Homework 3
due Wed, Jan 28

1. The 2 problem left from last week:

   3.3 — # 13 a, c # 17

2. Explain why the inverse of $\begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \\ 2 & 5 & 1 \end{pmatrix}$ does not exist.

3. Suppose that A is an $n\times n$-matrix in RREF. Show that $A = Id_n \iff$ the only solution to $A\vec{v} = \vec{0}$ is the zero vector $\vec{v} = \vec{0}$.

4. Assume that A, B are $n\times n$-matrices with $AB = Id_n$. Show that the only solution to $B\vec{v} = \vec{0}$ is $\vec{v} = \vec{0}$. Explain why the only RREF matrix row equivalent to B is Id_n.

5. This problem verifies that our method for finding the inverse of a matrix actually works. Let A be an $n\times n$-matrix and consider the augmented matrix

$$(A \mid Id_n)$$

Row reducing given a row-equivalent matrix

$$(C \mid B)$$

where C is in RREF.

a) Explain why if $C = Id_n$, then $AB = Id_n$.
(This means that B is the inverse of A.)

b) Explain why B cannot have a row of zeros.
Show that if $C \neq Id_n$ then it has a row of zeros.
Deduce that if $C \neq Id_n$ then A has no inverse.

6. Decide whether the following sets S are vector spaces.
If yes, show that it is a vector space by
① picking $u, v \in S$ and showing $u+v \in S$
② picking $r \in \mathbb{R}$ and $u \in S$, and showing $ru \in S$.
If no, find explicit $u, v \in S$ or $u \in S, r \in \mathbb{R}$ so that either $u+v \notin S$ or $r \cdot u \notin S$.

Example • $S = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x+y=0 \}$ is a vector space.

If $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in S$, then $y_1 = -x_1$
$y_2 = -x_2$

so $y_1 + y_2 = -(x_1 + x_2)$.

Therefore $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in S$ (since the 1st coord. + 2nd coord. = 0)

• $S = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 2 \}$ is not a vector space.

Note that $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \in S$ but $2 \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \notin S$.

a) $S = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x \neq 2 \}$.

b) The sol'n set of $\begin{cases} x+y=1 \\ z=5 \end{cases}$

c) The line $\begin{cases} x=0 \\ y+z=0 \end{cases}$ in $\mathbb{R}^3$.

d) The point $\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} \in \mathbb{R}^3$.

e) The set $\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid y=0 \}$

f) The set of all rational numbers $\mathbb{Q} \subseteq \mathbb{R}$.