Homework 8
due Wed, 9th Week

From Jacob:

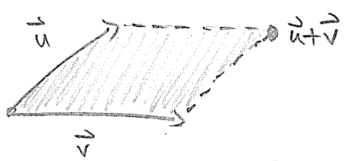
7.1 — #1 a,b; #2 a; #3; #6 c; #10

Not from Jacob: Complete the following investigation of the geometric nature of the determinant.

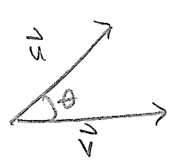
Def If $\vec{u}, \vec{v} \in \mathbb{R}^2$ are two vectors in the plane, the parallelogram spanned by $\vec{u}, \vec{v}$ is

$$P(\vec{u},\vec{v}) = \{ \alpha\vec{u} + \beta\vec{v} \mid 0 \le \underbrace{\alpha, \beta \le 1}_{\text{both are between } 0 + 1} \}.$$

1. Explain why the following geometric picture reflects the definition of $P(\vec{u},\vec{v})$.



2. Show that the area of $P(\vec{u},\vec{v})$ is $\sin\theta \cdot \|\vec{u}\| \, \|\vec{v}\|$, where θ is the angle between $\vec{u}$ and $\vec{v}$.



$2\frac{1}{2}$. The parallelogram problem from Homework 7.

3. Show that if $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then

$$\sin\theta \, \|\vec{u}\| \, \|\vec{v}\| = \left\langle \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle.$$

(Hint: use the geometric formula for $\langle , \rangle$.) Conclude that

$$\text{Area } P(\vec{u}, \vec{v}) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

4. If A is a 2×2-matrix, show that if

$$A\,P(\vec{u}, \vec{v}) = \{ A \cdot \vec{v} \mid \vec{v} \in P(\vec{u}, \vec{v}) \},$$

(this is the definition of $AP(\vec{u}, \vec{v})$)

then $A\,P(\vec{u}, \vec{v}) = P(A\vec{u}, A\vec{v})$. Explain why this makes sense geometrically.

5. Show that if A is a 2×2-matrix and $\vec{u}, \vec{v} \in \mathbb{R}^2$ then

$$\text{Area } P(\vec{u}, \vec{v}) = \det(A) \cdot \text{Area } AP(\vec{u}, \vec{v}).$$

This means that A distorts the area of parallelograms by a factor of $\det(A)$.

6. If $\det(A) = 0$, what do parallelograms of the form $AP(\vec{u}, \vec{v})$ look like?

7. Explain why it makes geometric sense that

$$\det(AB) = \det(A)\det(B).$$

(What happens if I distort area by a factor of 2 and then by a factor of 3?)