ENRICHED HODGE STRUCTURES

SPENCER BLOCH AND V. SRINIVAS

Abstract. A category of enriched Hodge structures is defined. Various examples are discussed. Deligne-Beilinson cycle classes for cycles on singular varieties as defined by M. Levine [8] are shown to coincide with extension classes in this category. The generalized Albanese of Esnault, Srinivas, Viehweg [7] admits a similar interpretation.

0. Introduction

As a first approximation, the theory of Hodge structures amounts to a very deep and powerful way to study "periods", i.e. the relation between the integral structure defined by topological cochains and the Hodge filtration defined by holomorphic and anti-holomorphic forms on a complex variety. Defined, in the first instance, for compact smooth varieties, the theory generalizes via the addition of a weight filtration (mixed hodge structures) to open and singular varieties as well. The category of Hodge structures is abelian, and Ext^1 groups in this category are frequently related to Chow groups of equivalence classes of algebraic cycles. For example, given X a compact Riemann surface, one has

$$(0.1)\quad \mathrm{Ext}^1(\mathbb{Z}, H^1(X,\mathbb{Z}(1))) \cong \mathrm{Pic}^0(X) := \text{Group of line bundles of degree 0.}$$

Here $H^1(X,\mathbb{Z}(1))$ is the first Betti cohomology of X, "Tate-twisted" by 1, and the ext group is computed in the category of mixed Hodge structures.

Although Hodge structures are defined for singular varieties, one has a sense that there is more information in the singular case. For example, when X is a cuspidal rational curve, the ext group is zero (because $H^1(X,\mathbb{Z}) = (0)$), but $\mathrm{Pic}^0(X) \cong H^1(X,\mathcal{O}) \neq 0$. This is a vector group. There are no periods, so Hodge structures give no information.

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In fact, there are a range of objects in algebraic geometry, including cycle groups on certain singular varieties, contravariant motives (one doesn't have even a conjectural description of these), and connections with irregular singular points, for the description of which periods do not suffice. We believe (though we cannot make our ideas precise) that interesting and unexpected relations exist between these.

To define enriched Hodge structures, let X be a complete but possibly singular complex variety. We work in the analytic topology with analytic sheaves. According to the basic construction of Deligne [4], we may find $\pi : \widetilde{X} \to X$ a hypercovering by a simplicial variety with proper, smooth strata such that π^* is an isomorphism on Betti cohomology and such that the Hodge structure coming from $\widetilde{X}$ is canonically defined depending only on X. On the other hand (this is the striking new idea which first appeared in [6] and was then developed in [8] and [7]) we can consider the naive analytic de Rham complex $\Omega^\bullet_X$ associated to the singular variety X. This is a strange object, but never mind. It is contravariant functorial and receives the constant sheaf $\mathbb{C}_X$ so we get a splitting in cohomology

$$H^*(X,\mathbb{C}) \xrightarrow{\rho} \mathbb{H}^*(X,\Omega^\bullet_X) \xrightarrow{\pi^*} \mathbb{H}^*(\widetilde{X},\Omega^\bullet_{\widetilde{X}}) \cong H^*(X,\mathbb{C}) \tag{0.2}$$

We define the Hodge filtration on $\mathbb{H}^*(X,\Omega^\bullet_X)$ in the evident way

$$F^p\mathbb{H}^*(X,\Omega^\bullet_X) := \mathrm{Image}\Big(\mathbb{H}^{*-p}(X,\Omega^p \to \dots) \to \mathbb{H}^*(X,\Omega^\bullet_X)\Big). \tag{0.3}$$

Note π^* in (0.2) is compatible with the Hodge filtration, but the map labeled ρ is not.

Fix an integer n and write $H := H^n(X,\mathbb{C})$, $V := \mathbb{H}^n(X,\Omega^\bullet)$. Write

$$V_p := \mathbb{H}^n(X,\mathcal{O} \to \dots \to \Omega^{p-1}); \quad H_p := H/F^pH. \tag{0.4}$$

We have a commutative diagram

$$\begin{tikzcd} H \arrow[r] & H_n \arrow[r] & \dots \arrow[r] & H_1 \\ V \arrow[r] \arrow[u, "\pi^*"] & V_n \arrow[r] \arrow[u, "\pi^*"] & \dots \arrow[r] & V_1 \arrow[u, "\pi^*"] \\ H \arrow[u, "\rho"] \end{tikzcd} \tag{0.5}$$

with $\pi^* \circ \rho = Id_H$. Note the maps $V \to V_i$ are not necessarily surjective. By definition, an enriched Hodge structure is a Hodge structure H, an integer n such that $F^{n+1}H = (0)$, $\mathbb{C}$-vector spaces V and V_i, and a diagram (0.5) satisfying $\pi^* \circ \rho = Id_H$.

The purpose of this note is to work out some simple properties of these enriched Hodge structures. We will give some examples and interpret cycle classes as ext classes.

Finally some disclaimers. Given the pathological nature of $\Omega^\bullet_X$, one is reluctant to insist too strongly that our method of associating an enriched Hodge structure to a singular variety is the best one. Indeed, variants are possible. One can e.g. replace $\Omega^\bullet_X$ in the above with $\Omega^\bullet_X/\mathrm{torsion}$ or with $\Omega^\bullet_X/F^{\dim X+1}\Omega^\bullet_X$. (It is perhaps not obvious that these map to the de Rham complex of the hypercovering computing the Hodge structure on X. See lemma 2.1 below.) The category EHS of enriched Hodge structures can also be criticized. One might want to work with triangulated categories and replace the V_i with $R\Gamma(X, \Omega^\bullet)$. Also, EHS is not self-dual, though the dual category is easily constructed. (Simply reverse all the arrows in (0.5).)

The notion of enriched Hodge structure is a categorical response to the cohomological work in [8]. The reader is referred to that paper for a careful development of the underlying geometric and cohomological theory.

1. Basic Properties

Let $\mathcal{C}$ denote the abelian category of all diagrams (not necessarily complexes) of $\mathbb{C}$-vector spaces

$$V_\bullet := \{\ldots = V_{a+1} = V_a \to V_{a-1} \to \ldots \to V_b \to 0 \to 0 \ldots\}$$

for $a \geq b \in \mathbb{Z}$. Objects are viewed as graded with V_i in degree i. In other words, objects of $\mathcal{C}$ are diagrams $\ldots \to V_i \to V_{i-1} \to \ldots$ indexed by $\mathbb{Z}$ which are eventually constant for $i >> 0$ and eventually 0 for $i << 0$. We write HS for the category of all (mixed, $\mathbb{Z}$-) Hodge structures. To an object $H \in HS$ we associate $\ldots \to H/F^aH \to H/F^{a-1}H \to \ldots \in \mathcal{C}$. (To simplify, we write $H = H_{\mathbb{C}}$.)

EHS will denote the category of enriched Hodge structures, i.e. the category of all objects $\{H, V_\bullet, \rho, \pi\}$ where H is a Hodge structure with

$$0 = F^aH \subset F^{a-1}H \subset \ldots \subset F^bH \subset F^{b-1}H = H,$$

$V_\bullet = \{\ldots = V_a \to \ldots \to V_b \to 0\ldots\} \in \mathcal{C}$, and ρ, π are maps fitting into a commutative diagram (with $H_c := H/F^cH$)

$$\begin{array}{ccccccccc} \ldots H_{a+1} & = & H_a & \longrightarrow & H_{a-1} & \longrightarrow & \ldots & \longrightarrow & H_b \\ & & \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} & & & & \uparrow{\scriptstyle \pi} \\ \ldots V_{a+1} & = & V_a & \longrightarrow & V_{a-1} & \longrightarrow & \ldots & \longrightarrow & V_b \\ & & \uparrow{\scriptstyle \rho} & & & & & & \\ & & H & & & & & & \end{array} \tag{1.1}$$

where we require that $\pi \circ \rho$ is the identity. (Sometimes we put a subscript ρ_i to indicate the composed map $H \to V_a \to V_i$.) Morphisms between objects are required to be compatible with all the structure. We will say that the EHS $(H, V_\bullet)$ is supported in the interval $[b, a]$.

Frequently, we will simplify notation and write $(H, V_\bullet)$ for an object in EHS.

Proposition 1.1. *The category EHS is abelian. The functor*

$$\nu : EHS \to HS; \quad \nu(H, V_\bullet) = H$$

has a right adjoint

$$\sigma : HS \to EHS; \quad \sigma(H) = \{H, H, Id, Id\}.$$

The functor

$$\mu : EHS \to \mathcal{C}; \quad \mu(H, V_a \to \ldots \to V_b, \rho, \pi) = W_a \to \ldots \to W_b,$$

where W_i is the kernel of $\pi : V_i \to H_i$, has a left adjoint

$$\tau : \mathcal{C} \to EHS; \quad \tau(V_\bullet) := \{0, V_\bullet, 0, 0\}.$$

All these functors are exact, and one has an exact sequence

$$0 \to \tau \circ \mu(H, V_\bullet) \to (H, V_\bullet) \to \sigma \circ \nu(H, V_\bullet) \to 0. \tag{1.2}$$

Proof. Straightforward. □

Remark 1.2. Let H be a Hodge structure. We will sometimes abuse notation and also write H for the enriched Hodge structure $\sigma(H) = \{H, H, Id, Id\}$. For example, $H = \mathbb{Z}(0)$ is given by

$$V_i = H_i = \begin{cases} \mathbb{C} & i \geq 1 \\ 0 & i \leq 0 \end{cases}.$$

Proposition 1.3. *Let $\mathcal{H} = (H, V_a \to \ldots \to V_b, \rho, \pi)$ be an enriched Hodge structure. Let $U_\bullet = \mu(\mathcal{H}) = \ker(\pi_\bullet : V_\bullet \to H_\bullet)$. Then there is an exact sequence*

$$F^0H \cap W_0H_{\mathbb{Z}} \xrightarrow{\partial} U_0 \to Ext^1_{EHS}(\mathbb{Z}(0), \mathcal{H}) \to Ext^1_{HS}(\mathbb{Z}(0), H) \to 0.$$

Here ∂ is induced from the map $\rho_0 : H \to V_0$.

Proof. We consider the set of enriched Hodge structures $\widetilde{\mathcal{H}} = (\widetilde{H}, \widetilde{V}, \tilde{\rho}, \tilde{\pi})$ which are extensions of $\mathbb{Z}(0)$ by $\mathcal{H}$ lifting a given extension $\widetilde{H}$ of H by $\mathbb{Z}(0)$ in the category of Hodge structures. For $i > j$ we write $\natural : V_j \to V_i$ for the given map. For $i \geq 1$ and $\iota : H \to \widetilde{H}$ we define

$$\widetilde{V}_i := V_i \coprod_H \widetilde{H} = (V_i \oplus \widetilde{H})/\{(\rho_i(h), \iota(h)) | h \in H\}. \tag{1.3}$$

One has an evident map $\tilde{\rho}_i : \widetilde{H} \to \widetilde{V}_i$. Since $\mathbb{Z}(0)_i = \mathbb{C}$ in this range, we have as well

$$\widetilde{H}_i = H_i \coprod_H \widetilde{H}. \tag{1.4}$$

Thus, for $i \geq 1$ we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & H_i \arrow[r] & \widetilde{H}_i \arrow[r] & \mathbb{Z}(0)_i \arrow[r] & 0 \\
0 \arrow[r] & V_i \arrow[r] \arrow[u, "\pi_i"'] & \widetilde{V}_i \arrow[r] \arrow[u, "\tilde{\pi}_i := \pi_i \amalg id"'] & \mathbb{Z}(0)_i \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[u, "\rho_i"'] & \widetilde{H} \arrow[r] \arrow[u, "\tilde{\rho}_i"'] & \mathbb{Z}(0) \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd} \tag{1.5}$$

For $i \leq 0$ define $\widetilde{V}_i := V_i$. The above diagram becomes

$$\begin{tikzcd}
0 \arrow[r] & H_i \arrow[r, equal] & \widetilde{H}_i \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & V_i \arrow[r, equal] \arrow[u, "\pi_i"'] & \widetilde{V}_i \arrow[r] \arrow[u, "\tilde{\pi}_i := \pi_i"'] & 0 \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[u, "\rho_i"'] & \widetilde{H} \arrow[r] \arrow[u, "\tilde{\rho}_i"'] & \mathbb{Z}(0) \arrow[r] \arrow[u] & 0.
\end{tikzcd} \tag{1.6}$$

With these definitions, one has natural maps

$$\tilde{\rho}_i : \widetilde{H} \to \widetilde{V}_i; \quad \tilde{\pi}_i : \widetilde{V}_i \to \widetilde{H}_i; \quad \tilde{\natural}_i : \widetilde{V}_i \to \widetilde{V}_{i-1}, \; i \neq 1.$$

It remains to define the map

$$\tilde{\natural}_1 : \widetilde{V}_1 = V_1 \coprod_H \widetilde{H} \to \widetilde{V}_0 = V_0. \tag{1.7}$$

Consider the diagram

$$
\begin{tikzcd}
 & V_0 = \widetilde{V}_0 \arrow[r, "\pi_0"] & H_0 = \widetilde{H}_0 & & \\
0 \arrow[r] & V_1 \arrow[u, "\natural_1"'] \arrow[r, "\alpha"] & V_1 \coprod_H \widetilde{H} = \widetilde{V}_1 \arrow[u, "\beta"'] \arrow[r] & \mathbb{C} \arrow[r] & 0 \\
0 \arrow[r] & H \arrow[u] \arrow[r, "\iota"] & \widetilde{H} \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, equal] \arrow[r] & 0.
\end{tikzcd}
\tag{1.8}
$$

The map α is a pushout of ι and hence is injective. The map β is induced by $\pi_0 \circ \natural_1$ on V_1 and the natural map $\widetilde{H} \to \widetilde{H}_0$. The map $\tilde{\natural}_1$ will be a lifting of β, viz. $\pi_0 \circ \tilde{\natural}_1 = \beta$. Choose $e \in \widetilde{V}_1$ mapping to $1 \in \mathbb{C}$. Choose $f \in V_0$ such that $\pi_0(f) = \beta(e)$. (This is possible since all the π_i are surjective.) Define $\tilde{\natural}_1 = \natural_1$ on $V_1 \subset \widetilde{V}_1$ and $\tilde{\natural}_1(e) = f$.

It follows that extensions of $\mathbb{Z}(0)$ by $\mathcal{H}$ lifting $\widetilde{H}$ are parametrized by $\pi_0^{-1}(\beta(e))$ which is a non-trivial $U_0 = \ker(V_0 \to H_0)$-torseur. In particular, the long exact sequence of Ext's obtained by applying $\mathrm{Hom}(\mathbb{Z}(0), \bullet)$ to (1.2) is surjective on the right at Ext^1.

Lemma 1.4. *$Ext^1(\mathbb{Z}(0), \tau(U_\bullet)) \cong U_0$*

proof of lemma. Let $\mathcal{U} = \tau(U_\bullet)$ and let $\widetilde{\mathcal{U}} = (\mathbb{Z}(0), \widetilde{U}_\bullet)$ be an extension of $\mathbb{Z}(0)$ by $\mathcal{U}$. Choose splittings $\widetilde{U}_i = U_i \oplus \mathbb{C}(0)_i$, and write $\tilde{\rho}_i(1) = (u_i, 1) \in \widetilde{U}_i$ for $i \geq 1$ (resp. $\tilde{\rho}_i(1) = u_i \in \widetilde{U}_i = U_i$ for $i \leq 0$). The maps $\tilde{\natural}_i : \widetilde{U}_i \to \widetilde{U}_{i-1}$ are uniquely determined by the requirement $\tilde{\rho}_{i-1}(1) = \tilde{\natural}_i \circ \tilde{\rho}_i(1)$, and $\tilde{\pi}_i$ is projection on the second factor. It is straightforward to check that the extension is split if and only if $\tilde{\rho}_0(1) = 0 \in \widetilde{U}_0 = U_0$. □

Note $\mathrm{Hom}(\mathbb{Z}(0), H) \cong F^0 H \cap W_0 H_{\mathbb{Z}}$. The ext sequence yields now

$$F^0 H \cap W_0 H_{\mathbb{Z}} \xrightarrow{\partial} U_0 \to \mathrm{Ext}^1_{EHS}(\mathbb{Z}(0), \mathcal{H}) \to \mathrm{Ext}^1_{HS}(\mathbb{Z}(0), H) \to 0,$$

which is the desired sequence. □

Corollary 1.5. *Assume $\mathcal{H} = (V, H)$ where $H = H^{2r-1}(X, \mathbb{Z}(r))$ and $V = \mathbb{H}^{2r-1}(X, \Omega^\bullet)(r)$ for a complete variety $X/\mathbb{C}$. Then we get a vector extension*

$$0 \to U_0 \to Ext^1_{EHS}(\mathbb{Z}(0), \mathcal{H}) \to Ext^1_{HS}(\mathbb{Z}(0), H) \to 0.$$

Proof. In this case the weights are ≤ -1 so $F^0 H = 0$. □

Remark 1.6. In geometric situations, e.g. when $\mathcal{H}$ is the enriched Hodge structure of a singular variety, it would be nice to be able to assert in some generality that the image of the map $\partial : F^0H \cap W_0H_{\mathbb{Z}} \to U_0$ is discrete so $\mathrm{coker}(\partial)$ would be a Lie group.

Example 1.7. One of the main results of this paper is to show that a codimension r cycle Z supported on the locus of smooth points of a variety X and having trivial de Rham class gives rise to an extension of enriched Hodge structures,

$$\{Z\} \in \mathrm{Ext}^1_{EHS}\Big(\mathbb{Z}(0), \big(H^{2r-1}_{\mathrm{Betti}}(X, \mathbb{Z}(r)), \mathbb{H}^{2r-1}(X, \Omega^\bullet)\big)\Big), \tag{1.9}$$

generalizing the familiar class in the intermediate jacobian.

We conclude this section with a question, which is an analogue for EHS of a result of Beilinson [3] for HS. Recall that the (Yoneda) Ext^n groups in an abelian category are defined to be equivalence classes of n-extensions. For a detailed discussion of equivalence of n-extensions, see [9], which considers the case of modules over a ring; the definitions and proofs are easily adapted to abelian categories.

Question 1.8. For any objects $\mathcal{A}$, $\mathcal{B}$ of EHS, is $\mathrm{Ext}^n(\mathcal{A}, \mathcal{B}) = 0$ for any $n \geq 2$?

2. Geometry

Let X be a proper variety (i.e. separated scheme of finite type) of dimension d over $\mathbb{C}$. Write ${}'\Omega^\bullet_X := \Omega^\bullet_X / F^{d+1}\Omega^\bullet_X$, and if X is reduced, ${}''\Omega^\bullet_X := \Omega^\bullet_X/\mathrm{torsion}$. We have $\Omega^\bullet_X \twoheadrightarrow {}'\Omega^\bullet_X \twoheadrightarrow {}''\Omega^\bullet_X$.

Lemma 2.1. *Let $\widetilde{X} \xrightarrow{\pi} X$ be a smooth proper hypercovering calculating the Hodge structure on $H^*(X, \mathbb{C})$. If X is reduced, then the pullback π^* on de Rham cohomology factors*

$$\mathbb{H}^*(X, \Omega^\bullet_X) \to \mathbb{H}^*(X, {}'\Omega^\bullet_X) \to \mathbb{H}^*(X, {}''\Omega^\bullet_X) \to \mathbb{H}^*(\widetilde{X}, \Omega^\bullet_{\widetilde{X}}) \cong H^*(X, \mathbb{C}).$$

If X is non-reduced, there is a similar factorization, omitting the term $\mathbb{H}^*(X, {}''\Omega^\bullet_X)$.

Proof. Write

$$\widetilde{X} = \ldots \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} \widetilde{X}_1 \begin{smallmatrix}\to\\ \to\end{smallmatrix} \widetilde{X}_0 \xrightarrow{\pi_0} X$$

We can take $\widetilde{X}_0 \xrightarrow{\pi_0} X$ to be a resolution of singularities of X_{red}. The map $\pi_0^* : \Omega^\bullet_X \to \Omega^\bullet_{\widetilde{X}_0}$ then factors through ${}'\Omega^\bullet_X$. If X is reduced, it kills torsion and hence further factors through ${}''\Omega^\bullet_X$. Since all the maps

$\widetilde{X}_i \to X$ factor through π_0, we get the same results on the level of the hypercovering. ☐

Corollary 2.2. *Taking* $V = \mathbb{H}^n(X, \Omega^\bullet)$ *(resp.* $V = \mathbb{H}^n(X, {}'\Omega^\bullet), V = \mathbb{H}^n(X, {}''\Omega^\bullet)$*) in* (0.5) *yields three (in general different) enriched Hodge structures associated to* X *and* n*.*

Note, of course, that these constructions are functorial insofar as the corresponding de Rham complexes are functorial. For example, ${}''\Omega^\bullet$ is functorial for dominant morphisms of reduced schemes, by Lemma 1 in [10].

Example 2.3. Let $D = \sum n_p(p)$ be an effective divisor on $\mathbb{P}^1$. Assume either $n_p = 0$ or $n_p \geq 2$. Define a singular curve X with $g : \mathbb{P}^1 \to X$, where $\mathcal{O}_X \subset g_*\mathcal{O}_{\mathbb{P}^1}$ is the subsheaf of functions f such that $f - f(p)$ vanishes to order $\geq n_p$ for all p. Then $X = \mathbb{P}^1$ as a topological space, $H^1(X, \mathbb{Q}) = (0)$, and

$$H^1(X, \mathcal{O}_X) \cong \bigoplus_{n_p \neq 0} \mathfrak{m}_p/\mathfrak{m}_p^{n_p}.$$

If we take the enriched Hodge structure associated to H^1 and the de Rham cohomology modulo torsion, we find $\{0, V_\bullet, 0, 0\}$ with $V_\bullet$ given by the diagram $(\cdots = H^1(\mathcal{O}) = H^1(\mathcal{O}) \to 0 = 0 = \cdots)$.

Example 2.4 (Compare [8], §5)**.** Let $\mathcal{X} \to \mathrm{Spec}\,(\mathbb{C}[\epsilon]/(\epsilon^2))$ be smooth and projective. Let $X = \mathcal{X}_{red}$. Let $\mathcal{H}$ be the enriched Hodge structure associated to $\mathbb{H}^n(\mathcal{X}, \Omega^\bullet_{\mathcal{X}})$ for some fixed n, and let $H = H^n(X, \mathbb{Z})$ be the ordinary Hodge structure. Define $W_p := H^{n-p+1}(X, \Omega^{p-1})$ and let $W_\bullet$ be the corresponding diagram with all maps zero. We claim there is an exact sequence of enriched Hodge structures

$$0 \to \tau(W_\bullet) \to \mathcal{H} \to H \to 0 \tag{2.1}$$

To see this, we need to show two things. First, one has an exact sequence

$$0 \to H^{n-p+1}(X, \Omega^{p-1}) \to \mathbb{H}^n(\mathcal{X}, \Omega^\bullet_{\mathcal{X}}/F^p\Omega^\bullet_{\mathcal{X}}) \to H/F^pH \to 0, \tag{2.2}$$

and second, that passing from F^p to F^{p-1} induces the zero map on the left in (2.2). Define

$$I\Omega^j_{\mathcal{X}} = \epsilon \cdot \Omega^j_{\mathcal{X}} + \Omega^{j-1}_{\mathcal{X}} \wedge d\epsilon = \ker(\Omega^j_{\mathcal{X}} \to \Omega^j_X) \cong \Omega^j_X \oplus \Omega^{j-1}_X. \tag{2.3}$$

The map $\epsilon a + b \wedge d\epsilon \mapsto a + db$ on $I\Omega^{p-1}$ gives a quasi-isomorphism $I\Omega^\bullet/F^pI\Omega^\bullet \to \Omega^{p-1}_X[1-p]$ so we get a distinguished triangle

$$\Omega^{p-1}_X[1-p] \to \Omega^\bullet_{\mathcal{X}}/F^p\Omega^\bullet_{\mathcal{X}} \to \Omega^\bullet_X/F^p\Omega^\bullet_X. \tag{2.4}$$

Passing to the relative de Rham complex yields a commutative diagram

$$
\begin{tikzcd}
\Omega_X^{p-1}[1-p] \arrow[r] \arrow[d] & \Omega^\bullet_{\mathcal{X}}/F^p\Omega^\bullet_{\mathcal{X}} \arrow[r] \arrow[d] & \Omega^\bullet_X/F^p\Omega^\bullet_X \arrow[d, equal] \\
\Omega^\bullet_X/F^p\Omega^\bullet_X \arrow[r] & \Omega^\bullet_{\mathcal{X}/\mathbb{C}[\epsilon]}/F^p\Omega^\bullet_{\mathcal{X}/\mathbb{C}[\epsilon]} \arrow[r] & \Omega^\bullet_X/F^p\Omega^\bullet_X
\end{tikzcd}
\tag{2.5}
$$

where the arrow on the left is the evident inclusion of the last sheaf of the complex. Degeneracy of the Hodge spectral sequence on X implies this map is injective on cohomology. Also, the boundary maps in cohomology from the long exact sequence associated to the bottom distinguished triangle in (2.5) are known to be 0 ([5], Théorme (5.5)), proving (2.2). Finally, the right hand isomorphism in (2.3) leads to a distinguished triangle

$$(\Omega_X^{p-1}\oplus\Omega_X^{p-2})[1-p]\to\Omega_X^{p-1}[1-p]\to\Omega_X^{p-2}[2-p]. \tag{2.6}$$

the left hand arrow is easily seen to be the identity on Ω^{p-1}, so the right hand arrow is zero on cohomology. This implies that the maps $\ker(\mathcal{H}\to H)_p\to\ker(\mathcal{H}\to H)_{p-1}$ are zero as claimed.

We claim that there is a natural map

$$H^1(X,T_X)\to\mathrm{Ext}^1(H,\tau(W_\bullet)) \tag{2.7}$$

carrying the Kodaira-Spencer deformation class of $\mathcal{X}$ to the extension class (2.1). In fact, using that the transition maps of $W_\bullet$ are 0, one computes easily that

$$\begin{aligned}\mathrm{Ext}^1(H,\tau(W_\bullet) &\cong \oplus_p\mathrm{Hom}(\mathrm{gr}^p_F H, W_{p-1})\\ &= \oplus_p\mathrm{Hom}(H^{n-p}(X,\Omega^p_X),H^{n-p+1}(X,\Omega_X^{p-1})).\end{aligned}$$

From the construction of $\mathcal{H}$, we can identify the class in

$$\mathrm{Hom}(H^{n-p}(X,\Omega^p_X),H^{n-p+1}(X,\Omega_X^{p-1}))$$

with the boundary map in cohomology associated to the exact sequence (obtained from (2.3))

$$0\to\Omega_X^{p-1}\to\Omega^p_{\mathcal{X}}\otimes\mathcal{O}_X\to\Omega^p_X\to 0,$$

which is induced by the cup product with the extension class of the corresponding sequence for $p=1$

$$0\to\mathcal{O}_X\to\Omega^1_{\mathcal{X}}\otimes\mathcal{O}_X\to\Omega^1_X\to 0.$$

The extension class of this latter sequence is the Kodaira-Spencer class.

Finally, taking $n=2r-1$ and twisting by r, proposition 1.3 yields a vector extension of the intermediate jacobian

(2.8)
$$0\to H^r(X,\Omega_X^{r-1})\to\mathrm{Ext}^1_{EHS}(\mathbb{Z}(0),\mathcal{H}(r))\to\mathrm{Ext}^1_{HS}(\mathbb{Z}(0),H(r))\to 0.$$

When $r = 1$, this sequence arises in the standard way from the exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}^*_{\mathcal{X}} \to \mathcal{O}^*_X \to 0.$$

Example 2.5. In [2], the authors consider the singular quartic surface X in $\mathbb{P}^3$ with homogeneous equation $w(x^3 - y^2 z) + f(x, y, z) = 0$, where f is a general quartic. They show that the Hodge structure $H^2(X, \mathbb{Z}(1))$ is just a direct sum of copies of $\mathbb{Z}(0)$, but that even $\otimes\mathbb{Q}$ this space is not spanned by classes of algebraic divisors. Because X is projective, any analytic line bundle on X can be algebrized, so the analytic exponential sequence shows that the image of the algebraic Cartier divisors in cohomology is given by $\ker(H^2(X, \mathbb{Z}(1)). \to H^2(X, \mathcal{O}_X))$. In other words,

$$\{\text{Cartier divisors in cohomology}\} \cong \mathrm{Hom}_{EHS}(\mathbb{Z}(0), \mathcal{H}(1)) \subset \mathrm{Hom}_{HS}(\mathbb{Z}(0), H^2(X, \mathbb{Z}(1))). \tag{2.9}$$

3. Cycle Classes

Let X be a proper complex variety, and let $Z \subset X$ be a closed subvariety. We assume throughout this section that Z is contained in the smooth locus of X, and we write $U = X - Z$. Let $h' : X' \to X$ be a proper, birational map such that $D' := h'^{-1}(Z)$ is a normal crossings divisor, and $h' : X' - D' \cong U$. Let $j' : X' - D' \hookrightarrow X'$, and write $j = h'j'(h'|_{X'-D'})^{-1} : U \hookrightarrow X$. Then $Rh'_*\Omega^\bullet_{X'}(\log D') \cong Rj_*\Omega^\bullet_U$ is canonical, independent of the choice of X', D'. Further we have maps

$$H^*(U, \mathbb{C}) \to \mathbb{H}^*(X, Rh'_*\Omega^\bullet_{X'}(\log D')) \cong \mathbb{H}^*(X', \Omega^\bullet_{X'}(\log D')). \tag{3.1}$$

Suppose now that $Z \subset X$ has pure codimension r. Using (3.1), we define an enriched Hodge structure

$$\mathcal{E} = \Big(H^{2r-1}(U, \mathbb{Z}(r)), \mathbb{H}^{2r-1}(X', \Omega^\bullet_{X'}(\log D'))(r)\Big) \tag{3.2}$$

Here $\mathbb{H}^{2r-1}(X', \Omega^\bullet_{X'}(\log D'))(r)$ refers to the diagram $V_\bullet$ with

$$V_p = \mathbb{H}^{2r-1}(X', \Omega^\bullet_{X'}(\log D')\big/F^{p+r}\Omega^\bullet_{X'}(\log D')).$$

Let $\mathcal{Z} = \sum n_i Z_i$ be a codimension r cycle on X which is supported on Z. Because points on Z are assumed smooth on X, we have defined a class in local cohomology $[\mathcal{Z}]_{loc} \in \mathbb{H}^{2r}_Z(X, \Omega^\bullet_X)$. Dropping the supports, we obtain $[\mathcal{Z}] \in \mathbb{H}^{2r}(X, \Omega^\bullet_X)$.

Assume $[\mathcal{Z}] = 0$. Write

$$\mathcal{H}^{2r-1}(r) = \Big(H^{2r-1}(X, \mathbb{Z}(r)), \mathbb{H}^{2r-1}(X, \Omega^\bullet)(r)\Big) \in EHS.$$

Our objective is to construct an extension of enriched Hodge structures $\{\mathcal{Z}\} \in \mathrm{Ext}^1(\mathbb{Z}(0), \mathcal{H}^{2r-1}(r))$:

$$\{\mathcal{Z}\}: \quad 0 \to \mathcal{H}^{2r-1}(r) \to \mathcal{E}_{\mathcal{Z}} \to \mathbb{Z}(0) \to 0. \tag{3.3}$$

Here $\mathcal{E}_{\mathcal{Z}}$ is a subquotient of $\mathcal{E}$, (3.2). Let $h'' : X'' \to X$ be a resolution of singularities. Since Z is supported on the locus of smooth points of X, h'' is an isomorphism over a neighborhood of Z. Consider the fibre product diagram ($\widetilde{X} = X' \times_X X''$)

$$\begin{tikzcd} X'' \arrow[d, "h''"] & \widetilde{X} \arrow[l, "k''"'] \arrow[d, "k'"] & \widetilde{D} \arrow[l] \arrow[d, "\cong"] \\ X & X' \arrow[l, "h'"'] & D'. \arrow[l] \end{tikzcd} \tag{3.4}$$

Let $\Omega^\bullet_{X'}(\log D') \to I'^{\bullet\bullet}$ be an injective resolution of the log complex, so $I'^{\bullet\bullet}$ is a double complex. Truncating yields injective resolutions for the Hodge subcomplexes, $F^p\Omega^\bullet_{X'}(\log D') \to F^pI'^{\bullet\bullet}$. Define

$$F^p\mathcal{C} := \mathrm{Cone}(F^p\Omega^\bullet_X \to h'_*F^pI'^{\bullet\bullet}); \quad \mathcal{C} := F^0\mathcal{C}.$$

In the same way, we can define objects $F^p\widetilde{\mathcal{C}}$ in the derived category of sheaves on X''. Let $\Omega^\bullet_{\widetilde{X}}(\log \widetilde{D}) \to \widetilde{I}^{\bullet\bullet}$ be an injective resolution and define

$$F^p\widetilde{\mathcal{C}} := \mathrm{Cone}(F^p\Omega^\bullet_{X''} \to k''_*F^p\widetilde{I}^{\bullet\bullet}); \quad \widetilde{\mathcal{C}} := F^0\widetilde{\mathcal{C}}.$$

One has quasi-isomorphisms

$$F^*\mathcal{C} \to Rh''_*F^*\widetilde{\mathcal{C}}$$

since both sides calculate the Borel-Moore homology (with some shift) $H_{2\dim(X)-\bullet}(Z, \mathbb{C})$.

Write $\mathcal{C}/F^p := \mathrm{Cone}(F^p\mathcal{C} \to \mathcal{C})$. We have a distinguished triangle

$$\Omega^\bullet_X/F^p \to Rh'_*(\Omega^\bullet_{X'}(\log D')/F^p) \to \mathcal{C}/F^p.$$

By purity, $\mathcal{C}/F^p = (0)$ for $p \le r$. We get a diagram

(3.5)

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{H}^{2r-1}(X, \Omega^\bullet) \arrow[r] \arrow[d] & \mathbb{H}^{2r-1}(X', \Omega^\bullet_{X'}(\log D')) \arrow[r, "s"] \arrow[d] & H_{top}(Z, \mathbb{C}) \\ 0 \arrow[r] & \mathbb{H}^{2r-1}(X, \Omega^\bullet/F^r) \arrow[r, "\cong"] & \mathbb{H}^{2r-1}(X', \Omega^\bullet_{X'}(\log D')/F^r) \arrow[r] & 0 \end{tikzcd}$$

Our cycle $\mathcal{Z}$ defines a map $[\mathcal{Z}] : \mathcal{C}(-r) \to H_{top}(Z, \mathbb{C})$, and the assumed vanishing of $[\mathcal{Z}]$ says that the image of that map falls in the image of

s. More precisely, we define the enriched Hodge structure $\mathcal{E}_{\mathcal{Z}} = (H, V_\bullet)$ where

$$H := s^{-1}([\mathcal{Z}] \cdot \mathbb{Z}) \subset H^{2r-1}(U, \mathbb{Z}(r)); \tag{3.6}$$

$$V_p := s^{-1}[\mathcal{Z}] \cdot \mathbb{C} \subset H^{2r-1}(X', \Omega^\bullet_{X'}(\log D')/F^{p-r}). \tag{3.7}$$

(In the definition of V_p, we think of $[\mathcal{Z}] \cdot \mathbb{C} \subset H^{2r-1}(X, \mathcal{C}/F^{p-r})$ which makes sense because this group receives $H^{2r-1}(X, \mathcal{C}) \cong H_{top}(Z, \mathbb{C})$. We have $V_p = \mathbb{H}^{2r-1}(X, \Omega^\bullet/F^{r+p})$ for $p \leq 0$.) It is straightforward to check that with this definition, one has an exact sequence (3.3), defining $\{\mathcal{Z}\}$.

Example 3.1. Take $X \subset \mathbb{P}^2$ to be the cuspidal rational curve with affine equation $y^2 = x^3$. Take $Z = \{z_1, z_2\}$ where the z_i are smooth, and take $\mathcal{Z} = (z_1) - (z_2)$. We claim that

$$\mathrm{Ext}^1(\mathbb{Z}(0), \mathcal{H}^1(1)) \cong H^1(X, \mathcal{O}) \cong \mathrm{Pic}(X) \tag{3.8}$$

and that $\{\mathcal{Z}\}$ is the class of $\mathcal{Z}$ in $\mathrm{Pic}(X)$.

To begin with, it is not totally obvious that $\{\mathcal{Z}\}$ is defined in this case because one must check that $[\mathcal{Z}] = 0$ in $\mathbb{H}^2(X, \Omega^\bullet)$. Ω^2_X is torsion, and the identity $2ydy = 3x^2dx$ shows it is spanned by $dx \wedge dy$, $xdx \wedge dy$. the substitution $x \mapsto t^2$, $y \mapsto t^3$ shows that $3ydx - 2xdy$ is a torsion 1-form. But $d(3ydx - 2xdy) = -5dx \wedge dy$ and $d(x(3ydx - 2xdy)) = -7xdx \wedge dy$, so $d : \Omega^1_{tors} \twoheadrightarrow \Omega^2_{tors}$. It follows from this that

$$\mathbb{H}^2(X, \Omega^\bullet) \cong \mathbb{H}^2(X, \Omega^\bullet/tors).$$

Let $\pi : \mathbb{P}^1 \to X$ be the normalization. There is a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "d"] & \pi_*\mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d, "d"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_X/tors \arrow[r] & \pi_*\Omega^1_{\mathbb{P}^1} \arrow[r] & \mathbb{C} \arrow[r] & 0.
\end{tikzcd}$$

We conclude $\mathbb{H}^2(X, \Omega^\bullet/tors) \cong \mathbb{C}$, and the cycle class is given by the degree so $[\mathcal{Z}] = 0$. In this case $\mathcal{E}_{\mathcal{Z}}$ is given ((3.6), (3.7)) by

$$V_{-1} = H^1(X, \Omega^\bullet_X(\log Z)) \cong H^1(X, \mathcal{O}_X) = V_0$$

and the diagram (3.3) becomes

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H^1(X, \Omega^\bullet_X(\log Z)) \arrow[r, "c"] \arrow[d, "\alpha", "\wr"'] & \mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] & 0 &
\end{tikzcd}$$

The obstruction to splitting is just $\alpha \circ c^{-1}(1) \in H^1(\mathcal{O}_X)$. (The reader should check that $H^0(X, \Omega^1(\log \mathcal{Z})/tors) = (0)$ so α above is an isomorphism and that the cycle class is the class in the Picard group as claimed.)

Proposition 3.2. *With notation as in* (3.3),

$$Ext^1_{EHS}(\mathbb{Z}(0), \mathcal{H}^{2r-1}(r)) \cong \mathbb{H}^{2r-1}(X, \Omega^\bullet_X/F^r)/Image\big(H^{2r-1}(X, \mathbb{Z}(r))\big).$$

Proof. The proof is straightforward using proposition 1.3. □

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Dept. of Mathematics, University of Chicago, Chicago, IL 60637, USA

E-mail address: bloch@math.uchicago.edu

School of mathematics, Tata Institute of Fundamental Research, Homi Bhabha Rd., Mumbai 400005, India

E-mail address: srinivas@math.tifr.res.in