1. Prove the Peirce decomposition of an associative algebra: let $e_1, \ldots, e_n$ be elements of a (not necessarily commutative) algebra $A$ with identity 1 such that
\begin{enumerate}
\item $e_i e_j = \delta_{ij} e_i$ (where $\delta_{ij}$ is the delta function, 1 if $i = j$ and 0 otherwise).
\item $\sum_i e_i = 1$.
\end{enumerate}
We call this a complete, orthonormal collection of idempotents. Prove that, as an abelian group, $A = \bigoplus_{i,j} e_i A e_j$.

2. Let $\{f_1, \ldots, f_n\}$ be a basis of a finite-dimensional vector space $V$, and let $\{f_1^*, f_2^*, \ldots, f_n^*\}$ be the dual basis. Prove that the elements
\[ v \mapsto f_i^*(v) \cdot f_i \in \text{End}(V) \]
form a complete, orthonormal collection of idempotents of $\text{End}(V)$. Use the Peirce decomposition to prove that, after choosing a basis of $V$, every endomorphism of $V$ can be written as a matrix.

3. Let $V$ be a finite-dimensional vector space $V$ over an algebraically closed field $k$, $\text{End}(V)$ the ring of $k$-linear maps from $V$ to $V$, and $\text{Aut}(V)$ the set of invertible elements of $\text{End}(V)$. $\text{Aut}(V)$ acts on the ring $\text{End}(V)$ by conjugation. We say a function $\varphi : \text{End}(V) \to k$ is algebraic if it is in the algebra generated by functions of the form $A \mapsto \phi(Av)$ (monomials) where $\phi \in V^*$, the dual space of $V$, and $v \in V$.
\begin{enumerate}
\item We say that an element of $\text{End}(V)$ is regular if it is conjugate to a linear transformation which is diagonal with all eigenspaces of dimension 1. Prove that there is an algebraic function $\varphi : \text{End}(V) \to k$ such that an element $A$ is not regular if and only if it is a zero of $\varphi$. Prove that we may take this algebraic function to have integer coefficients with respect to a suitable generating set of monomials (hint: matrix coefficients). Deduce that any algebraic function is determined by its values on the regular elements of $\text{End}(V)$.
\end{enumerate}
(b) Let $\lambda_1, \ldots, \lambda_\ell$ be a finite set of algebraically independent elements of a commutative $k$-algebra (that is, there are no nontrivial polynomial relations between them with coefficients in $k$). The $m$-th **elementary symmetric polynomial** $E_m(\lambda_1, \ldots, \lambda_\ell)$ (for $1 \leq m \leq \ell$) is defined by:

$$
\prod_{i=1}^{\ell} (x - \lambda_i) = \sum_{i=0}^{\ell} E_m(\lambda_1, \ldots, \lambda_\ell)x^{\ell-m}.
$$

Prove that any symmetric polynomial in the $\lambda_i$ (any polynomial invariant under changing the indices of $i$) is a polynomial in the elementary symmetric polynomials.

(c) Let $B$ be a basis of $V$. Prove that a conjugation-invariant function on $\text{End}(V)$ is determined by its values on elements of $\text{End}(V)$ regular and diagonalizable with respect to $B$. Prove that the restriction homomorphism of the algebra of conjugation-invariant functions to regular elements of $\text{End}(V)$ diagonalizable on $B$ is an isomorphism onto the symmetric functions of the eigenvalues.