The Chern character does not commute with proper pushforward. In other words, let $f : X \to Y$ be a proper morphism of nonsingular varieties. Then the square

$$
\begin{array}{ccc}
K(X) & \xrightarrow{f_*} & K(Y) \\
\downarrow^{\text{ch}_X} & & \downarrow^{\text{ch}_Y} \\
A(X) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{f_*} & A(Y) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}
$$

doesn’t commute, where $A(X)$ denotes the Chow ring and ch is the Chern character. The Grothendieck-Riemann-Roch theorem states that

$$
\text{ch}(f_*\alpha) \cdot \text{td}(T_Y) = f_* (\text{ch}(\alpha) \cdot \text{td}(T_X)),
$$

where td denotes Todd genus. We describe the proof when $f$ is a projective morphism.

## 1 Statement of the theorem

Fix a field $k$. In this document the word ‘scheme’ will mean ‘$k$-scheme of finite type.’ Let $X$ be a scheme. $K^\circ(X)$ denotes the Grothendieck group of vector bundles on $X$. $K_c(X)$ denotes the Grothendieck group of coherent sheaves on $X$. If $X$ is quasiprojective nonsingular, the canonical homomorphism

$$
K^\circ(X) \to K_c(X)
$$

is an isomorphism. This is because the local rings of $X$ are regular, and hence of global dimension equal to their finite Krull dimension, which is bounded above by the dimension of $X$. Therefore any coherent sheaf $\mathcal{F}$ on $X$ admits a finite locally free resolution

$$
\text{ch}(f_*\alpha) \cdot \text{td}(T_Y) = f_* (\text{ch}(\alpha) \cdot \text{td}(T_X)),$$
0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathcal{F} \to 0,

yielding an inverse of the above homomorphism which takes \([\mathcal{F}]\) to \(\sum_{i=0}^{\infty}(-1)^i[E_i]\).

So, when we are studying a nonsingular variety \(X\), we can write \(\mathcal{K}(X)\) with no ambiguity. The notation \(H^i(X, \mathcal{F})\) denotes the \(i\)th right derived functor of the global sections functor \(\Gamma\) on \(X\) with coefficients in the sheaf \(\mathcal{F}\).

Let \(X, Y\) be schemes. For any morphism \(f: Y \to X\) there is an induced homomorphism
\[
f^*: \mathcal{K}(X) \to \mathcal{K}(Y),
\]
taking a vector bundle \([E]\) to \([f^*E]\) where \(f^*E = Y \times_X E\) is the pullback bundle. For any proper morphism of schemes \(f: X \to Y\) there is a homomorphism
\[
f_*: \mathcal{K}(X) \to \mathcal{K}(Y)
\]
which takes \([F]\) to \(\sum_{i \geq 0}(-1)^i[R^i f_* \mathcal{F}]\), where \(R^i f_* \mathcal{F}\) denotes \(i\)th higher direct image. For the remainder of this document, \(X\) will denote a smooth quasiprojective algebraic variety.

We consider for the moment the situation when \(X\) is moreover a complex variety. Then, we have the usual resolution of the constant sheaf \(\mathcal{Z}\) by the complex of singular cochains, and characteristic classes of vector bundles on \(X\) lying in \(H^*(X, \mathcal{Z})\).

The Chern character \(\text{ch}(E)\) of a vector bundle \(E\) on \(X\) is defined by the formula
\[
\text{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i).
\]
Here \(\alpha_i\) are Chern roots for \(E\). When \(E\) has a filtration with line bundle quotients \(L_i\), then \(\alpha_i = c_1(L_i) \in H^2(X, \mathcal{Z})\). The Todd class \(\text{td}(E)\) of a vector bundle \(E\) is defined by the formula
\[
\text{td}(E) = \prod_{i=1}^{r} Q(\alpha_i), \quad \text{where} \quad Q(x) = \frac{x}{1 - e^{-x}}.
\]
Since Chern roots are additive on exact sequences of bundles, \(\text{td}\) is multiplicative and \(\text{ch}\) additive. Moreover, if \(E\) and \(E'\) are vector bundles, \(\text{ch}(E \otimes E') = \text{ch}(E) \cdot \text{ch}(E')\). Therefore, \(\text{ch}\) descends to a homomorphism
\[
\text{ch}: \mathcal{K}(X) \to H^*(X, \mathcal{Z}) \otimes \mathcal{Q} \cong H^*(X, \mathcal{Q}).
\]
Note that the image of \(\text{ch}\) is contained in even cohomology.

Let \(f: X \to Y\) be a proper morphism of smooth quasiprojective complex varieties. Then the Grothendieck-Riemann-Roch theorem states that for \(\alpha \in \mathcal{K}(X)\),
\[
\text{ch}(f_* \alpha) \cdot \text{td}(T_Y) = f_*(\text{ch}(\alpha) \cdot \text{td}(T_X))
\]
in the ring \(H^*(Y, \mathcal{Q})\). The map \(f_*\) on cohomology can be described in the following way. The class \(\text{ch}(\alpha) \cdot \text{td}(T_X) \in H^*(X, \mathcal{Q})\) can be represented by an algebraic cycle
$W$ on $X$. This cycle admits a locally finite triangulation, i.e. such that a compact subset of $X$ intersects only finitely many simplices. This triangulation defines the class of $W$ in Borel-Moore homology of $X$, which is by definition the homology of the complex of locally finite singular chains. The functoriality of these chains for a proper map $f : X \rightarrow Y$ is evident, since if $C \subset Y$ is compact, $f^{-1}C$ is also, and hence only finitely many (singular) simplices have image in $Y$ intersecting $C$. By assumption, $X$ and $Y$ are smooth quasiprojective complex varieties. Poincaré duality extends to give an isomorphism

$$H^i(X, \mathbb{Z}) \cong H_{2n-i}^{BM}(X, \mathbb{Z}),$$

where $X$ has algebraic dimension $n$ and $H_i^{BM}$ denotes Borel-Moore homology (likewise for $Y$). This defines the map $f_*$. By taking the theorem in the special case of $f : X \rightarrow \{\cdot\}$, one recovers the theorem of Hirzebruch-Riemann-Roch (HRR), which in our case says, for $E$ a vector bundle on a nonsingular complex projective variety $X$,

$$\chi(X, E) = \int_X \text{ch}(E) \cdot \text{td}(T_X).$$

Here, the notation $\int_X$ means to take the cohomology in the highest degree, represent it as a linear combination of points via Poincaré duality, and count these points with multiplicity. Let us recover from this the statement of classical Riemann-Roch, which applies when $X$ is a complete nonsingular curve of genus $g$. The geometric genus of a curve is by definition $\dim_k H^0(X, \omega_X)$, the dimension of the global sections of the canonical sheaf $\omega_X = \Omega_X/k$ (our remarks so far restrict us in what follows to the case $k = \mathbb{C}$). The arithmetic genus of a curve is $\dim_k H^1(X, O_X)$. It happens that $\omega_X$ is a dualizing sheaf on $X$, and by Serre duality the vector spaces $H^0(X, \omega_X)$ and $H^1(X, O_X)$ are dual to one another. Their dimension can be taken as the definition of the genus of a (complete nonsingular) curve. In any event, since $H^0(X, \omega_X) = k$, this, together with HRR and the computation of the first two terms of the Todd class of a line bundle

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2} x + \sum_{k=1}^{m} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k},$$

reveals that

$$1 - g = \chi(X, O_X) = \frac{1}{2} \int_X c_1(T_X).$$

If $E$ is a vector bundle of rank $e$ on $X$, then since ch is additive on short exact sequences and $c_1(E)$ is simply the sum of the Chern roots of $E$, $\text{ch}(E) = e + c_1(E)$, and we have

$$\chi(X, E) = \int_X c_1(E) + e(1 - g).$$

In particular, when $E = O(D)$ is a line bundle,
The Chow ring. Let $X$ be a smooth scheme. The Chow group $A_k^*(X)$, resp. $A^k(X)$ denotes the group of algebraic cycles of dimension, resp. codimension, $k$ on $X$ modulo rational equivalence. We denote algebraic cycles of dimension (resp. codimension) $k$ on $X$ by $Z_k^X$, resp. $Z^k_X$. Since $X$ is smooth, the intersection product gives $A^*(X)$ the structure of commutative, graded ring with unit $[X]$. The notation $A(X)_\mathbb{Q}$ denotes $A(X) \otimes \mathbb{Z}[X]$.

Characteristic classes can be defined very easily as operators on the Chow ring. When $L$ is a line bundle on $X$, find a Cartier divisor on $X$ with $\mathcal{O}(D) \cong L$. Then $c_1(L) \cdot \alpha = [D] \cdot \alpha$ for $\alpha \in A^*(X)$; i.e. the action of $c_1(L)$ on the Chow ring of $X$ is simply intersection with $D$. The first Chern class of a bundle $E$ of rank $r$ can be defined simply in terms of determinants, as $c_1(E) = c_1(\wedge^r E)$. To define the higher classes, we mention the splitting construction.

Given a finite collection of vector bundles $\mathcal{S}$ of vector bundles on a scheme $X$, there is a flat morphism $f : X' \to X$ such that

1. $f^* : A(X) \to A(X')$ is injective, and
2. for each $E$ in $\mathcal{S}$, $f^*E$ has a filtration by subbundles

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$$

with line bundle quotients $L_i = E_i/E_{i-1}$.

The flag varieties of vector bundles provide the desired $X'$.

Now, with $f$ as in the splitting construction, $f^*E$ is filtered with line bundle quotients $L_i$. Define the Chern polynomial

$$c_i(f^*E) = \prod_{i=1}^r (1 + c_1(L_i)t).$$

Then $c_i(f^*E)$ is simply the coefficient of $t^i$ in $c_j(f^*E)$. By insisting that the $c_i(E)$ are natural under flat pullback, we determine the $c_i(E)$ completely.

We then define Chern character $\text{ch}$ and Todd class $\text{td}$ identically to as before. Defined algebraically in this way, the Chern character actually induces an isomorphism

$$\text{ch} : K(X)_\mathbb{Q} \to A(X)_\mathbb{Q}$$

of $\mathbb{Q}$-algebras. To see this, one passes to associated graded groups, giving $A_\mathbb{X}$ its natural filtration and $K_\mathbb{X}$ its topological filtration defined by letting $F_iK_\mathbb{X}$ be the subgroup generated by coherent sheaves whose support has dimension at most $k$. There is a surjection $A_k^X \to Gr_k K_\mathbb{X}$, which, composed with $\text{ch}$, gives the natural inclusion of $A_\mathbb{X}$ in $A(X)_\mathbb{Q}$. Since, after tensoring with $\mathbb{Q}$, $\text{ch}$ determines an isomorphisms on associated graded groups, the same must hold on the original groups.
The Grothendieck-Riemann-Roch theorem remains true if you replace ordinary cohomology with the Chow ring. Namely, for $\alpha \in K^*(X)$, $f: X \to Y$ a projective morphism of nonsingular schemes (over any field),

$$\text{ch}(f_*\alpha) \cdot \text{td}(T_Y) = f_*(\text{ch}(\alpha) \cdot \text{td}(T_X))$$

in the ring $A^*(Y)$. Here $f_*$ is the proper pushforward of algebraic cycles.

2 Proof of the theorem

Let $X$ be a nonsingular scheme. The proof of the theorem is organized in the following way. First we consider the toy case of the zero-section imbedding of $X$ in a vector bundle on it. After turning briefly to discuss the $K$-theory of a projective bundle on $X$, we discuss the deformation to the normal cone of a closed imbedding. In the final subsection, we use the results for the $K$-theory of a projective bundle to prove the main theorem in the case of a projection, and the deformation to the normal cone to prove the theorem in the case of a closed imbedding. Together, these constitute the proof of Grothendieck-Riemann-Roch in the case of a projective morphism.

2.1 The toy case

Let us first consider the special case of a closed imbedding $f: X \to Y$ where $Y = P(N \oplus 1)$ for $N$ an arbitrary vector bundle of rank $d$ on $X$; in particular, $f$ is the zero section imbedding of $X$ in $N$, followed by the canonical open imbedding of $N$ in $P(N \oplus 1)$. Let $p$ denote bundle projection $Y \to X$, and let $Q$ be the universal quotient bundle, of rank $d$, on $Y$. Let $s$ denote the section of $Q$ determined by the projection of the trivial factor in $p^* (N \oplus 1)$ to $Q$. Then $s$ is a regular section, and

$$f_* (f^* \alpha) = c_d(Q) \cdot \alpha. \quad (1)$$

Additionally, the Koszul complex

$$0 \to \wedge^d Q^\vee \to \ldots \to \wedge^2 Q^\vee \to Q^\vee \xrightarrow{s^\vee} \mathcal{O}_Y \to f_* \mathcal{O}_X \to 0$$

is a resolution of the sheaf $f_* \mathcal{O}_X$. For any vector bundle $E$ on $X$, we therefore have the explicit resolution of $E$

$$0 \to \wedge^d Q^\vee \otimes p^* E \to \ldots \to Q^\vee \otimes p^* E \to p^* E \to f_* E \to 0.$$

Hence,

$$\text{ch}_{f_*}[E] = \sum_{p=0}^d (-1)^p \text{ch}(\wedge^p Q^\vee) \cdot \text{ch}(p^* E). \quad (2)$$
Chern character $\text{ch}$ and Todd class $\text{td}$ are related by the formula
\[
\sum_{p=0}^{d} (-1)^p \text{ch}(\wedge^p Q) = c_d(Q) \cdot \text{td}(Q)^{-1}.
\] (3)

Combining (1), (2), and (3), we write
\[
\text{ch}_f^* E = c_d(Q) \cdot \text{td}(Q)^{-1} \cdot \text{ch}(p^* E) = f_*(f^* \text{td}(Q)^{-1} \cdot f^* \text{ch}(p^* E)).
\]

Since $f^* Q = N$ and $f^* p^* E = E$, this can be rewritten as
\[
\text{ch}_f^* E = f_*(\text{td}(N)^{-1} \cdot \text{ch}(E)).
\] (4)

By the multiplicativity of $\text{td}$ and the exact sequence of vector bundles arising from a regular imbedding of a nonsingular subvariety in a nonsingular variety
\[
0 \to T_X \to f^* T_Y \to N_X Y \to 0,
\]
we find
\[
\text{td}(N)^{-1} = f^* \text{td}(T_Y)^{-1} \cdot \text{td}(T_X).
\]

The right side of (4) is therefore
\[
f_*(f^* \text{td}(T_Y)^{-1} \cdot \text{td}(T_X) \cdot \text{ch} E) = \text{td}(T_Y)^{-1} \cdot f_*(\text{td}(T_X) \cdot \text{ch} E),
\]
and (4) can be rewritten as
\[
\text{ch}(f_! E) \cdot \text{td}(T_Y) = f_*(\text{ch}(E) \cdot \text{td}(T_X)).
\] (5)

2.2 $K(P)$

**Theorem 1.** Let $X$ be a nonsingular scheme, $E$ a vector bundle on $X$ of rank $n + 1$, $q : P = P(E) \to X$ the projection. Then, $K(P)$ is a free $K(X)$-module generated by the classes of $\mathcal{O}(-i)$, $i = 0, \ldots, n$.

**Proof (of Theorem).** There are two steps: first, showing that the classes of $\mathcal{O}(-i)$, $i = 0, \ldots, n$ generate a free submodule of $K(P)$ over $K(X)$; second, showing that these classes generate $K(P)$ as a module over $K(X)$. The below commutative diagram establishes notation.
For the first step, it suffices to write down projection maps $K(P) \to K(X)$. For $i = 0, 1, \ldots, n,$

$$R^i q_* (\Omega^j_{P/X} (j - i)) = \begin{cases} \mathcal{O}_X & \text{if } a = i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $H = \Omega^1_{P/X}(1)$, and $e_i : K(P) \to K(X)$ is given by

$$e_i(?) = (-1)^i q_* (? \otimes \wedge^i H),$$

then $e_i$ assumes the value $[\mathcal{O}_X]$ on $[\mathcal{O}(-i)]$ and 0 on $[\mathcal{O}(-j)]$ for $0 \leq j \neq i \leq n$. Hence the classes of $\mathcal{O}(-i), i = 0, \ldots, n$ generate a free module over $K(X)$.

For the second step, we must show that every coherent sheaf on projective space is equal to a linear combination of the $\mathcal{O}(-i), i = 0, \ldots, n$, in $K(P)$. The Koszul complex

$$0 \to \mathcal{O}(-n) \boxtimes \wedge^n H \to \cdots \to \mathcal{O}(-2) \boxtimes \wedge^2 H \to \mathcal{O}(1) \boxtimes H \to \mathcal{O}_{P \times P} \to \mathcal{O}_\Delta \to 0$$

is in fact a resolution of the diagonal $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_P \subset P \times X P$ for projective space. Therefore, for a coherent sheaf $?_1$ on $P$,

$$?_1 = p_1_*(\mathcal{O}_\Delta \otimes p_2^*(?)$$

$$= p_1_*(\sum_{i=0}^{n} (-1)^i \mathcal{O}(-i) \otimes \wedge^i H \otimes p_2^*(?)$$

$$= p_1_*(\sum_{i=0}^{n} (-1)^i \mathcal{O}(-i) \otimes (\wedge^i H \otimes ?))$$

$$= \sum_{i,j=0}^{n} (-1)^{i+j} \mathcal{O}(-i) \otimes_{\mathcal{O}_X} R^i q_* (\mathcal{P}, \wedge^i H \otimes ?)$$

in $K(P)$, where we have written simply $?$, etc. for the class $[?]$ in $K(P)$, and the last equality is by K"unneth. This proves step 2, and the theorem.
2.3 Deformation to the normal cone

Let $X$ be a closed subscheme of $Y$. The claim is that there is a scheme $M = M_X Y$, a closed imbedding $X \times \mathbb{P}^1 \hookrightarrow M$, and a flat morphism $\rho : M \to \mathbb{P}^1$ so that

$$
\begin{array}{c}
X \times \mathbb{P}^1 \\
\downarrow \text{pr} \\
\mathbb{P}^1
\end{array} \quad \quad \quad \quad \quad \quad \begin{array}{c}
\quad M \\
\downarrow \rho \\
\mathbb{P}^1
\end{array}
$$

commutes, and such that

1. Over $\mathbb{P}^1 - \{\infty\} = \mathbb{A}^1$, $\rho^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$ and the imbedding is the trivial one

$$X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1.$$

2. Over $\infty$, the divisor $M_\infty = \rho^{-1}(\infty)$ is the sum of two effective divisors

$$M_\infty = P(C \oplus 1) + \tilde{Y}$$

where $\tilde{Y}$ is the blowup of $Y$ along $X$. The imbedding of $X = X \times \{\infty\}$ in $M_\infty$ is the zero-section imbedding of $X$ in $C$ followed by the canonical open imbedding of $C$ in $P(C \oplus 1)$. The divisors $P(C \oplus 1)$ and $\tilde{Y}$ intersect in the scheme $P(C)$, which is imbedded as the hyperplane at infinity in $P(C \oplus 1)$, and as the exceptional divisor in $\tilde{Y}$. In particular, the image of $X$ in $M_\infty$ is disjoint from $\tilde{Y}$. Letting $M^0 = M_X^0 Y$ be the complement of $\tilde{Y}$ in $M$, one has a family of imbeddings of $X$:

$$
\begin{array}{c}
X \times \mathbb{P}^1 \\
\downarrow \text{pr} \\
\mathbb{P}^1
\end{array} \quad \quad \quad \quad \quad \quad \begin{array}{c}
\quad M^0 \\
\downarrow \rho^0 \\
\mathbb{P}^1
\end{array}
$$

which deforms the given imbedding of $X$ in $Y$ to the zero-section imbedding of $X$ in $C$.

Such an $M$ is found by blowing up $Y \times \mathbb{P}^1$ along $X \times \{\infty\}$.

2.4 Proof of Riemann-Roch for a projective morphism

**Theorem 2.** Let $f : X \to Y$ be a projective morphism of nonsingular varieties. Then for all $\alpha \in K(X)$,

$$
\text{ch}(f_* \alpha) \cdot \text{td}(T_Y) = f_*(\text{ch}(\alpha) \cdot \text{td}(T_X))
$$

in $A(Y)_Q$.

Let
\[ \tau_X : K(X) \to A(X)_Q \]

be defined by
\[ \tau_X(\alpha) = \ch(\alpha) \cdot \td(T_X). \]

Then the theorem can be reformulated as ‘\( \tau \) commutes with pushforward under a projective morphism’; i.e. \( f_\ast \circ \tau_X = \tau_Y \circ f_\ast \). It follows that if the theorem is valid for a closed imbedding \( g : X \to Y \times \mathbb{P}^m \) and for the projection \( p : Y \times \mathbb{P}^m \to Y \), then it is valid for the projective morphism \( g \circ p \).

**Riemann-Roch for closed imbeddings** The name of the game is to reduce the case of \( f : X \to Y \) a closed imbedding to the toy case. Let \( N \) denote the normal bundle to \( X \) in \( Y \). We shall use the deformation to the normal bundle to deform the imbedding \( f \) into the imbedding \( \tilde{f} : X \to P(N \oplus 1) \) discussed at the beginning of this section.

We have a diagram

\[
\begin{array}{c}
  X \xrightarrow{\tilde{f}} P(N \oplus 1) + Y \xrightarrow{\tau} M_\infty \to \{\infty\} \\
  \downarrow i_\infty \quad \downarrow k \quad \downarrow f_\infty \\
  X \times \mathbb{P}^1 \xrightarrow{f} M \xrightarrow{\tau} \mathbb{P}^1 \\
  \downarrow i_0 \quad \downarrow j_0 \\
  X \xrightarrow{f} Y \xrightarrow{\tau} M_0 \to \{0\}
\end{array}
\]

where \( M \) is the blowup of \( Y \times \mathbb{P}^1 \) along \( X \times \{\infty\} \). We may assume \( \alpha = [E] \), with \( E \) a vector bundle on \( X \). Let \( \tilde{E} = p^* E \), where \( p \) is the projection from \( X \times \mathbb{P}^1 \) to \( X \).

Choose a resolution \( G \), of \( F_\ast(\tilde{E}) \) on \( M \):
\[
0 \to G_n \to G_{n-1} \to \ldots \to G_0 \to F_\ast(\tilde{E}) \to 0. \tag{*}
\]

Since \( X \times \mathbb{P}^1 \) and \( M \) are both flat over \( \mathbb{P}^1 \), the restrictions of the sequence (*) to the fibers \( M_0 \) and \( M_\infty \) remains exact. Therefore \( j_0^\ast G \) resolves \( j_0^\ast(F_\ast(\tilde{E})) \) and \( j_\infty^\ast G \) resolves \( j_\infty^\ast(F_\ast(\tilde{E})) \). Since \( j_0^\ast F_\ast \tilde{E} = f_\ast i_0^\ast \tilde{E} = f_\ast(E) \),

(i) \( j_0^\ast G \) resolves \( f_\ast(E) \) on \( Y = M_0 \).

Similarly, \( j_\infty^\ast G \) resolves \( f_\ast(E) \) on \( M_\infty \). But, \( f(X) \) is disjoint from \( \tilde{Y} \). Therefore

(ii) \( k^\ast G \) resolves \( f_\ast(E) \) on \( P(N \oplus 1) \), and

(iii) \( l^\ast G \) is acyclic.

For a complex \( F \), of vector bundles, we write \( \ch(F) \) for the alternating sum \( \sum (-1)^i \ch(F_i) \). We compute the image of \( \ch(F_\ast E) \) in \( A(M)_Q \) (writing \( \ch(F_\ast E) \) in lieu of \( \ch(f_\ast E) \cdot [Y] \)):

\[
\begin{align*}
  j_0 \ast (\ch(F_\ast E)) &= j_0 \ast (\ch(j_0^\ast G),) \quad &\text{by (i)} \\
  &= \ch(G_\ast) \cdot j_0 \ast [Y] \quad &\text{(projection formula for Chern classes)} \\
  &= \ch(G_\ast) \cdot (k_\ast[P(N \oplus 1)] + l_\ast[\tilde{P}])
\end{align*}
\]
Proof of the theorem

(by the basic fact that \([M_0] - [M_\infty] = [\div \rho] = 0\) in \(A(M)_Q\))

\[ = k_*(\ch(k^*G_1)) + l_*(\ch(l^*G_2)) \quad \text{(projection formula)} \]

\[ = k_*(\ch(\overline{f}_*E)) + 0 \quad \text{by (ii) and (iii)}. \]

The morphism \(\overline{f}\) was precisely the object of study at the beginning of this section. Equation (4) of that section allows us to write

\[ (iv) \quad j_0* \ch(f_*E) = k_*((f_*((\td(N)^{-1} \cdot \ch(E)))) \quad \text{in} \quad A(M)_Q. \]

Let \(q: M \to Y\) be the composite of the blowdown \(M \to Y \times \mathbb{P}^1\) followed by the projection. By construction of \(M\), \(q \circ j_0 = \text{id}_Y\), and \(q \circ k \circ \overline{f} = f\). Applying \(q_*\) to (iv), we find

\[ \ch(f_*E) = f_*((\td(N)^{-1} \cdot \ch(E))). \]

The theorem now follows from the same manipulations as were used to pass from (4) to (5) in the toy case.

Riemann-Roch for the projection  Consider first more generally the projection \(f: Y \times Z \to Y\), with \(Z\) nonsingular. There is a commutative diagram

\[
\begin{array}{ccc}
K(Y) \otimes K(Z) & \xrightarrow{\gamma \otimes \tau} & A(Y)_Q \otimes A(Z)_Q \\
\downarrow \times & & \downarrow \times \\
K(Y \times Z) & \xrightarrow{\gamma \times \tau} & A(Y \times Z)_Q.
\end{array}
\]

Since the Todd class is multiplicative, \(\td(T_{Y \times Z}) = \td(T_Y) \times \td(T_Z)\). If \(Z = \mathbb{P}^m\), the left vertical map is surjective, and \(K(\mathbb{P}^m)\) is generated by \([\mathcal{O}(-i)]\), \(i = 0, 1, \ldots, m\), both statements following from Theorem 1. It suffices therefore to verify the theorem for the projection from \(\mathbb{P}^m\) to a point and \(\alpha = [\mathcal{O}(-i)]\); i.e. to verify the formula

\[ \int_{\mathbb{P}^m} \ch(\mathcal{O}(-i)) \cdot \td(T_{\mathbb{P}^m}) = \chi(\mathbb{P}^m, \mathcal{O}(-i)). \]

Here, if \(p: \mathbb{P}^m \to \text{Spec} k\) is the projection, the notation \(\int_{\mathbb{P}^m}\) denotes the extension of the proper pushforward \(p_*: A_0(\mathbb{P}^m) \to A_0(\text{Spec} k)\) by zero to the whole Chow ring \(A(\mathbb{P}^m)\). As both \(\ch\) and \(\chi\) are homomorphisms of rings, in particular it suffices to verify the formula after flipping sign

\[ \int_{\mathbb{P}^m} \ch(\mathcal{O}(n)) \cdot \td(T_{\mathbb{P}^m}) = \chi(\mathbb{P}^m, \mathcal{O}(n)), \]

\(n = 0, 1, \ldots, m\).

Now, \(\td(T_{\mathbb{P}^m}) = (x/1 - e^{-x})^{m+1}\), where \(x = c_1(\mathcal{O}_{\mathbb{P}^m}(1))\), and compute

\[ \int_{\mathbb{P}^m} e^{nx} x^{m+1} / (1 - e^{-x})^{m+1} = \binom{n+m}{n}. \]
To see this, note that the integrand is a power series in $x$, for which we want the coefficient of $x^m$. Dividing the integrand by $x^m$, this is the same as computing the residue of $e^{nx}/(1 - e^{-x})^{m+1}$. Changing variables $y = 1 - e^{-x}$ this is the same as asking for the residue of $(1 - y)^{-n-1}y^{-m-1}$, or the coefficient of the term of degree $m$ in $(1 - y)^{-n-1} = (1 + y + y^2 + \cdots)^{n+1}$, which is $\binom{n+m}{n}$. On the other hand, the sheaves $\mathcal{O}(n)$ for $n = 0, 1, \ldots, m$ are generated by global sections and have no higher cohomology; hence

$$\chi(P^n, \mathcal{O}(n)) = \dim_k \text{Sym}^n k^{m+1} = \binom{n+m}{m}.$$ 

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References
