Large Gaps Between Zeros of GL(2) $L$-Functions

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http://web.williams.edu/Mathematics/sjmiller/public_html/

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Introduction
The Random Matrix Theory Connection

Philosophy: Critical-zero statistics of $L$-functions agree with eigenvalue statistics of large random matrices.

- Montgomery - pair-correlations of zeros of $\zeta(s)$ and eigenvalues of the Gaussian Unitary Ensemble.

- Hejhal, Rudnick and Sarnak - Higher correlations and automorphic $L$-functions.

- Odlyzko - further evidence through extensive numerical computations.
Consecutive zero spacings of $\zeta(s)$ vs. GUE predictions (Odlyzko).
Let $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_i \leq \cdots$ be the ordinates of the critical zeros of an $L$-function.

**Conjecture**

Gaps between consecutive zeros that are arbitrarily large, relative to the average gap size, appear infinitely often.
Let $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_i \leq \cdots$ be the ordinates of the critical zeros of an $L$-function.

**Conjecture**

Gaps between consecutive zeros that are arbitrarily large, relative to the average gap size, appear infinitely often.

Letting

$$\Lambda = \limsup_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{\text{average spacing}},$$

this conjecture is equivalent to $\Lambda = \infty$.

- Best unconditional result for the Riemann zeta function is $\Lambda > 2.69$. 
Dedekind zeta functions in quadratic number fields

Higher degree $L$-functions are mostly unexplored. First nontrivial quantitative lower bound for an $L$-function of degree greater than 1:

**Theorem (Turnage-Butterbaugh ’14)**

Let $T \geq 2$, $\varepsilon > 0$, $\zeta_K(s)$ the Dedekind zeta function attached to a quadratic number field $K$ with discriminant $d$ with $|d| \leq T^\varepsilon$, and $S_T := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ be the distinct zeros of $\zeta_K \left( \frac{1}{2} + it, f \right)$ in the interval $[T, 2T]$. Let $\kappa_T$ denote the maximum gap between consecutive zeros in $S_T$. Then

$$\kappa_T \geq \sqrt{6} \frac{\pi}{\log \sqrt{|d|} T} \left(1 + O(d^\varepsilon \log T)^{-1}\right).$$

- Assuming GRH, this means $\Lambda \geq \sqrt{6} \approx 2.449$. 
Dedekind zeta functions in quadratic number fields

Using a similar argument with the added flexibility of smoothed mean-value estimates, been improved to

\[ \kappa_T \geq 2.866 \frac{\pi}{\log \sqrt{|d|} T} \left( 1 + O(d^\varepsilon \log T)^{-1} \right). \]

Assuming GRH, this means \( \Lambda \geq 2.866 \).
A Lower Bound on Large Gaps

We proved the following unconditional theorem for an $L$-function associated to a holomorphic cusp form $f$ on $GL(2)$.

**Theorem (BMMRTW ’14)**

Let $S_T := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ be the set of distinct zeros of $L\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let $\kappa_T$ denote the maximum gap between consecutive zeros in $S_T$. Then

$$\kappa_T \geq \frac{\sqrt{3}\pi}{\log T} \left(1 + O\left(\frac{1}{c_f (\log T)^{-\delta}}\right)\right),$$

where $c_f$ is the residue of the Rankin-Selberg convolution $L(s, f \times \overline{f})$ at $s = 1$.

Assuming GRH, there are infinitely many normalized gaps between consecutive zeros at least $\sqrt{3}$ times the mean spacing, i.e.,

$$\Lambda \geq \sqrt{3} \approx 1.732.$$
An Upper Bound on Small Gaps

Theorem (BMMRTW ’14)

$L$ in Selberg class primitive of degree $m_L$. Assume GRH for

$$\log L(s) = \sum_{n=1}^{\infty} b_L(n)/n^s, \sum_{n \leq x} |b_L(n) \log n|^2 = (1 + o(1)) x \log x.$$  

Have a computable nontrivial upper bound on $\mu_L$ (liminf of smallest average gap) depending on $m_L$.

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($m_L = 1$ due to Carneiro, Chandee, Littmann and Milinovich).

Key idea: use pair correlation analysis.
Results on Gaps and Shifted Second Moments
Shifted Moment Result

To prove our theorem, use a method due to R.R. Hall and the following shifted moment result.

**Theorem (BMMRTW ’14)**

\[
\int_T^{2T} L \left( \frac{1}{2} + it + \alpha, f \right) L \left( \frac{1}{2} - it + \beta, f \right) \, dt = c_f T \sum_{n \geq 0} \frac{(-1)^n 2^{n+1} (\alpha + \beta)^n (\log T)^{n+1}}{(n+1)!} + O \left( T (\log T)^{1-\delta} \right),
\]

where \( \alpha, \beta \in \mathbb{C} \) and \(|\alpha|, |\beta| \ll 1/\log T\).

Key idea: differentiate wrt parameters, yields formulas for integrals of products of derivatives.
Approximate functional equation:

\[ L(s + \alpha, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{s+\alpha}} e^{-\frac{n}{X}} + F(s) \sum_{n \leq X} \frac{\lambda_f(n)}{n^{1-s-\alpha}} + E(s), \]

where \( \lambda_f(n) \) are the Fourier coefficients of \( L(s, f) \), \( F(s) \) is a functional equation term, and \( E(s) \) is an error term.

We have an analogous expression for \( L(1 - s + \beta, f) \).
Shifted Moments Proof Technique

- Analyze product

\[ L(s + \alpha, f)L(1 - s + \beta, f), \]

where each factor gives rise to four products (so sixteen products to estimate).

- Use a generalization of Montgomery and Vaughan’s mean value theorem and contour integration to estimate product and compute the resulting moments.
Shifted Moment Result for Derivatives

- Shifted moment result yields lower order terms and moments of derivatives of $L$-functions by differentiation and Cauchy’s integral formula.

- Derive an expression for

$$\int_{T}^{2T} L^{(\mu)} \left( \frac{1}{2} + it, f \right) L^{(\nu)} \left( \frac{1}{2} - it, f \right) \, dt,$$

where $T \geq 2$ and $\mu, \nu \in \mathbb{Z}^+$. Use this in Hall’s method to obtain the lower bound stated in our theorem.

- Need $(\mu, \nu) \in \{(0, 0), (1, 0), (1, 1)\}$; other cases previously done (Good did $(0, 0)$ and Yashiro did $\mu = \nu$).
Using Hall’s method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg.

**Lemma (Bredberg)**

Let $y : [a, b] \to \mathbb{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then

$$
\int_{a}^{b} |y(x)|^2\,dx \leq \left( \frac{b - a}{\pi} \right)^2 \int_{a}^{b} |y'(x)|^2\,dx.
$$
Proving our Result

For $\rho$ a real parameter to be determined later, define

$$g(t) := e^{i\rho t \log T} L \left( \frac{1}{2} + it, f \right),$$

Fix $f$ and let $\tilde{\gamma}_f(k)$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.
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- $g(t)$ has same zeros as $L(s, f)$ (at $t = \tilde{\gamma}_f(k)$). Use in the modified Wirtinger’s inequality.
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  $g(t)$ has same zeros as $L(s, f)$ (at $t = \tilde{\gamma}_f(k)$). Use in the modified Wirtinger’s inequality.

- For adjacent zeros have
  \[
  \sum_{n=1}^{N-1} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g(t)|^2 \, dt \leq \sum_{n=1}^{N-1} \frac{\kappa^2 T}{\pi^2} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g'(t)|^2 \, dt.
  \]

- Summing over zeros with $n \in \{1, \ldots, N\}$ and trivial estimation yields integrals from $T$ to $2T$.
Proving our Result

- \(|g(t)|^2 = |L(1/2 + it, f)|^2\) and
  
  \[|g'(t)|^2 = |L'(1/2 + it, f)|^2 + \rho^2 \log^2 T \cdot |L(1/2 + it, f)|^2\]
  
  \[+ 2\rho \log T \cdot \text{Re} \left( \frac{L'(1/2 + it, f) \overline{L(1/2 + it, f)}}{L(1/2 + it, f)} \right).\]

- Apply sub-convexity bounds to \(L(1/2 + it, f)\):
  
  \[\int_T^{2T} |g(t)|^2 \, dt \leq \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |g'(t)|^2 \, dt + O \left( T^{2/3} (\log T)^{5/6} \right).\]

- As \(g(t)\) and \(g'(t)\) may be expressed in terms of \(L\left(\frac{1}{2} + it, f\right)\) and its derivatives, can write our inequality explicitly in terms of formulas given by our mixed moment theorem.
After substituting our formula, we have

\[
\frac{\kappa_T^2}{\pi^2} \geq \frac{3}{3\rho^2 - 6\rho + 4} (\log T)^{-2} \left( 1 + O(\log T)^{-\delta} \right).
\]

The polynomial in \( \rho \) is minimized at \( \rho = 1 \), yielding

\[
\kappa_T \geq \frac{\sqrt{3\pi}}{\log T} \left( 1 + O \left( \frac{1}{c_f} (\log T)^{-\delta} \right) \right).
\]
Essential GL(2) properties
For primitive \( f \) on \( GL(2) \) over \( \mathbb{Q} \) (Hecke or Maass) with

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \quad \Re(s) > 1,
\]

we isolate needed crucial properties (all are known).

1. \( L(s, f) \) has an analytic continuation to an entire function of order 1.

2. \( L(s, f) \) satisfies a function equation of the form

\[
\Lambda(s, f) := L(s, f_\infty)L(s, f) = \epsilon_f \Lambda(1 - s, \bar{f})
\]

with \( L(s, f_\infty) = Q^s \Gamma \left( \frac{s}{2} + \mu_1 \right) \Gamma \left( \frac{s}{2} + \mu_2 \right) \).
**Properties (continued)**

3. Convolution $L$-function $L(s, f \times \bar{f})$,  
\[ \sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^s}, \quad \Re(s) > 1, \]

is entire except for a simple pole at $s = 1$.

4. The Dirichlet coefficients (normalized so that the critical strip is $0 \leq \Re(s) \leq 1$) satisfy  
\[ \sum_{n \leq x} |a_f(n)|^2 \ll x. \]

5. For some small $\delta > 0$, we have a subconvexity bound  
\[ \left| L \left( \frac{1}{2} + it, f \right) \right| \ll |t|^\frac{1}{2} - \delta. \]
Mœglin and Waldspurger prove the needed properties of $L(s, f \times \overline{f})$ (in greater generality).

Dirichlet coefficient asymptotics follow for Hecke forms essentially from the work of Rankin and Selberg, and for Maass by spectral theory.

Michel and Venkatesh proved a subconvexity bound for primitive $GL(2)$ $L$-functions over $\mathbb{Q}$.

Other properties are standard and are valid for $GL(2)$. 
References
References

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Thank you!