

ALGEBRAIC CYCLES AND ADDITIVE CHOW GROUPS

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ABSTRACT. A report on recent results and outstanding problems concerning additive chow groups.

“Mighty oaks from little acorns grow.”

1. INTRODUCTION

Additive algebraic K -theory means roughly “replace the algebraic group GL by the lie algebra \mathfrak{gl} where ever you see it”, [L]. Given the central role of algebraic cycles in motivic cohomology, one may ask for an algebraic cycle interpretation of additive K -theory. More than a simple restatement of the theory, such a geometric reformulation suggests new problems. What are motivic sheaves over $k[t]/(t^2)$? What is the tangent space to the space of motives?

One should, I believe, have the following picture in mind. An algebraic circle is represented by the pair $\mathbb{A}^1, \{0, t\}$ for any $t \neq 0$. It is natural, geometrically, to think of the limiting situation $t \rightarrow 0$ as represented by

$$(1.1) \quad \mathbb{A}^1, 2(0)$$

As a simple example, for k a field

$$(1.2) \quad H_M^1(k, \mathbb{Z}(1)) = \text{Pic}(\mathbb{A}_k^1, \{0, t\}) = \mathbb{G}_m(k) = k^\times,$$

It is natural to write for the corresponding additive group

$$(1.3) \quad TH_M^1(k, \mathbb{Z}(1)) := \text{Pic}(\mathbb{A}_k^1, 2(0)) = \mathbb{G}_a(k) = k.$$

A word of warning. The notation TH_M suggests tangent space, but this is perhaps not the precise analogy. A better picture would be a sort of non-semi-stable degeneration, with a group like \mathbb{G}_m which is constant in the parameter t degenerating to an additive group for $t = 0$.

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2. HYPERBOLIC VERSUS EUCLIDEAN

Another analogy is the picture of the hyperbolic metric on a disk of expanding radius. As the radius tends to ∞ , the metric tends to the euclidean metric. The link with motivic cohomology is summarized by the following 4-term exact sequence

$$(2.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & H_M^1(\mathbb{C}, \mathbb{Z}(2)) & \rightarrow & B_2(\mathbb{C}) & \xrightarrow{\delta} & \mathbb{C}^\times \otimes \mathbb{C}^\times \rightarrow K_2(\mathbb{C}) \rightarrow 0 \\ & & \downarrow \text{reg} & & \downarrow \text{vol} & & \\ & & \mathbb{R} & \xrightarrow{=} & \mathbb{R} & & \end{array}$$

Here $B_2(\mathbb{C})$ has generators $[x]_2$, $x \in \mathbb{C} - \{0, 1\}$, and $\text{vol}[x]_2$ is the hyperbolic volume of the tetrahedron in hyperbolic 3-space with vertices at infinity at the points $0, 1, \infty, x \in \mathbb{P}^1(\mathbb{C})$. One defines $\delta[x]_2 = x \otimes (1-x)$, and the symbol $[x]_2 \in B_2$ satisfies the classical 5-term relation.

The additive analogue of 2.1 was worked out in [BE] using a K -theoretic definition of the additive motivic group $TH_M^1(k, \mathbb{Z}(2))$ involving the relative K -group $K_2(\mathcal{O}_{\mathbb{A}^1, 0}, (t^2))$:

$$(2.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & TH_M^1(k, \mathbb{Z}(2)) & \rightarrow & TB_2(k) & \xrightarrow{T\delta} & k \otimes k^\times \xrightarrow{a \otimes b \rightarrow adb/b} \Omega_k^1 \rightarrow 0 \\ & & \downarrow \rho & & \downarrow \rho & & \\ & & k & \xrightarrow{=} & k & & \end{array}$$

By definition, $TB_2(k)$ is a k^\times -module (not a k -vector space!) with generators $\langle x \rangle_2$ which satisfy the 4-term information-theory equation first identified in this context by Kontsevich

$$(2.3) \quad \langle x \rangle_2 - \langle y \rangle_2 + x \star \langle y/x \rangle_2 + (1-x) \star \langle (1-y)/(1-x) \rangle_2 = 0$$

The regulator map ρ in (2.2) is defined by $\rho \langle x \rangle_2 = x(1-x)$. One might hope that $\rho \langle x \rangle_2$ represents some Euclidean polyhedron with Euclidean volume $x(1-x)$. The action of k^\times on the target of ρ is by the cube of the standard character, suggesting an interpretation in terms of volume in \mathbb{R}^3 .

3. CYCLES

We consider algebraic cycles on $\mathbb{A}^1 \times (\mathbb{P}^1)^n$, where \mathbb{A}^1 has parameter t and the \mathbb{P}^1 have parameters t_i , $1 \leq i \leq n$. We have divisors $\sigma_i : t_i = 1$ and $\delta_i = (t_i)_0 - (t_i)_\infty$. Fix $m \geq 2$. A closed subvariety $Z \subset \mathbb{A}^1 \times (\mathbb{P}^1)^n$ will be said to be congruent to 1 mod t^m if scheme-theoretically

$$(3.1) \quad Z \cdot \{t^m = 0\} \subset \bigcup_{i=1}^n Z \cdot \sigma_i$$

We consider algebraic cycles $z = \sum n_\nu Z_\nu$ such that all faces $Z_\nu \cdot \delta_I$ (I multi-index) have $\text{codim.} \geq \#(I)$ in Z_ν and are congruent to 1 mod m . Write

$$(3.2) \quad T_m CH^p(k, q) = T_m H_M^{2p-q}(k, \mathbb{Z}(p))$$

for the resulting higher Chow groups, generated by cycles of codim. p on $\mathbb{A}^1 \times (\mathbb{P}^1)^{q-1}$. The case of 0-cycles has been worked out by K. Rülling.

Theorem 3.1 (K. Rülling, [R]). $T_m H_M^n(k, \mathbb{Z}(n)) \cong W_{m-1} \Omega^{n-1}$, the de Rham-Witt groups built from the “big” Witt ring on k .

Remark 3.2. Rülling uses a slightly different version of the congruence condition (3.1). I have checked casually that his results hold under (3.1), but this should perhaps be verified more carefully.

A cycle-theoretic version of (2.2) involves cycles of dimension 1. This has been considered by J. Park [P]. His main result is the construction of a non-trivial regulator map

$$(3.3) \quad \rho_{m,n} : T_m CH^{n-1}(k, n) \rightarrow \Omega_k^{n-3}.$$

(Of particular interest is the case $n = 3$, $\rho_{m,3} : T_m CH^2(k, 3) \rightarrow k$.)

4. AN ABSTRACT REGULATOR CONSTRUCTION

In trying to generalize Park’s regulator construction to cycles of dim. $r > 1$, one is led to consider meromorphic differential forms

$$(4.1) \quad \frac{t_{i_0} - 1}{t^{m+1}} dt_{i_1}/t_{i_1} \wedge \dots \wedge dt_{i_r}/t_{i_r}$$

on $\mathbb{A}_t^1 \times (\mathbb{P}_{t_i}^1)^n$. The most subtle aspect of his work, the careful control of signs necessary to verify that the regulator he constructs is trivial on boundaries of two-dimensional cycles on $\mathbb{A}^1 \times (\mathbb{P}^1)^{n+1}$, will not be attempted here; but let me sketch the construction of a generalized regulator on cycles.

Let X be a smooth, projective variety, and let D_0, \dots, D_r be effective Cartier divisors on X . Let $Z \subset X$ be a closed subvariety of dimension r . We assume all intersections $Z \cdot D_I$ are either empty or have the correct dimension. In particular, $Z \cap \bigcap_0^r D_i = \emptyset$.

For $z \in Z$ a closed point, we have surjections

$$(4.2) \quad H_z^r(Z, \Omega_{Z/k}^r) \twoheadrightarrow H^r(Z, \Omega_{Z/k}^r) \xrightarrow{\text{deg}} k$$

(The group on the left is local cohomology. We do not assume Z smooth.)

Let ω be a meromorphic Kähler r -form on X which is regular on $X - \bigcup_0^r D_i$. (For example, if $X = \mathbb{A}_t^1 \times (\mathbb{P}_{t_i}^1)^n$ one might take $\omega = \frac{t_{i_0}-1}{t_{i_0}^{m+1}} dt_{i_1}/t_{i_1} \wedge \dots \wedge dt_{i_r}/t_{i_r}$ with $D_0 : t = 0$ and $D_j = (t_{i_j})_0 + (t_{i_j})_\infty$, $1 \leq j \leq r$.)

Choose $i \neq j \in [0, r]$ and define $X(ij) = (D_i + D_j) \cap \bigcap_{h \neq i, j} D_h$ (resp. $Z(ij) = X(ij) \cap Z$). The r open sets $X - (D_i + D_j)$, $X - D_h$, $h \neq i, j$ cover $X - X(ij)$, and we may view ω as a Čech $r-1$ cocycle representing a class $\omega(ij) \in H^{r-1}(X - X(ij), \Omega_{X/k}^r)$ (resp. by restriction $\omega_Z(ij) \in H^{r-1}(Z - Z(ij), \Omega_{Z/k}^r)$.)

$Z(ij)$ is a finite set of points which we can write as a disjoint union $Z(ij) = Z_j(i) \amalg Z_i(j)$, where $Z_j(i) \subset Z - Z \cap D_j$. Write $\deg_j(i) \in k$ for the image of $\omega_Z(ij)$ under the composition

$$(4.3) \quad H^{r-1}(Z - Z(ij), \omega^r) \xrightarrow{\partial} H_{Z(ij)}^r(Z, \omega^r) \xrightarrow{\text{proj}} H_{Z_j(i)}^r(Z, \omega^r) \xrightarrow{(4.2)} k$$

Clearly, $\deg_j(i) = -\deg_i(j)$.

Now take $\nu \in [0, r]$ with ν, i, j all distinct. Note the sets $Z_j(\nu)$ and $Z_j(i)$ coincide. (They are the intersection of Z with all the D_h , $h \neq j$.) Furthermore, the open coverings

$$(4.4) \quad \begin{aligned} X - (D_i + D_j), X - D_h, h \neq i, j \\ X - (D_\nu + D_j), X - D_h, h \neq \nu, j \end{aligned}$$

agree upto reordering as open coverings of $X - X(ij) - D_j = X - X(\nu j) - D_j$. It follows that

$$(4.5) \quad \deg_i(j) = -\deg_j(i) \stackrel{(*)}{=} \pm \deg_j(\nu) = \mp \deg_\nu(j).$$

But now, for a fourth index μ , the identity $(*)$ above gives

$$(4.6) \quad \deg_i(j) = \pm \deg_\nu(j) = \pm \deg_\nu(\mu)$$

We conclude that, upto a sign which depends on i, j and the ordering of the divisors D_h , the quantity

$$(4.7) \quad \deg_j(i) \in k$$

is independent of the choice of i, j .

Example 4.1. When Z is a curve ($r = 1$) the poles of $\omega|_Z$ are supported on $Z \cap (D_0 + D_1)$. Our construction then amounts to taking residues along those poles lying on $Z \cap D_0$.

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