \beta\frac{db}{b} & (2\alpha+2\beta+1)\gamma\theta - 2\beta(\alpha+1)\theta \\ \frac{\theta}{2} & (\alpha+\beta+2)\theta - \gamma\theta - (\alpha+1)\frac{da}{a} - \beta\frac{db}{b} \end{pmatrix}.$$

Here of course $\theta = \frac{da}{a} + \frac{db}{b} - 2\frac{dz}{z}$.

Note

$$\begin{aligned} \operatorname{Tr}(A_{\text{new}}) &= (\alpha+\beta+2)\theta - 2(\alpha+1)\frac{da}{a} - 2\beta\frac{db}{b} \\ &\equiv (\beta-\alpha)\frac{da}{a} + (\alpha-\beta)\frac{db}{b} - 2(\alpha+\beta)\frac{dz}{z} \mod d\log(K^\times). \end{aligned}$$

At $z = 0$ the polar part of A_{new} looks like

$$\begin{pmatrix} \frac{\sqrt{ab}}{z}(\frac{da}{a} + \frac{db}{b} - 2\frac{dz}{z}) & (2\beta+2\alpha+1)\frac{\sqrt{ab}}{z}(\frac{da}{a} + \frac{db}{b} - 2\frac{dz}{z}) + 4\beta(\alpha+1)\frac{dz}{z} \\ \frac{-dz}{z} & \frac{-\sqrt{ab}}{z}(\frac{da}{a} + \frac{db}{b} - 2\frac{dz}{z}) - 2(\alpha+\beta+2)\frac{dz}{z} \end{pmatrix}.$$

Writing $A_{\text{pol},0} = g_0\frac{dz}{z^2} + \frac{\eta_0}{z}$, the matrix for g_0 with coefficients in $\mathbb{C}[z]/(z^2)$ is

$$g_0 = \begin{pmatrix} -2\sqrt{ab} & -2(2\alpha+2\beta+1)\sqrt{ab} + 4\beta(\alpha+1)z \\ -z & 2\sqrt{ab} - 2(\alpha+\beta+2)z \end{pmatrix}.$$

Also

$$\eta_0 = \begin{pmatrix} \sqrt{ab}(\frac{da}{a}+\frac{db}{b}) & (2\alpha+2\beta+1)\sqrt{ab}(\frac{da}{a}+\frac{db}{b}) \\ 0 & -\sqrt{ab}(\frac{da}{a}+\frac{db}{b}) \end{pmatrix}.$$

With respect to the trivialization $d\log(z^{-1})$ the matrix g at $z=\infty$ is

$$g_\infty = \begin{pmatrix} 0 & 4\beta(\alpha+1) \\ -1 & -2(\alpha+\beta+2) \end{pmatrix}.$$

Notice that the matrices for g are invertible both at 0 and ∞. Writing $A_{\mathrm{pol},\infty} = g_\infty \frac{dz}{z} + \eta_\infty$ the contribution res $\mathrm{Tr} dg_\infty g_\infty^{-1}\eta_\infty$ is of course vanishing, as well as the contribution at ∞ obtained by changing the trivialization $\frac{dz}{z}$ to unit $\cdot \frac{dz}{z}$. At 0 we get ($\bar g := g \mod (z)$)

$$\text{(4.3)} \qquad \text{res Tr}(dgg^{-1}\frac{\eta}{z}) = \text{res Tr}(d\bar g\bar g^{-1}\frac{\eta}{z}) = \frac{1}{2}(\frac{da}{a}+\frac{db}{b}) \times$$

$$\mathrm{Tr}\Big[\begin{pmatrix} 0 & 4\beta(\alpha+1) \\ -1 & -2(\alpha+\beta+2) \end{pmatrix} \begin{pmatrix} -1 & -(2\beta+2\alpha+1) \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2\beta+2\alpha+1 \\ 0 & 1 \end{pmatrix}\Big]$$

$$= \frac{1}{2}(\frac{da}{a}+\frac{db}{b})(2\beta+2\alpha+4) \equiv (\alpha+\beta)(\frac{da}{a}+\frac{db}{b}) \mod d\log(K^\times).$$

Now we compare with the conjectural formula 2.7

$$\det(H^*_{DR}(\mathcal{E}))^{-1} = c_1(\omega(\mathcal{D}), s)\cdot \det(\nabla) - \text{res Tr}(dgg^{-1}\frac{\eta}{z}).$$

Here s can be taken to be the divisor defined by the trivializing section $\frac{dz}{z^2} - \frac{dz}{z}$ of the sheaf $\omega(2\cdot 0+\infty)$, that is $z=1$. Thus one has

$$\mathrm{Tr} A_{\mathrm{new}} = (\beta-\alpha)\frac{da}{a} + (\alpha-\beta)\frac{db}{b}.$$

Further, we have to write

$$s = \frac{dz}{z^2}w,$$

where $w = 1-z \in \mathcal{O}^\times_{X,(20)}$. Since $\mathrm{Tr}\eta_0 = 0$, res $\mathrm{Tr}\frac{dw}{w}\frac{\eta_0}{z} = 0$ as well, thus the local contribution at 0 is given by (4.3).

The conjecture gives (writing $\delta_2 : \mathbb{P}^1 \to \mathbb{P}^1,\ x \mapsto x^2$)

$$\det(H^*_{DR}(\delta_2^*\mathcal{E}))^{-1} \stackrel{?}{=} (\beta-\alpha)\frac{da}{a} + (\alpha-\beta)\frac{db}{b} - (\alpha+\beta)(\frac{da}{a}+\frac{db}{b})$$
$$= -2\alpha\frac{da}{a} - 2\beta\frac{db}{b}.$$

Notice we have adjoined $\sqrt{ab}$ to K so we have lost some 2-torsion. Bearing in mind that $\mathcal{E} = R^1\pi_{*,DR}$ which introduces a minus sign in the determinant calculations and comparing with our earlier calculation (4.2) above, we find that what we need to finish is

Proposition 4.5. *The Gauß-Manin determinant for de Rham cohomology of $\delta_2^*\mathcal{E}$ is twice the corresponding determinant for $\mathcal{E}$.*

Proof. Again we use sheaf notation for working with modules. Recall for the pullback we substituted $z^2 = w = v^{-1}$. We can write $\delta_2^*\mathcal{E} = \mathcal{E} \oplus z\mathcal{E}$. We have

$$\delta_2^*\nabla(ze) = z\nabla(e) + ze \otimes \frac{dv}{-2v},$$

so with respect to the above decomposition we can write

$$(\delta_2^*\mathcal{E}, \delta_2^*\nabla)) = (\mathcal{E}, \nabla) \oplus (\mathcal{E}, \nabla - \frac{1}{2}\frac{dv}{v})$$

The second term on the right is the connection obtained by tensoring $\mathcal{E} = R^1\pi_{*,DR}(\mathcal{L}_1 \boxtimes \mathcal{L}_2)$ with $(\mathcal{O}, -\frac{1}{2}\frac{dv}{v})$. Using the projection formula and the invariance of the latter connection, this is the same as the connection on $R^1\pi_{*,DR}((\mathcal{L}_1 - \frac{1}{2}\frac{dt}{t}) \boxtimes (\mathcal{L}_2 - \frac{1}{2}\frac{du}{u}))$, i.e. it amounts to replacing α, β by $\alpha - \frac{1}{2}, \beta - \frac{1}{2}$. Using (1), this changes the Gauß-Manin determinant by $d\log(\sqrt{ab})$ which is trivial. It follows that the Gauß-Manin determinant of $\delta_2^*\mathcal{E}$ is twice that of $\mathcal{E}$, which is what we want. $\square$

This concludes the proof of Theorem 4.3. $\square$

5. Periods

Let $X/\mathbb{C}$ be a smooth, complete curve. We consider a connection (relative to $\mathbb{C}$) $\nabla : E \to E \otimes \omega_X(\mathcal{D})$. Let $\mathcal{E}$ be the corresponding local system on $X(\mathbb{C}) - \mathcal{D}$. Notice that we do not assume ∇ has regular singular points, so $\mathcal{E}$ does not determine (E, ∇). For example, it can happen that $\mathcal{E}$ is a trivial local system even though ∇ is highly non-trivial. In this section, we consider the question of associating periods to $\det H^*_{DR}(X - \mathcal{D}, E)$. We work with algebraic de Rham cohomology in order to capture the irregular structure. The first remark is that it should be possible using Stokes structures [8] to write down a homological dual group $H_1(X^*, \mathcal{E})$ and perfect pairings ($\mathcal{E}^\vee$ is the dual local system)

$$H_1(X^*, \mathcal{E}^\vee) \times H^1_{DR}(X - \mathcal{D}, E) \to \mathbb{C}. \tag{5.1}$$

Here X^* is some modification of the Riemann surface X. (The point is that e.g. in the example we give below the de Rham group can be large while the local system $\mathcal{E}$ is trivial and $X - D = \mathbb{A}^1$.) Let $F \subset \mathbb{C}$ be a subfield, and assume we are given (i) an F-structure on $\mathcal{E}$, i.e. an F-local system $\mathcal{E}_F$ and an identification $\mathcal{E}_F \otimes \mathbb{C} \cong \mathcal{E}$. (ii) A triple $(X_0, \mathcal{D}_0, E_0)$ defined over F and an identification of the extension to $\mathbb{C}$

of these data with $(X, \mathcal{D}, E)$. When e.g. (E, ∇) satisfies the condition of proposition 2.3, one has

$$(5.2) \quad \det H^*_{DR}(X - \mathcal{D}, E) \cong \mathbb{C} \otimes_F \det H^*(X_0, E_0) \otimes \det H^*(X_0, E_0 \otimes \omega(\mathcal{D}_0))^{-1}$$

so the determinant of de Rham cohomology gets an F-structure, even if ∇ is not necessary itself defined over F. Of course, (i) determines an F-structure on $H_*(X^*, \mathcal{E})$. Choosing bases $\{p_j\}$, $\{\eta_k\}$ compatible with the F-structure and taking the determinant of the matrix of periods $\int_{p_j} \eta_k$ (5.1) yields an invariant

$$(5.3) \qquad Per(E_0, \nabla, \mathcal{E}_F) \in \mathbb{C}^\times / F^\times.$$

(More generally, one can consider two subfields $k, F \subset \mathbb{C}$ with a reduction of $\mathcal{E}$ to F and a reduction of E to k. The resulting determinant lies in $F^\times \backslash \mathbb{C}^\times / k^\times$.) In the case of regular singular points these determinants have been studied in [9].

Notice that the period invariant depends on the choice of an F-structure on the local system $\mathcal{E} = \ker(\nabla^{\mathrm{an}})$. When (E, ∇) are "motivic", i.e. come from the de Rham cohomology of a family of varieties over X, the corresponding local system of Betti cohomology gives a natural $\mathbb{Q}$-structure on $\mathcal{E}$. By a general theorem of Griffiths, the connection ∇ in such a case necessarily has regular singular points. In a non-geometric situation, or even worse, in the irregular case, there doesn't seem to be any canonical such $\mathbb{Q}$ or F-structure. For example, the equation $f' - f = 0$ has solution space $\mathbb{C} \cdot e^x$. Is the $\mathbb{Q}$-reduction $\mathbb{Q} \cdot e^x$ more natural than $\mathbb{Q} \cdot e^{x+1}$? Of course, in cases like this where the monodromy is trivial, the choice of $\mathcal{E}_F$ is determined by choosing an F-point $x_0 \in X_0 - \mathcal{D}_0$ and taking $\mathcal{E}_F = E_{x_0}$.

Even if there is no canonical F-structure on $\mathcal{E}$, one may still ask for a formula analogous to conjecture 2.7 for $Per(E_0, \nabla, \mathcal{E}_F)$. In this final section we discuss the very simplest case

$$(5.4) \quad X = \mathbb{P}^1,\ \mathcal{D} = m \cdot \infty,\ E = \mathcal{O}_X,\ \nabla(1) = df = d(a_{m-1}x^{m-1} + \ldots + a_1 x).$$

Period determinants in this case (and more general confluent hypergeometric cases) were computed by a different argument in [10]. We stress that our objective here is not just to compute the integral, but to exhibit the analogy with formula (2.9). We would like ultimately to find a formula for periods of higher rank irregular connections which bears some relation to conjectures 2.7 and 2.11. We consider the situation (5.4) with $a_{m-1} \neq 0$. Then $H^0_{DR} = (0)$ and $H^1_{DR} \cong \mathbb{H}^1(\mathbb{P}^1, \mathcal{O} \to \omega(m \cdot \infty))$ has as basis the classes of $z^i dz$, $0 \leq i \leq m-3$. One has

$\mathcal{E}^\vee = \mathbb{C}\cdot\exp(f(z))$ (trivial local system) so we take the obvious $\mathbb{Q}$-structure with basis $\exp(f)$. We want to compute the determinant of the period matrix

$$\Big(\int_{\sigma_i}\exp(f(z))z^{j-1}dz\Big)_{i,j=1,\dots,m-2} \tag{5.5}$$

for certain chains σ_i on some X^*. Let S be a union of open sectors about infinity on $\mathbb{P}^1$ where $Re(f)$ is positive, (i.e. S is a union of sectors of the form (here $N >> 1$ and $\epsilon << 1$ are fixed)

$$S_k := \{re^{i\theta}\mid N<r<\infty,\ \frac{-\arg(a_{m-1})+(2k-\frac{1}{2}-\epsilon)\pi}{m-1}\} < \theta_k < \frac{-\arg(a_{m-1})+(2k+\frac{1}{2}+\epsilon)\pi}{m-1}\}$$

so $X^* := \mathbb{P}^1 - S \sim \mathbb{P}^1 - \{p_1,\dots,p_{m-1}\}$ where the p_k are distinct points. In particular, $H_1(X^*) = \mathbb{Z}^{m-2}$. Define $\sigma_k := \gamma_k - \gamma_0$, where

$$\gamma_k := \Big\{r\exp(i\theta)\ |0\le r<\infty;\ \theta = \frac{-\arg(a_{m-1})+(2k+1)\pi}{m-1}\Big\} \tag{5.6}$$

The σ_k, $1\le k\le m-2$ form a basis for $H_1(X^*,\mathbb{Z})$.

Write $P_{ij} = \int_{\sigma_i}\exp(f(z))z^{j-1}dz$.

Lemma 5.1. *We have*

$$\det(P_{ij})_{1\le i,j\le m-2} = \int_{\sigma_1\times\cdots\times\sigma_{m-2}}\exp(f(z_1)+\ldots+f(z_{m-2}))\prod_{i<j}(z_j-z_i)dz_1\wedge\ldots\wedge dz_{m-2} \tag{5.7}$$

Proof. The essential point is the expansion

$$\prod_{i<j}(z_j-z_i) = \sum_a(-1)^{\mathrm{sgn}(a)}z_1^{a(1)-1}z_2^{a(2)-1}\cdots z_{m-2}^{a(m-2)-1}, \tag{5.8}$$

where a runs through permutations of $\{1,\dots,m-2\}$. □

We will evaluate (5.7) by stationary phase considerations precisely parallel to the techniques described in section 2 and [2]. Indeed, the degree $m-2$ part $J^{m-2}(\mathbb{P}^1, m\cdot\infty)\subset J(\mathbb{P}^1, m\cdot\infty)$ of the generalized jacobian is simply the $\mathcal{O}^\times_{m\cdot\infty}$-torsor $\omega^\times_{m\cdot\infty}$ of trivializations of $\omega_{m\cdot\infty} = \omega(m\cdot\infty)/\omega$ modulo multiplication by a constant in $\mathbb{C}^\times$. Writing $u = z^{-1}$, we may identify this torsor with

$$\{b_0\frac{du}{u}+\ldots+b_{m-1}\frac{du}{u^m}\mid b_{m-1}\neq 0\}. \tag{5.9}$$

The quotient of such trivializations up to global isomorphism is

(5.10)
$$\omega^\times_{m\cdot\infty}/\mathbb{C}^\times = \{s_{m-1}\frac{du}{u} + \ldots + s_1\frac{du}{u^{m-1}} + \frac{du}{u^m}\} = \{(s_{m-1},\ldots,s_1)\}.$$

Let $B \subset \omega^\times_{m\cdot\infty}/\mathbb{C}^\times$ be defined by $s_{m-1} = 0$. Let $\Gamma(\mathbb{P}^1, \omega(m\cdot\infty))^\times$ denote the space of sections which generate $\omega(m\cdot\infty)$ at ∞. We have

(5.11)
$$\mathbb{A}^{m-2} \twoheadrightarrow \mathrm{Sym}^{m-2}(\mathbb{A}^1) \overset{div}{\cong} \Gamma(\mathbb{P}^1, \omega(m\cdot\infty))^\times/\mathbb{C}^\times \cong B \subset \omega^\times_{m\cdot\infty}/\mathbb{C}^\times.$$

Let $z_1,\ldots,z_{m-2}$ be as in (5.7), and add an extra variable z_{m-1}. Take $s_k(z_1,\ldots,z_{m-1})$ to be the k-th elementary symmetric function, so e.g. $s_{m-1} = z_1z_2\cdots z_{m-1}$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{m-2} & \longrightarrow & B \\ \big\downarrow{\scriptstyle z_m-1=0} & & \big\downarrow \\ \mathbb{A}^{m-1} & \xrightarrow{z\mapsto(s_{m-1}(z),\ldots,s_1(z))} & \omega^\times_{m\cdot\infty}/\mathbb{C}^\times. \end{array} \tag{5.12}$$

Notice that

$$\prod_{i<j}(z_j - z_i)dz_1\wedge\ldots\wedge dz_{m-2} = ds_1\wedge\ldots\wedge ds_{m-2}. \tag{5.13}$$

Let $p_k(z_1,\ldots,z_{m-2}) = z_1^k + \ldots + z_{m-2}^k$ be the k-th power sum (or k-th Newton class). Define

(5.14)
$$F(s_1,\ldots,s_{m-2}) := f(z_1) + \ldots + f(z_{m-2}) = a_1p_1 + \ldots + a_{m-1}p_{m-1}.$$

Notice that, although the righthand expression makes sense on all of $\omega^\times_{m\cdot\infty}/\mathbb{C}^\times$, we think of F as defined only on $B : s_{m-1} = 0$. Let Ψ be the direct image on B of the chain $\sigma_1\times\cdots\times\sigma_{m-2}$ on $\mathbb{A}^{m-2}$. The integral (5.7) becomes

$$\int_\Psi \exp(F(s_1,\ldots,s_{m-2}))ds_1\wedge\ldots\wedge ds_{m-2}. \tag{5.15}$$

Lemma 5.2. *Let $b \in B = Sym^{m-2}(\mathbb{A}^1)$ correspond to the divisor of zeroes of $df = f'dz = (a_1 + 2a_2z + \ldots + (m-1)a_{m-1}z^{m-2})dz$. Then dF vanishes at b and at no other point of B.*

Proof. The differential form

$$\eta := a_1dp_1(z_1,\ldots,dz_{m-1}) + \ldots + a_{m-1}dp_{m-1}(z_1,\ldots,z_{m-1})$$

on $\omega^\times_{m\cdot\infty}/\mathbb{C}^\times$ is translation invariant. Indeed, to see this we may trivialize the torsor and take the point $s_1 = \ldots = s_{m-1} = 0$ to be the identity.

Introducing a formal variable T with $T^m = 0$, the group structure is then given by $s \oplus s' =: s''$ with

(5.16)
$$(1-s_1T+\ldots+(-1)^{m-1}s_{m-1}T^{m-1})(1-s'_1T+\ldots+(-1)^{m-1}s'_{m-1}T^{m-1}) \\ = (1-s''_1T+\ldots+(-1)^{m-1}s''_{m-1}T^{m-1}).$$

Since $-\log(1-s_1T+\ldots+(-1)^{m-1}s_{m-1}T^{m-1}) = p_1T+\ldots+p_{m-1}T^{m-1}$ it follows that the p_i are additive, whence η is translation invariant.

Note that $dF = \eta|_B$. Define $\pi : \mathbb{A}^1 \to B \subset \omega^\times_{m\cdot\infty}/\mathbb{C}^\times$ by $\pi^* p_k = z^k$, $k \le m-2$, $\pi^* s_{m-1} = 0$. Then $\pi^*\eta = df$. In particular, $\pi^*\eta$ vanishes at the zeroes of $df = f'dz$. It follows that since $b = (f') \in \mathrm{Sym}^{m-2}(\mathbb{A}^1)$, we have $DF|_b = 0$ as well. The proof that b is the unique point where $\eta|_B$ vanishes is given in [2], lemma 3.10. We shall omit it here. $\square$

Note that

$$b = (b_1, \ldots, b_{m-2}) = (\frac{a_1}{(m-1)a_{m-1}}, \frac{2a_2}{(m-1)a_{m-1}}, \ldots, \frac{(m-2)a_{m-2}}{(m-1)a_{m-1}})$$

in the s-coordinate system on B. Set $t_i = s_i - b_i$, and write $F(s) = F(b) + G(t)$, so $G(t_1, \ldots, t_{m-2})$ has no constant or linear terms.

Lemma 5.3. *There exists a non-linear polynomial change of variables of the form*

$$t'_j = t_j + B_j(t_1, \ldots, t_{j-1})$$

such that $B(0, \ldots, 0) = 0$ and $G(t) = Q(t')$ where Q is homogeneous of degree 2.

Proof. The proof is close to [2], lemma 3.10. We write (abusively) $p_k(s)$ for the power sum $z_1^k + \ldots + z_{m-1}^k$, taken as a function of the elementary symmetric functions $s_1, \ldots, s_k$. The quadratic monomials $s_i s_{m-1-i}$ all occur with nonzero coefficient in p_{m-1}. By construction, $F(s) = a_1p_1(s) + \ldots + a_{m-1}p_{m-1}(s)$ with $a_{m-1} \neq 0$. If we think of s_i as having weight i, $p_k(s)$ is pure of weight k, so s_is_{m-1-i} occurs with nonzero coefficient in $F(s)$. Since the weight $m-1$ is maximal, t_it_{m-1-i} will occur with nonzero coefficient in $G(t)$ as well. Thus, we have

$$G(t) = Q(t_1, \ldots, t_{m-1}) + H(t)$$

where Q is quadratic and contains t_it_{m-1-i} with nonzero coefficient, and H has no terms of degree < 3. Further, H has no terms of weight $> m-1$. In particular, the variable t_{m-2} does not occur in H. If we replace t_{m-2} by $t'_{m-2} := t_{m-2} + A_{m-2}(t_1, \ldots, t_{m-3})$ for a suitable polynomial A_{m-2}, we can eliminate t_1 from H completely, $G(t) = Q(t_1, \ldots, t_{m-3}, t'_{m-2}) + \tilde{H}(t_2, \ldots, t_{m-3})$. The weight and degree conditions on $\tilde{H}$ are the same as those on H, so we conclude that $\tilde{H}$ does not involve t_{m-3}. Also, this change does not affect the monomials $t_i t_{m-1-i}$ in Q for $i \geq 2$. Thus we may write

$$G = (*)t_1 t'_{m-2} + (**)t_2 t_{m-3} + \tilde{Q}(t_3, \dots, t_{m-4}) + \tilde{H}(t_2, \dots, t_{m-4})$$

since $(**) \neq 0$, we may continue in this fashion, writing $t'_{m-3} = t_{m-3} + A_{m-3}(t_1, \dots, t_{m-4})$, etc. □

The constant term can be written

$$F(b) = \sum_{\substack{\beta \\ f'(\beta)=0}} f(\beta). \tag{5.17}$$

The nonlinear change of variables $t \mapsto t'$ has jacobian 1. Also, the quadratic form above is necessarily nondegenerate (otherwise F would have more that one critical point). One has

Proposition 5.4.
(5.18)

$$\det(P_{ij})_{1 \leq i,j, \leq m-2} = \prod_{\substack{\beta \\ f'(\beta)=0}} \exp(f(\beta)) \int_\Theta \exp(Q(t)) dt_1 \wedge \dots \wedge dt_{m-2},$$

where

$$Q = t_1^2 + \dots + t_{m-2}^2 \tag{5.19}$$

is the standard, nondegenerate quadric on B, and Θ is some $n-2$-chain on B. Moreover, $\int_\Theta$ is determined up to $\mathbb{Q}^\times$-multiple on purely geometric grounds.

Since the shape of the integral is obviously coming from the change of coordinates $t \mapsto t'$, we have to understand the meaning of $\int_\Theta$.

Let $W \subset \mathbb{P}^n \times \mathbb{P}^1$ be the family of quadrics over $\mathbb{P}^1$ defined by

$$UQ(S_1, \dots, S_{m-2}) - VT^2 = 0. \tag{5.20}$$

Here $S_1, \dots, S_{m-2}, T$ are homogeneous coordinates on $\mathbb{P}^{m-2}$, U, V are homogeneous coordinates on $\mathbb{P}^1$, and $Q(S_1, \dots, S_{m-2})$ is a nondegenerate quadric. We have Weil divisors $Y : U = T = 0$; $Z : Q = T = 0$ in W. Note Y and Z are smooth, and $W_{\text{sing}} = Y \cap Z$. Let $\pi : W' \to W$ be the blowup of W along the Weil divisor Y. Let $Y' \subset W'$ be the exceptional divisor.

Lemma 5.5. i. *W' is smooth.*

ii. *The strict transform Z' of Z in W' is isomorphic to Z and*

$$Y' \cap Z' \subset Y'_{smooth}.$$

Proof. Let $P' = \mathrm{BL}(Y \subset \mathbb{P}^n \times \mathbb{P}^1)$ be the blowup. Then W' is the strict transform of W in P'. Since $Y \cap Z$ is the Cartier divisor $U = 0$ in Z, it follows that the strict transform of Z in P' or W' is isomorphic to Z.

We consider the structure of W' locally around the exceptional divisor. We may assume some S_i is invertible and write $s_j = S_j/S_i,\ t = T/S_i,\ u = U/V,\ q = Q/S_i^2$. The local defining equation for W is $uq(s) - t^2 = 0$. Thinking of W' as $\mathrm{Proj}(\bigoplus_{p\geq 0} I^p\mathcal{O}_W)$ with $I = (u,t)$, we have open sets $\mathcal{U}_1 : \tilde{t} \neq 0$ and $\mathcal{U}_2 : \tilde{u} \neq 0$. (The tilde indicates we view these as projective coordinates on the Proj.) We have the following coordinates and equations for W' and Y':

$$\begin{aligned}(5.21)\qquad \mathcal{U}_1; \qquad u't = u; \qquad W' : u'q(s) - t = 0; \qquad Y' : t = 0.\\ \mathcal{U}_2; \qquad t'u = t; \qquad W' : q(s) - ut'^2 = 0; \qquad Y' : u = 0.\end{aligned}$$

The strict transform Z' of Z lies in the locus $\tilde{t} = 0$ and so doesn't meet $\mathcal{U}_1$. Both defining equations for W' are smooth, and Y' is smooth on $\mathcal{U}_2$. Finally, $Z' \cap Y' : q = t' = u = 0$ is also smooth. □

Write

$$W^0 := W' - Z'; \quad Y^0 = Y' - Y' \cap Z'.$$

We want to show that the chains over which we integrate can be understood as chains on the topological pair $(W^0 - U, Y^0)$ for some open U (cf. lemma 5.6 below). In z-coordinates, we deal with chains γ_k, (5.6), which are parametrized $\alpha_k = r_k e^{i\theta_k}, 0 \leq r_k < \infty$ for fixed θ_k. Note that for $r >> 1$ the real part of f on γ_k will $\to -\infty$. By abuse of notation, we write γ_j also for the closure of this chain on $\mathbb{P}^1$, i.e. including the point $r = \infty$.

Write $F(z) = F(b) + Q(S_1, \ldots, S_{m-2})$. It is easy to check by looking at weights that $|S_k| = O(|r|^k)$ as $|r| \to \infty$. On the other hand, because the paths are chosen so the real parts of $a_{m-1}\alpha_k^{m-1}$ are all negative we find there exist positive constants C, C' such that $C|r|^{m-1} \leq |F(z)| = |Q(S(z))| \leq C'|r|^{m-1}$. In homogeneous coordinates

$$(S_k, T), (U, V),$$

the point associated to a point with coordinates z on our chain is

$$\begin{aligned}(5.22)\quad S_k = S_k(z) = O(|r|^k),\ 1 \leq k \leq m-2;\ T = 1;\\ U = 1;\ V = Q(S(z)) \geq C|r|^{m-1}.\end{aligned}$$

With reference to the coordinates in (5.21) we see that

$$\begin{aligned}(5.23)\qquad |u| = |U/V| \leq C^{-1}|r|^{1-m},\ |t| = |T/S_i| \geq C_1|r|^{-i},\\ |t'| = |t/u| \geq C_2|r|^{m-1-i}.\end{aligned}$$

In particular, the limit as $|r| \to \infty$ does not lie on $\mathcal{U}_2$. Since, near ∞ $Z \subset \mathcal{U}_2$, we conclude our chains stay away from Z at infinity.

We fix $\epsilon << 1$ and $N >> 1$ and define a connected, simply connected domain $D \subset \mathbb{A}^1 \subset \mathbb{P}^1$ by

$$D = \{re^{i\theta} \mid r > N,\ -\frac{\pi}{2} - \epsilon < \theta < \frac{\pi}{2} + \epsilon\}$$

thus, D is an open sector at infinity, and $\exp(z)$ is rapidly decreasing as $|z| \to \infty$ in the complement of D. In what follows, let $g : W \to \mathbb{P}^1$ be the projection.

Lemma 5.6. *The assignment*

$$\gamma \mapsto \int_\gamma \exp(Q(S_1/T, \dots, S_{m-2}/T)d(S_1/T) \wedge \dots \wedge d(S_{m-2}/T)$$

defines a functional $H_{m-2}(W^0 - g^{-1}(D), Y^0; \mathbb{Q}) \to \mathbb{C}$.

Proof. Write τ for the above integrand. Let M be some neighborhood of Z. Then τ is rapidly decreasing on $W^0 - g^{-1}(D)$ near $Y^0 - M \cap Y^0$, where the size is defined by some metric on the holomorphic $m-2$ forms on W^0. Since the chains are compact, a chain γ on W^0 will be supported on $W - M$ for a sufficiently small neighborhood M of Z. Thus, integration defines a functional

$$C_{m-2}(W^0 - g^{-1}(D), Y^0) \to \mathbb{C}.$$

It remains to show $\int_{\partial\Gamma} \tau = 0$ for an $m-1$ chain Γ. Let M be an open neighborhood of Z not meeting Γ. Let R be an open neighborhood of Y^0 in $W - M$. Write $\Gamma = \Gamma_1 + \Gamma_2$ where $\Gamma_1 \subset \bar{R}$ and $\Gamma_2 \cap R = \emptyset$. Since τ is closed, $\int_{\partial\Gamma_2} \tau = 0$. On the other hand, the volume of $\partial\Gamma_1$ can be taken to be bounded independent of R. It follows that $\int_{\partial\Gamma} \tau = 0$. $\square$

Lemma 5.7. $H_{m-2}(W^0 - g^{-1}(D), Y^0; \mathbb{Q}) \cong \mathbb{Q}$.

Proof. Let $p \in D$ be a point. We first show

$$H_{m-2}(W^0 - g^{-1}(p), Y^0; \mathbb{Q}) \cong \mathbb{Q}. \tag{5.24}$$

We calculate $H^*(W^0 - g^{-1}(p))$ using the Leray spectral sequence. For $x \neq 0, \infty$, $g^{-1}(x)$ is a smooth, affine quadric of dimension $m-3$. So $R^p g_* \mathbb{Q} = (0)$ away from $0, \infty$ for $p \neq 0, m-3$, and $R^{m-3} g_* \mathbb{Q}|_{\mathbb{P}^1 - \{0,\infty\}}$ is a rank 1 local system. The monodromy about 0 and ∞ is induced by $(S, T) \mapsto (S, -T)$ and $(S, T) \mapsto (-S, T)$, respectively. Both actions give -1 on the fibres. It follows that, writing $j : \mathbb{P}^1 - \{0, \infty\} \hookrightarrow \mathbb{P}^1$, we have

(5.25)

$$j_! R^{m-3} g_* \mathbb{Q}|_{\mathbb{P}^1 - \{0,\infty\}} \cong j_* R^{m-3} g_* \mathbb{Q}|_{\mathbb{P}^1 - \{0,\infty\}} \cong Rj_* R^{m-3} g_* \mathbb{Q}|_{\mathbb{P}^1 - \{0,\infty\}}.$$

It follows that the natural map $R^{n-1}g_*\mathbb{Q} \to j_*R^{n-1}g_*\mathbb{Q}|_{\mathbb{P}^1-\{0,\infty\}}$ is surjective and we get a distinguished triangle in the derived category

$$\mathcal{P} \to R^{n-1}g_*\mathbb{Q} \to Rj_*R^{n-1}g_*\mathbb{Q}|_{\mathbb{P}^1-\{0,\infty\}} \tag{5.26}$$

where $\mathcal{P}$ is a sheaf supported over $0, \infty$. In particular,

$$H^1(\mathbb{P}^1, R^{m-3}g_*\mathbb{Q}) \cong H^1(\mathbb{P}^1 - \{0,\infty\}, R^{n-1}g_*\mathbb{Q}) = (0), \tag{5.27}$$

where the vanishing comes by identifying with group cohomology of $\mathbb{Z}$ acting on $\mathbb{Q}$ with the generator acting by -1. An easy Gysin argument yields

$$H^1(\mathbb{P}^1 - \{p\}, R^{m-3}g_*\mathbb{Q}) \cong H^1(\mathbb{P}^1 - \{0,\infty,p\}, R^{m-3}g_*\mathbb{Q}) = \mathbb{Q}. \tag{5.28}$$

It also follows from (5.26) that $H^2(\mathbb{P}^1, R^{m-3}g_*\mathbb{Q}) = (0)$. The spectral sequence thus gives

$$H^{m-2}(W^0, \mathbb{Q}) \cong (R^{m-2}g_*\mathbb{Q})_{\{0,\infty\}}. \tag{5.29}$$

To compute these stalks, note the fibre of $W' \to \mathbb{P}^1$ over 0 is a singular quadric with singular point $S_1 = \ldots = S_{m-2} = 0$, $T = 1$ away from Z. Thus the fibre of $g : W^0 \to \mathbb{P}^1$ over 0 is the homogeneous affine quadric $Q(S) = 0$ which is contractible. Further, because Z meets the fibre of W' smoothly, one has basechange for the non-proper map g, so $(R^{m-2}g_*\mathbb{Q})_0 = (0)$. At infinity, we have seen again that Z meets the fibre smoothly, so again one has basechange for g. Let $h : W^0 - Y^0 \hookrightarrow W^0$. It follows that

$$(R^{m-2}g_*(h_!\mathbb{Q}))_{\{0,\infty\}} = (0). \tag{5.30}$$

Combining (5.28),(5.29),(5.30) yields (5.24).

To finish the proof of the lemma, we must show the inclusion

$$(W^0 - g^{-1}(D), Y^0) \hookrightarrow (W^0 - g^{-1}(p), Y^0)$$

is a homotopy equivalence. We can define a homotopy from $D - \{p\}$ to ∂D by flowing along an outward vector field v. E.g. if $p = 0$ and one has cartesian coordinates x, y, one can take $v = x\frac{d}{dx} + y\frac{d}{dy}$. Since $W'/\mathbb{P}^1$ is smooth over D, one can lift v to a vector field w on $g'^{-1}(D)$. Since Z' meets the fibres of g' smoothly over some larger $D_1 \supset D$, we can arrange for w to be tangent to Z' along Z'. Let h be a smooth function on $\mathbb{P}^1$ which is positive on D and vanishes on $\mathbb{P}^1 - D$. We view $g'^*(h)w$ as a vector field on $W' - g'^{-1}(p)$. Flowing along $g'^*(h)w$ lifts the flow along hv, carries $g'^{-1}(\bar{D})$ into $g'^{-1}(\partial D)$ and stabilizes $W' - Z'$ over $\bar{D}$. This is the desired homotopy equivalence. $\square$

Remark 5.8. *It follows from theorem 2.3.3 in* [10] *that*

$$\int_\Theta \exp(Q)dt \in \Big(\frac{2\pi}{(m-1)a_{m-1}}\Big)^{\frac{m-2}{2}} \cdot \mathbb{Q}.$$

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