Research Statement

Aaron Brown

1. Introduction

My primary research area is smooth dynamical systems, particularly dynamical systems with some degree of hyperbolicity. In discrete time, a dynamical system is a generated by a self-diffeomorphism

$$f: M \to M$$

of a manifold M. One then studies properties of f under iteration. Some general aims of the field are to understand the asymptotic and statistical properties of the orbits of f, identify and classify invariant geometric structures, and understand changes in dynamical behavior under small perturbations.

Iteration of a diffeomorphism $f: M \to M$ generates an action of the 1-parameter group $\mathbb{Z}$. Given an infinite, finitely-generated discrete group Γ, we may similarly consider **actions** of Γ by diffeomorphisms on a manifold M; that is, we may consider homomorphisms

$$\alpha: \Gamma \to \mathrm{Diff}^r(M)$$

from Γ to the group of C^r diffeomorphisms of M. Typically one only considers actions $\alpha: \Gamma \to \mathrm{Diff}^r(M)$ with infinite image $\alpha(\Gamma)$.

My recent research focuses primarily on actions of large discrete groups Γ. A common approach in my research is to study actions of large groups by restricting to a 1-parameter subgroup of Γ and using techniques and tools developed to study classical 1-parameter dynamical systems. For instance, two key tools used in my work on Zimmer's conjecture are **Lyapunov exponents** and **metric entropy**—objects from classical dynamical systems used to measure asymptotic exponential divergence and exponential complexity of a diffeomorphism.

I am particularly interested in **rigidity** phenomena for group actions; many questions in this research program are summarized by the following problems. In full generality, such problems are intractable. My research focuses on such problems under additional dynamical hypotheses on the action or geometric hypotheses on the acting group.

Problems. Consider an infinite, finitely-generated group Γ and a compact manifold M.

(1) (Existence) Determine if there exist actions $\alpha: \Gamma \to \mathrm{Diff}^r(M)$ with infinite image $\alpha(\Gamma)$.
(2) (Local rigidity/structural stability) Show that two sufficiently close actions $\alpha, \alpha': \Gamma \to \mathrm{Diff}^r(M)$ coincide up to a small change of coordinates.
(3) (Global rigidity) Classify all actions $\alpha: \Gamma \to \mathrm{Diff}^r(M)$ up to a change of coordinates.
(4) (Existence and classification of invariant structures) Given an action $\alpha: \Gamma \to \mathrm{Diff}^r(M)$, find and classify invariant geometric structures (probability measures, Riemannian metrics, foliations, etc.).

In Section 2, I give some standard examples of actions by lattices in $\mathrm{SL}(n, \mathbb{R})$—an important family of discrete groups in my research. Motivated by Problem (1), I then discuss my recent work related to Zimmer's conjecture. In Section 3, I discuss a version of Problem (3) and my recent work on rigidity of Anosov actions. In Section 4, I outline a common philosophy used in the proofs of Theorem A and B. In Section 5, I discuss some work related to Problem (4) studying all invariant and stationary measures for a group action under certain dynamical hypotheses.

2. Lattice actions and work on Zimmer's conjecture

Let $G = \mathrm{SL}(n, \mathbb{R})$ be the group of $n \times n$ matrices with determinant 1. The **rank** of $G = \mathrm{SL}(n, \mathbb{R})$ is $n-1$; typically, I consider the case $n \geq 3$ to ensure that G is **higher-rank**. A **lattice** in G is a discrete subgroup $\Gamma \subset G$ such that the quotient G/Γ has a finite G-invariant measure. A lattice $\Gamma \subset G$ is **cocompact** if the quotient G/Γ is a compact manifold; otherwise Γ is said to be **nonuniform**.

As an example, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$—the group of integer valued $n \times n$ matrices with determinant 1—is a nonuniform lattice in $G = \mathrm{SL}(n, \mathbb{R})$. While $\mathrm{SL}(n, \mathbb{Z})$ is not cocompact, $\mathrm{SL}(n, \mathbb{R})$ does possess cocompact lattices.

Examples. Given a lattice $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ consider the following standard actions.

(1) (Trivial actions.) Given a homomorphism

$$\alpha_0 \colon \Gamma \to \mathrm{Diff}^r(M),$$

if the image $\alpha_0(\Gamma)$ is a finite subgroup of $\mathrm{Diff}^r(M)$, the action α_0 is said to be **trivial**.

(2) (Projective actions.) Consider $M = S^{n-1}$ as the set of unit vectors in $\mathbb{R}^n$. Given a lattice $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, let

$$\alpha_1 \colon \Gamma \to \mathrm{Diff}^\infty(S^{n-1})$$

be the action induced by the action on rays: given a matrix $\gamma \in \Gamma$ and a unit vector $x \in S^{n-1}$, let

$$\alpha_1(\gamma)(x) = \frac{\gamma x}{\|\gamma x\|}.$$

(3) (Linear actions.) Let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ and take M to be the n-torus $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $\alpha_2 \colon \Gamma \to \mathrm{Aut}(\mathbb{T}^n)$ be the induced action by toral automorphisms:

$$\alpha_2(\gamma)(x + \mathbb{Z}^n) = \gamma x + \mathbb{Z}^n.$$

More generally, given a homomorphism $\rho \colon \Gamma \to \mathrm{SL}(d,\mathbb{Z})$ we have $\tilde\rho \colon \Gamma \to \mathrm{Aut}(\mathbb{T}^d)$ given by

$$\tilde\rho(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma)x + \mathbb{Z}^d.$$

2.1. **Zimmer's conjecture.** Given the prototype actions in Examples (2) and (3) above, **Zimmer's conjecture** asserts the following.

Conjecture 1 (Zimmer's conjecture for $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$)**.** *For $n \geq 3$, let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$. Let M be a compact manifold.*

(1) *If $\dim(M) < n-1$ then every action $\alpha \colon \Gamma \to \mathrm{Diff}^\infty(M)$ is trivial; that is, the image $\alpha(\Gamma)$ is finite.*
(2) *If $\dim(M) < n$ then every volume-preserving action $\alpha \colon \Gamma \to \mathrm{Diff}^\infty_{\mathrm{vol}}(M)$ is trivial.*

Here, $\mathrm{Diff}^\infty_{\mathrm{vol}}(M)$ denotes the group of diffeomorphisms of M preserving the volume form vol on M. The volume-preserving case (2) of the conjecture is due to Zimmer and appears in [Zim2, Zim3] for actions by C^∞ diffeomorphisms. The non-volume-preserving case in (1) is first stated by Farb and Shalen in their work on analytic actions [FS]. Most previous work on Zimmer's conjecture concerns actions in extremely low dimensions. See [Wit, BM, Ghy] for results on actions on the circle and [FH, Pol] for results on volume-preserving actions on surfaces. Above dimension 2, very few results were known before Theorem A below.

Recently, D. Fisher, S. Hurtado, and I verified Conjecture 1 for C^2-actions when Γ is assumed cocompact.[1]

Theorem A ([BFH, Theorem 1.1])**.** *For $n \geq 3$, let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a cocompact lattice. Let M be a compact manifold.*

(1) If $\dim(M) < n-1$ then any action $\alpha \colon \Gamma \to \mathrm{Diff}^2(M)$ is trivial.
(2) If $\dim(M) = n-1$ then any action $\alpha \colon \Gamma \to \mathrm{Diff}^2_{\mathrm{vol}}(M)$ is trivial.

The proof uses a number of well-known results, particularly Zimmer's cocycle superrigidity theorem [Zim1] and Ratner's measure classification theorem [Rat]. Two new ingredients in the proof are Strong property (T) due to V. Lafforgue [Laf, dLdlS] and work of myself, F. Rodriguez Hertz, and Z. Wang on smooth ergodic theory of $\mathbb{Z}^d$-actions appearing [BRH2, Bro, BRHW3] as applied to actions of higher-rank lattices in [BRHW2]. A key idea used in the proof of Theorem A and in [BRHW2] is outlined in Section 4.

2.2. **Future work on Zimmer's conjecture.** Theorem A is a significant breakthrough in a conjecture dating to the early 1980s. However, Theorem A and the more general results in [BFH] fall short of solving the most general version of Zimmer's conjecture. In the near future, I hope to improve the techniques in [BFH] to show that Theorem A holds for actions of $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ and other nonuniform lattices in $\mathrm{SL}(n,\mathbb{R})$.

[1] More generally, analogous results to those in Theorem A(1) holds for cocompact lattices in all higher-rank, split simple Lie groups G with finite center. The analog of Theorem A(2) holds for symplectic groups but misses the conjectured dimension by 1 for actions by lattices in the split orthogonal groups $\mathrm{SO}(n,n)$ and $\mathrm{SO}(n,n+1)$. We also obtain partial results for non-split Lie groups.

A more challenging project involves solving the analogous version of Conjecture 1 for actions of lattices in $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SO}(p,q)$, $q \geq p+2$, and other higher-rank, non-split Lie groups. For instance, from [BFH] we obtain for $n \geq 3$ and any cocompact lattice Γ in $\mathrm{SL}(n,\mathbb{C})$ that every homomorphism $\alpha\colon \Gamma \to \mathrm{Diff}^2(M)$ is finite when $\dim(M) \leq n-2$. It is conjectured that this should hold when $\dim(M) \leq 2n-3$. A planned project is to strengthen the main arguments from [BRHW2]—particularly the invariance principle described in Section 4—in order to obtain the conjectured critical dimension in the most general version of Zimmer's conjecture.

3. Rigidity of Anosov actions by higher-rank lattices

Zimmer's conjecture fits into a larger research program known as the **Zimmer program**. For $n \geq 3$, lattices Γ in $\mathrm{SL}(n,\mathbb{R})$ have many rigidity properties relative to linear representations $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$, particularly local rigidity [Sel, Wei], global rigidity [Mar1, Mos, Pra], and superrigidity and arithmeticity [Mar2]. The Zimmer program roughly aims to answer to what extent rigidity phenomena for linear representations $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ extend to non-linear (possibly volume-preserving) representations $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$. When Γ is a higher-rank lattice, Problems (1)–(3) above summarize a number of conjectures in the Zimmer program.

Within the Zimmer program are a number of global rigidity conjectures which suggest that all actions $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(M)$ are constructed from algebraic examples using standard modifications. In a recent work [BRHW1], F. Rodriguez Hertz, Z. Wang, and I established such a global rigidity result for actions satisfying certain dynamical properties—namely, **Anosov actions** on tori and nilmanifolds.

3.1. Background.

Consider a matrix $A \in \mathrm{GL}(d,\mathbb{Z})$. Since A has integer entries and $\det A = \pm 1$, A preserves the subgroup $\mathbb{Z}^d$ and induces an automorphism of the quotient torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Let $L_A\colon \mathbb{T}^d \to \mathbb{T}^d$ be the induced action:

$$L_A(x+\mathbb{Z}^d) := Ax + \mathbb{Z}^d.$$

If no eigenvalue of A has modulus 1, then $L_A\colon \mathbb{T}^d \to \mathbb{T}^d$ is an example of an **Anosov** or **uniformly hyperbolic** diffeomorphism. More generally, a diffeomorphism $f\colon M \to M$ if **Anosov** if the tangent space splits as

$$T_xM = E^s(x) \oplus E^u(x)$$

and vectors in $E^s(x)$ (resp. $E^u(x)$) are contracted exponentially under forwards (resp. backwards) iteration. It is conjectured that the only manifolds admitting Anosov diffeomorphisms are tori and (infra)nilmanifolds.

The following classical global rigidity theorem gives a complete classification of all Anosov diffeomorphisms on tori (and nilmanifolds) up to a continuous change of coordinates.

Theorem 1 (Franks–Manning [Fra, Man])**.** *Let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov diffeomorphism. Then there exist $A \in \mathrm{GL}(d,\mathbb{Z})$ and a homeomorphism $h\colon \mathbb{T}^d \to \mathbb{T}^d$ such that*

$$h \circ f \circ h^{-1} = L_A. \tag{1}$$

The map h satisfying (1) is always Hölder continuous but in general need not by C^1.

3.2. Global rigidity of Anosov actions.

Given a discrete group Γ, an action $\alpha\colon \Gamma \to \mathrm{Diff}^r(M)$ is **Anosov** if $\alpha(\gamma_0)$ is Anosov for some $\gamma_0 \in \Gamma$. One may ask to what extent Theorem 1 generalizes to actions of larger discrete groups: if $\alpha\colon \Gamma \to \mathrm{Diff}^r(\mathbb{T}^d)$ is an Anosov action, is there a linear action $\tilde\rho\colon \Gamma \to \mathrm{Aut}(\mathbb{T}^d)$ as in Example (3) above and a change of coordinates $h\colon \mathbb{T}^d \to \mathbb{T}^d$ with

$$h \circ \alpha(\gamma) \circ h^{-1} = \tilde\rho(\gamma) \qquad \text{for all } \gamma \text{ in a finite index subgroup of } \Gamma? \tag{2}$$

While a linear map L_A and a map h satisfying (1) may exist for individual elements $\alpha(\gamma)$ of the action, for an arbitrary discrete group Γ, a single homeomorphism $h\colon \mathbb{T}^d \to \mathbb{T}^d$ intertwining the non-linear action α with a linear action $\tilde\rho$ need not hold. For instance, when Γ is a free group with at least 2 generators, there exist Anosov actions $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ for which no such $\tilde\rho$ and h as in (2) exists. On the other hand, in [BRHW1] we established the existence of such $\tilde\rho$ and h when Γ is a lattice in a higher-rank simple Lie group.

Theorem B ([BRHW1, Theorem 1.7])**.** *For* $n \geq 3$*, let* $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ *be a lattice and let* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ *be an Anosov action. Assume that* Γ *is cocompact or, if* Γ *is nonuniform, that the action* $\alpha\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^d)$ *lifts*[2] *to an action* $\tilde{\alpha}\colon \Gamma \to \mathrm{Diff}^\infty(\mathbb{R}^d)$.

Then there exist a linear representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{Z})$ *inducing a linear action* $\tilde{\rho}\colon \Gamma \to \mathrm{Aut}(\mathbb{T}^d)$ *and a* C^∞ *diffeomorphism* $h\colon \mathbb{T}^d \to \mathbb{T}^d$ *such that*

$$h \circ \alpha(\gamma) \circ h^{-1} = \tilde{\rho}(\gamma) \qquad \text{for all } \gamma \text{ in a finite index subgroup } \Gamma' \subset \Gamma.$$

Note that in addition to finding a change of coordinates h satisfying (2), we also obtain that h is C^∞.

3.3. **Future work on Anosov actions.** A key innovation in [BRHW1] over earlier approaches to studying rigidity of Anosov actions is the construction of a topological conjugacy between a non-linear Anosov action and a linear Anosov action without the assumption that the non-linear action preserves a measure. Our technique should have a number of other applications which I plan to explore in future projects. For instance, local rigidity for affine Anosov actions of lattices $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$ was established in [KS, Theorem 15]. The method of proof heavily uses that Γ satisfies Property (T) of Kazhdan and that Zimmer's cocycle superrigidity theorem holds. On the other hand, the approach in [BRHW1] together with [FKS, RHW] recovers the local rigidity result in [KS, Theorem 15] without using Property (T) or cocycle superrigidity. Using the approach in [BRHW1], I expect to establish new local and global rigidity results for Anosov actions by lattices that fail to have Property (T), particularly irreducible lattices in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

4. An invariance principle and new ideas in the proofs of Theorems A and B

Theorems A and B concern an action of a lattice $\Gamma \subset G = \mathrm{SL}(n,\mathbb{R})$ on a manifold M. The first step to prove either theorem is to **induce** a G-action on an auxiliary manifold—denoted by M^α—from the Γ-action α. The manifold M^α has the structure of a fiber bundle with fibers diffeomorphic to M and base G/Γ.

Consider the subgroup A of $G = \mathrm{SL}(n,\mathbb{R})$ of diagonal matrices with positive entries:

$$A = \left\{\mathrm{diag}(e^{s_1}, \dots, e^{s_n}) : s_i \in \mathbb{R}\right\}.$$

Since we require the determinant to be 1, we have $s_1 + \cdots + s_n = 0$ and the map $(s_1, \dots, s_n) \mapsto \mathrm{diag}(e^{s_1}, \dots, e^{s_n})$ identifies A with $\mathbb{R}^{n-1}$.

For the group $G = \mathrm{SL}(n,\mathbb{R})$, the **roots** of G are the linear functionals $\beta_{i,j}\colon A \to \mathbb{R}$ for $i \neq j$ where, given $s = \mathrm{diag}(e^{s_1}, \dots, e^{s_n}) \in A$,

$$\beta_{i,j}(s) = s_i - s_j.$$

Each root $\beta_{i,j}$ corresponds to the 1-dimensional **root group** $U^{\beta_{i,j}}$ in G with ones on the diagonal, real entries in the (i,j)-entry, and zeros elsewhere. For instance, in $\mathrm{SL}(3,\mathbb{R})$ we have

$$U^{\beta_{2,3}} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

To prove either Theorem A or Theorem B, we study the dynamics of the higher-rank abelian group $A \simeq \mathbb{R}^{n-1}$ on the auxiliary space M^α. A common philosophy is employed in both [BFH] and [BRHW1]: *non-resonance implies invariance.* Roughly, the philosophy is as follows: Given an A-invariant (or equivariant) object O, we associate to O a class of linear functionals $\mathcal{O}$ on A. A root $\beta_{i,j}$ of G is **non-resonant** with $\mathcal{O}$ if it is not positively proportional to any element of $\mathcal{O}$. In certain situations, given a non-resonant root $\beta_{i,j}$ of G, the object O will automatically be invariant (or equivariant) under the root group $U^{\beta_{i,j}}$. Given enough such non-resonant roots $\beta_{i,j}$, the object O is automatically G-invariant (or equivariant).

In the proof of Theorem A, a key step is to promote a certain A-invariant measure μ on M^α to a G-invariant measure. The relevant linear functionals $\mathcal{O}$ are the **fiberwise Lyapunov exponents** $\lambda_i^F\colon A \to \mathbb{R}$ of the measure μ for the action of A on the fiber bundle M^α. When the measure μ projects to the G-invariant

[2] A sufficient condition that guarantees the lifting property is the vanishing of a certain second group cohomology $H^2_\rho(\Gamma, \mathbb{R}^d)$. This condition only depends on the action induced on homology by α and holds, for instance, whenever Γ is cocompact [GH, Bor].

measure on G/Γ, it was established earlier in [BRHW2, Proposition 5.1] that μ is automatically $U^{\beta_{i,j}}$-invariant for each non-resonant root $\beta_{i,j}\colon A \to \mathbb{R}$. In the setting of Theorem A, constraints on the dimension of M then force μ to be G-invariant.

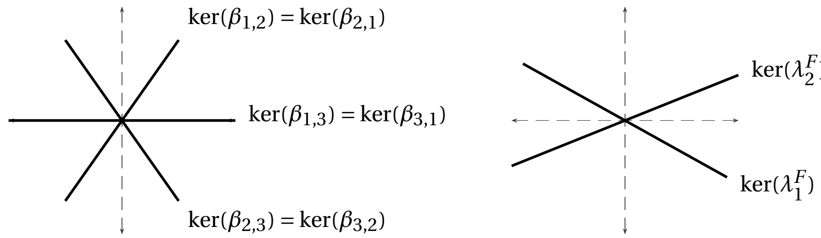

FIGURE 1. Kernels corresponding to the 6 roots $\beta_{i,j} = -\beta_{j,i}$ and two non-zero fiberwise Lyapunov exponents λ_1^F and λ_2^F for $A \simeq \mathbb{R}^2$ in $\mathrm{SL}(3,\mathbb{R})$.

To illustrate, consider the case where $G = \mathrm{SL}(3,\mathbb{R})$, $\dim(M) \geq 2$, and μ is an A-invariant measure that projects to the G-invariant measure on G/Γ and has two distinct, non-zero fiberwise Lyapunov exponents λ_j^F. In Figure 1, the kernels of the 6 roots $\beta_{i,j} = -\beta_{j,i}$ of G and the two non-zero fiberwise Lyapunov exponents λ_j^F are shown in $A \simeq \mathbb{R}^2$. Because the kernel of each λ_j^F is distinct from the kernel of each $\beta_{i,j}$, the measure μ on M^α is automatically G-invariant.

In the setting of Theorem B there exist two spaces M^α and M_ρ, each with a G-action. The first step is to construct a map $\tilde{h}$ intertwining the A-actions on M^α and M_ρ. Here, the relevant functionals $\mathcal{O}$ are the weights of a certain representation $\rho\colon G \to \mathrm{SL}(d,\mathbb{R})$. After verifying that the roots of G are non-resonant with the weights of ρ, the map $\tilde{h}$ automatically intertwines the entire G-actions on M^α and M_ρ which allows us to construct h as in Theorem B.

5. RIGIDITY OF STATIONARY MEASURES

Consider a compact manifold M and let $\Gamma \subset \mathrm{Diff}^\infty(M)$ be a subgroup of the group of C^∞ diffeomorphisms. It may be that there is no Borel probability measure on M preserved by the entire group Γ. Indeed, this is the case for the action in Example (2) above. On the other hand, given a probability measure ν on the group $\mathrm{Diff}^\infty(M)$ there always exist **ν-stationary** measures on M—that is, measures satisfying

$$\mu(B) = \int \mu(f^{-1}(B))\, d\nu(f)$$

for all B. Given a probability measure ν on the group $\mathrm{Diff}^\infty(M)$, let Γ_ν denote the subgroup generated by all diffeomorphisms contained in the support of ν.

The work described here is motivated by the following questions: for a measure ν on $\mathrm{Diff}^r(M)$

(1) is it possible to classify all ν-stationary measures, and
(2) is every ν-stationary measure in fact Γ_ν-invariant?

In a number of homogeneous settings, answers to such questions are obtained in [BQ1, BQ2] for measures ν satisfying certain algebraic hypotheses. In an affine but inhomogeneous setting, related results are obtained in [EM] for the $\mathrm{SL}(2,\mathbb{R})$-action on strata of Abelian differentials on a surface. In a completely non-affine, inhomogeneous setting, in [BRH1] F. Rodriguez Hertz and I answer the above questions for hyperbolic stationary measures satisfying a certain "local irreducibility" criterion.

Theorem C ([BRH1, Theorem 3.4])**.** *Let ν be a compactly supported measure on $\mathrm{Diff}^2_{\mathrm{vol}}(S)$ and let μ be an ergodic, hyperbolic, ν-stationary measure. Assume there are no μ-measurable line fields preserved by ν-almost every diffeomorphism f. Then either*

(1) μ is finitely supported, or
(2) μ coincides with an ergodic component of the invariant volume.

In particular, all such stationary measures are Γ_ν-invariant.

5.1. **Future work.** The methods used to establish Theorem C in [BRH1] do not generalize directly to groups of diffeomorphisms of higher-dimensional manifolds. With A. Eskin, I have begun a project that would provide results analogous to those in Theorem C in arbitrary dimensions by adapting the arguments of [EM] to a non-affine setting. Long-term outcomes of this project would include characterizations of stationary measures in a number of concrete situations including perturbations of homogenous systems and for actions of the mapping class group on character varieties.

REFERENCES

[BQ1] Y. Benoist and J.-F. Quint. Mesures stationnaires et fermés invariants des espaces homogènes. *Ann. of Math. (2)*, 174(2):1111–1162, 2011.

[BQ2] Y. Benoist and J.-F. Quint. Stationary measures and invariant subsets of homogeneous spaces (II). *J. Amer. Math. Soc.*, 26(3):659–734, 2013.

[Bor] A. Borel. Stable real cohomology of arithmetic groups. II. In *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, volume 14 of *Progr. Math.*, pages 21–55. Birkhäuser, Boston, Mass., 1981.

[Bro] A. Brown. Smooth ergodic theory of $\mathbb{Z}^d$-actions part 2: Entropy formulas for rank-1 systems. *In preparation*, 2016.

[BFH] A. Brown, D. Fisher, and S. Hurtado. Zimmer's conjecture: Subexponential growth, measure rigidity, and strong property (T). *Preprint*, 2016. arXiv:1608.04995.

[BRH1] A. Brown and F. Rodriguez Hertz. Measure rigidity for random dynamics on surfaces and related skew products. *J. Amer. Math. Soc.*, accepted for publication, 2016.

[BRH2] A. Brown and F. Rodriguez Hertz. Smooth ergodic theory of $\mathbb{Z}^d$-actions part 1: Lyapunov exponents, dynamical charts, and coarse Lyapunov manifolds. *In preparation*, 2016.

[BRHW1] A. Brown, F. Rodriguez Hertz, and Z. Wang. Global smooth and topological rigidity of hyperbolic lattice actions. *Preprint*, 2015. arXiv:1512.06720.

[BRHW2] A. Brown, F. Rodriguez Hertz, and Z. Wang. Invariant measures and measurable projective factors for actions of higher-rank lattices on manifolds. *Preprint*, 2016. arXiv:1609.05565.

[BRHW3] A. Brown, F. Rodriguez Hertz, and Z. Wang. Smooth ergodic theory of $\mathbb{Z}^d$-actions part 3: Product structure of entropy. *In preparation*, 2016.

[BM] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002.

[dLdlS] T. de Laat and M. de la Salle. Strong property (T) for higher-rank simple Lie groups. *Proc. Lond. Math. Soc. (3)*, 111(4):936–966, 2015.

[EM] A. Eskin and M. Mirzakhani. Invariant and stationary measures for the SL(2,R) action on Moduli space. 2013. arXiv:1302.3320.

[FS] B. Farb and P. Shalen. Real-analytic actions of lattices. *Invent. Math.*, 135(2):273–296, 1999.

[FKS] D. Fisher, B. Kalinin, and R. Spatzier. Global rigidity of higher rank Anosov actions on tori and nilmanifolds. *J. Amer. Math. Soc.*, 26(1):167–198, 2013. *With an appendix by James F. Davis.*

[Fra] J. Franks. Anosov diffeomorphisms on tori. *Trans. Amer. Math. Soc.*, 145:117–124, 1969.

[FH] J. Franks and M. Handel. Distortion elements in group actions on surfaces. *Duke Math. J.*, 131(3):441–468, 2006.

[GH] H. Garland and W.-C. Hsiang. A square integrability criterion for the cohomology of arithmetic groups. *Proc. Nat. Acad. Sci. U.S.A.*, 59:354–360, 1968.

[Ghy] É. Ghys. Actions de réseaux sur le cercle. *Invent. Math.*, 137(1):199–231, 1999.

[KS] A. Katok and R. J. Spatzier. Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions. *Tr. Mat. Inst. Steklova*, 216(Din. Sist. i Smezhnye Vopr.):292–319, 1997.

[Laf] V. Lafforgue. Un renforcement de la propriété (T). *Duke Math. J.*, 143(3):559–602, 2008.

[Man] A. Manning. There are no new Anosov diffeomorphisms on tori. *Amer. J. Math.*, 96:422–429, 1974.

[Mar1] G. A. Margulis. Non-uniform lattices in semisimple algebraic groups. In *Lie groups and their representations (Proc. Summer School on Group Representations of the Bolyai János Math. Soc., Budapest, 1971)*, pages 371–553. Halsted, New York, 1975.

[Mar2] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[Mos] G. D. Mostow. *Strong rigidity of locally symmetric spaces*. Princeton University Press, Princeton, N.J., 1973. *Annals of Mathematics Studies, No. 78.*

[Pol] L. Polterovich. Growth of maps, distortion in groups and symplectic geometry. *Invent. Math.*, 150(3):655–686, 2002.

[Pra] G. Prasad. Strong rigidity of $\mathbf{Q}$-rank 1 lattices. *Invent. Math.*, 21:255–286, 1973.

[Rat] M. Ratner. Invariant measures and orbit closures for unipotent actions on homogeneous spaces. *Geom. Funct. Anal.*, 4(2):236–257, 1994.

[RHW] F. Rodriguez Hertz and Z. Wang. Global rigidity of higher rank abelian Anosov algebraic actions. *Invent. Math.*, 198(1):165–209, 2014.

[Sel] A. Selberg. On discontinuous groups in higher-dimensional symmetric spaces. In *Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960)*, pages 147–164. Tata Institute of Fundamental Research, Bombay, 1960.

[Wei] A. Weil. On discrete subgroups of Lie groups. II. *Ann. of Math. (2)*, 75:578–602, 1962.

[Wit] D. Witte. Arithmetic groups of higher **Q**-rank cannot act on 1-manifolds. *Proc. Amer. Math. Soc.*, 122(2):333–340, 1994.

[Zim1] R. J. Zimmer. Strong rigidity for ergodic actions of semisimple Lie groups. *Ann. of Math. (2)*, 112(3):511–529, 1980.

[Zim2] R. J. Zimmer. Arithmetic groups acting on compact manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 8(1):90–92, 1983.

[Zim3] R. J. Zimmer. Actions of semisimple groups and discrete subgroups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1247–1258. Amer. Math. Soc., Providence, RI, 1987.