Problem Set 1
DUE: To Shanshan’s mailbox, January 18, 1 pm. No extensions.

These problems are intended to be straightforward with not much computation.

1. Solve all of the following equations. [Note that the left sides of these equations are identical.]
   a). $2x + 5y = 5$
   b). $2x + 5y = 0$
   c). $2x + 5y = 1$
   d). $2x + 5y = 2$
   $x + 3y = -1$
   $x + 3y = -2$
   $x + 3y = 0$
   $x + 3y = 1$

2. [Bretscher, Sec.2.1 #13]
   a) Let $A := \begin{pmatrix} 1 & 2 \\ c & 6 \end{pmatrix}$. With your bare hands (not using anything about determinants) show that $A$ is invertible if and only if $c \neq 3$.
   b) Let $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. With your bare hands (not using anything about determinants) show that $M$ is invertible if and only if $ad - bc \neq 0$. [Hint: Treat the cases $a \neq 0$ and $a = 0$ separately.]

3. Let $A$ and $B$ be $2 \times 2$ matrices.
   a) If $B$ is invertible and $AB = 0$, show that $A = 0$.
   b) Give an example where $AB = 0$ but $BA \neq 0$.
   c) Find an example of a $2 \times 2$ matrix with the property that $A^2 = 0$ but $A \neq 0$.
   d) Find all invertible $n \times n$ matrices $A$ with the property $A^2 = 3A$.

4. [Bretscher, Sec.2.3 #19] Find all the matrices that commute with $A := \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$.

5. a) Find a real $2 \times 2$ matrix $A$ (other than $A = \pm I$) such that $A^2 = I$.
    b) Find a real $2 \times 2$ matrix $A$ such that $A^4 = I$ but $A^2 \neq I$.

6. Let $L$, $M$, and $P$ be linear maps from the $(x_2, x_2)$ plane to the $(y_1, y_2)$ plane:
   $L$ is rotation by 90 degrees counterclockwise.
   $M$ is reflection across the line $x_1 = x_2$.
   $N\vec{v} := -\vec{v}$ for any vector $\vec{v} \in \mathbb{R}^2$.
   a) Find matrices representing each of the linear maps $L$, $M$, and $N$.
   c) Which pairs of these maps commute?
d) Which of the following identities are correct—and why?

1) \( L^2 = N \)  
2) \( N^2 = I \)  
3) \( L^4 = I \)  
4) \( L^5 = L \)  
5) \( M^2 = I \)  
6) \( M^3 = M \)  
7) \( MNM = N \)  
8) \( NMN = L \)