Research statement

Asaf Hadari

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My work focuses on Riemann surfaces and their self mappings. Two objects that have been central to my research are mapping class groups and various moduli spaces associated to Riemann surfaces. I often study the interplay between these objects. In the course of my work I have employed tools from the fields of dynamics, combinatorical group theory, Teichmüller theory and the theory of mapping class groups (see [15] for much of the information on the last two of these fields that is used in this report).
I am currently pursuing several projects in varied aspects of this field. In four of these projects I have results, three of which have led to submitted papers.

1 Character varieties and algebraic entropy

Let $\Sigma_{g,b}$ be a surface of genus g with b punctures. Denote its fundamental group by π. Let G be an algebraic group. The G-character variety of π, which we denote by $\mathfrak{X}(\pi,G)$ is a variety which parameterizes representations of π into G up to conjugacy. For many choices of G, this variety also parameterizes important geometric structures on $\Sigma_{g,b}$. For example, if $G = SL_2(\mathbb{R})$ then one component of $\mathfrak{X}(\pi,G)$ is Teichmüller space, and if $G = SU(2)$ we get the moduli space of flat connections on $\Sigma_{g,b}$.
The mapping class group $\mathrm{Mod}(\Sigma_{g,b})$ acts on the space $\mathfrak{X}(\pi,G)$ (see [19] for a survey of character varieties and of this action). Measuring the complexity of this action has been a source of much research. For instance, it is known that when G is compact then the action is ergodic ([18], [30]). A natural question to ask is the following.

Question 1. Given $f \in \mathrm{Mod}(\Sigma_{g,b})$, and a group G, give a measure of the complexity of the the action of f on $\mathfrak{X}(\pi,G)$.

There are several common measures of complexity in dynamics which are called entropy. The most commonly studied ones are measure theoretic entropy and topological entropy. The problem with using these to answer Question 1 is that they require the space in question to be either compact or have a finite invariant measure. In the case at hand neither

condition is necessarily true. As an alternative for answering question 1, I opted instead to use a concept called algebraic entropy, which makes use of the fact that $\mathfrak{X}(\pi, G)$ is a variety on which $\mathrm{Mod}(\Sigma_{g,b})$ acts algebraically.
Algebraic entropy was first introduced by Bellon and Viallet in [3] for maps $f : \mathbb{C}^n \to \mathbb{C}^n$ whose coordinates are all polynomials. It measures the exponential growth rate of the degrees of iterates of f. Richard Brown was the first to calculate algebraic entropy for mapping classes by calculating the algebraic entropy of the action of elements in $\mathrm{Mod}(\Sigma_{1,1})$ on a specific embedding of $\mathfrak{X}(\pi, SU(2))$ into $\mathbb{C}^3$ ([8]).

Two immediate complications arise when attempting to generalize this result. The first is that algebraic entropy was originally only defined for maps of affine space. General character varieties can be embedded in $\mathbb{C}^n$ for some n, but this embedding is by no means canonical. In order to avoid this problem, I extended the definition of algebraic entropy to include maps $f : V \to V$ where V is an affine variety. I denote this quantity $e_{alg}(f)$. The second difficulty is finding a suitable candidate for the algebraic entropy. I define such a candidate called the spectral radius of a mapping class, which is a generalization of the dilatation of a pseudo-Anosov. I denote it by $\rho(f)$. Using completely different methods than those of Brown's, I was able to prove the following very general result:

Theorem 1.1. *[22] Let $K = \mathbb{R}$ or $\mathbb{C}$ and G be one of the following groups:*

$$SL_N(K), GL_N(K), O_N(\mathbb{R})(N \geq 3), SO_N(\mathbb{R})(n \geq 3), U_N, SU_2, Sp_{2N}(\mathbb{R})$$

Let S be a surface with free non-abelian fundamental group, and let $f \in Mod(S)$. The algebraic entropy of the action of f on $\mathfrak{X}(\pi, G)$ is given by

$$e_{alg}(f) = \rho(f).$$

Aside from its interest in the study of character varieties and mapping class groups, Theorem 1.1 is an noteworthy result in the study of algebraic entropy. The examples provided in the theorem are of a different nature than all formerly known examples (aside from the ones calculated by Brown). Furthermore, the theorem provides one the most extensive and complicated families of explicit examples of algebraic entropy calculations (see [1], [2], [4], [12] for further examples of algebraic entropy calculations).

It is interesting to know when algebraic entropy equals topological entropy. There are several theorems comparing these two measures, often showing that they are equal (see [20], [31], [32] for some examples). A future direction I wish to explore in this project is to calculate the topological entropy of this action in the cases where it is applicable. This has been calculated by Fried for the case $S = S_{1,1}$ and $G = SU(2)$ ([17]) and by Cantat and Loray for reduced character varieties (these are character varieties where the traces of boundary components are fixed) in the case $S = S_{0,4}$, $G = SL_2(\mathbb{C})$ ([11]). In all these cases the toplogical entropy agrees with the algebraic entropy I calculate in Theorem 1.1, which leads to the following question.

Question 2 Suppose we are given a character variety (or reduced character variety) $\mathfrak{X}$, and an element f of a mapping class group which acts on $\mathfrak{X}$ for which the topological entropy can be defined. Denote this entropy by $\mathrm{e}_{\mathrm{top}}(f)$. Is it necessarily true that:

$$\mathrm{e}_{\mathrm{alg}}(f) = \mathrm{e}_{\mathrm{top}}(f)?$$

2 The geometry of surface groups with infinite generating sets

Given a surface with a finitely generated fundamental group, there is no canonical choice of generating set. If one wishes to define a suitably canonical generating set of a geometric nature then it becomes necessary to consider infinite generating sets. One such set is the set of all elements whose conjugacy class can be represented by a simple closed curve. These are in some sense the simplest elements of the fundamental group, and are thus a natural choice for a generating set.

Benson Farb posed the question whether the fundamental group, endowed with the word metric given by this set, has finite diameter. This question was answered negatively by Danny Calegari [10]. In a joint work with Khalid Bou Rabee, I investigated the same question for some quotients of the fundamental group. In contrast with Calegari's result, we found the following.

Theorem 2.1. *[5] Let Σ be a surface of finite type, $\pi = \pi_1(\Sigma)$, and $\mathcal{S} \subset \pi$ be any generating set containing at least one element in each conjugacy class that is represented by a nonseparating simple closed curve. Let $\rho : \pi \to N$ be a homomorphism into any nilpotent group. Then $\rho(\pi)$ has finite diameter in the word metric given by the set $\rho(\mathcal{S})$.*

We say that a group G is *nilpotent-bounded with respect to the set S* if any nilpotent quotient of G has finite diameter with respect to the word metric given by the image of S. As part of the proof, we showed a far more general result.

Theorem 2.2. *[5] Let G be a finitely generated group, and let $S \subset G$ be a generating set such that $G/[G,G]$ has finite diameter with respect to the word metric given by S. Then G is nilpotent-bounded with respect to S.*

Theorem 2.2 suggests that there might be many more sets for which π is nilpotent-bounded. This raises the following question.

Question 1 Are there smaller sets for which π is nilpotent-bounded? Is there an easy to check geometric criterion that distinguishes the sets for which π is nilpotent-bounded?

As a first step toward answering this question, we showed the following: suppose Σ is a surface and $S \subset \Sigma$ is a collection of disjoint curves that cuts Σ into subsurfaces $\Sigma_1, \dots, \Sigma_k$.

Then any subset of $\pi_1(\Sigma)$ that contains an element in each conjugacy class of a simple closed curve that lies in one of $\Sigma_1, \dots, \Sigma_k$ satisfies the conclusion of Theorem 2.2.

Note that in surfaces of genus > 1, the group π has many nilpotent quotients of every degree of nilpotency. Furthermore, it is residually nilpotent, that is for every $x \in \pi$ there is some nilpotent quotient $q : \pi \to N$ such that $q(x) \neq 1$. Thus, it is natural to think of nilpotent quotients of fundamental groups as approximating the group itself. This further highlights the contrast between our result and Calegari's result which says that π has infinite diameter with respect to $\mathcal{S}$. In order to investigate this contrast, we are currently pursuing the following question:

Question 2 Let π_n be the n-th term in the lower central series of π and $L_n = \pi/\pi_n$. By Theorem 2.1, the group L_n has finite diameter with respect to $\mathcal{S}$. Call this diameter d_n. The sequence $\{d_n\}_{n=1}^{\infty}$ is nondecreasing. Is this sequence bounded? If so, by what value? If not, what is its asymptotic growth rate?

Note that if the sequence $\{d_n\}_{n=1}^{\infty}$ was indeed unbounded then it would imply that π has infinite diameter with respect to $\mathcal{S}$. However, the converse implication is not necessarily true. To see this, consider the following example: Suppose that π is a free group. Choose a free generating set for π, and let $|.|$ be the word metric given by this set. The set $\bigcup_{i=1}^{\infty} L_i$ is countable. Choose an enumeration of all of its elements: $\{\ell_i\}_{i=1}^{\infty}$. Each of the ℓ_i's is a coset of an infinite subgroup of π. For each i, choose an element $l_i \in \ell_i$ such that $|l_{i+1}| > 2^{|l_i|}$. Let $\mathcal{L} = \{l_i\}_{i=1}^{\infty}$. The group π is nilpotent-bounded with respect to the set $\mathcal{L}$. Indeed, by construction $\mathcal{L}$ surjects onto every nilpotent quotient, and thus generates each nilpotent quotient with diameter 1. However, by using the triangle inequality for $|.|$, it is simple to see that $\mathcal{L}$ cannot generate π with finite diameter.

In order to tackle this question we are attempting to assess the size of the image of $\mathcal{S}$ in various characteristic quotients of π, by using the fact that $\mathcal{S}$ is contained in a $\mathrm{Aut}(\pi)$ orbit.

3 Teichmüller disks with small Veech groups

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, and let $\mathcal{QM}_g \to \mathcal{M}_g$ its cotangent bundle. The points of $\mathcal{QM}_g$ can be given a geometric interpretation: they can be thought of as Riemann surfaces equipped with certain kinds of flat structures. More specifically, each point of $\mathcal{QM}_g$ is given by a pair (X, ω) where X is a Riemann surface and ω is a quadratic differential on that surface.

There is a natural $PSL_2(\mathbb{R})$ action on $\mathcal{QM}_g$, which makes it a very useful tool for studying $\mathcal{M}_g$ (see [33] for a survey). The projection of a $PSL_2(\mathbb{R})$-orbit to $\mathcal{M}_g$ is called a Teichmüller disk.

The $PSL_2(\mathbb{R})$-stabilizer of a point on a Teichmüller disk is called a Veech group. These groups have a strong influence on the geometry of the disk. For instance, a Teichmüller disk is an algebraic curve in $\mathcal{M}_g$ if and only if its Veech group is a lattice. In this case the disk is called a Teichmüller curve. It is known that every Veech group must be discrete, but not every discrete subgroup of $PSL_2(\mathbb{R})$ is a Veech group (c.f. [21]). This leads to the following question:

Question 1 Which Fuchsian groups are Veech groups?

For the most part, relatively little is known about this question, partly due to the difficulty of explicitly calculating Veech groups. The largest body of knowledge on this subject concerns Teichmüller curves. For instance, it is known that every finite index subgroup of $SL_2(\mathbb{Z})$ is a Veech group ([13]), and a complete list of which triangle groups are Veech groups is also known ([6], [24]). However, Teichmüller curves are very rare amongst Teichmüller disks. While most Teichmüller disks have trivial Veech groups, there are many cases of disks with Veech groups that are neither lattices nor trivial. While some examples of such disks have been constructed, almost none of the Veech groups of this nature have been calculated precisely (see [25], [26], [29] for some constructions of such disks).
Among the nontrivial discrete subgroups of $PSL_2(\mathbb{R})$, the simplest are the ones which don't contain non-abelian free groups - a list which includes finite and virtually cyclic subgroups. Call these small groups. Amongst the small groups, the simplest are the finite groups. As part of a project of looking for Teichmüller disks with small Veech groups, I proved the following theorem.

Theorem 3.1. *[23] Every finite subgroup of $PSL_2(\mathbb{R})$ arise as Veech group.*

To the best of my knowledge, aside from the cyclic group of order 2, no other finite group had been explicitly realized as a Veech group prior to this project.

The next groups to investigate after the finite ones are cyclic groups. It is simple to find disks with cyclic Veech groups that are generated by parabolic elements. A famous question in this field is the following.

Question 2 Are there cyclic Veech groups that are generated by an hyperbolic element?

In order to study this question, I have attempted to follow a method inspired by the parabolic case. Parabolic elements of the Veech group preserve a decomposition of the surface into cylinders - regions in the surface swept out by a collection of parallel closed curves. Thus is analogous to the fact that parabolic matrices preserve a collection of parallel lines in the plane. Since hyperbolic matrices preserve a family of hyperbolas in the plane, it is natural to search for such hyperbolas on the surface. My research has led me to identify such a collection of hyperbolas, and to study their properties. This is in keeping with the line of research initiated by S. Allen Broughton and Chris Judge of studying conics on translation surfaces.

4 Comparing metrics on Moduli space

Let $\mathcal{A}_g$ the moduli space of isomorphism classes of g dimensional principally polarized abelian varieties. The map which sends a Riemann surface to its Jacobian variety descends to a map $p : \mathcal{M}_g \to \mathcal{A}_g$, called the period mapping. By a theorem of Torelli, this map is injective.

The image $p(\mathcal{M}_g)$ is called the Schottky locus in $\mathcal{A}_g$ and is a well studied object (see [9] for an example of a geometric study of the Schottky locus). The space $\mathcal{A}_g$ is covered by a symmetric space called the Siegel upper half plane, and it inherits a distinguished metric from this space. This metric can be restricted to the Schottky locus, giving a metric on $\mathcal{M}_g$. Call this metric d_{Sc}. There are several well known metric on $\mathcal{M}_g$. One of the most important of these is the Teichmüller metric. Call this metric d_T. My goal in this project, which is a work in progress is to provide some comparisons between these two metrics.

Before stating some theorems, recall the concept of metric distortion. Given a set X equipped with two metrics: d_1 and d_2, say that a function $f : (0,\infty) \to (0,\infty)$ is a *distortion function from* d_1 *to* d_2 if

$$\forall x, y \in X : d_1(x,y) \leq f(d_2(x,y))$$

In [14], Farb posed the problem of finding a distortion function from d_T to d_{Sc}. Recently, Leuzinger and Ji gave an heuristic argument suggesting that such a function could not exist ([27]). The first theorem I can prove is the following.

Theorem 4.1. *Given any* $M, \varepsilon > 0$ *and* $r \geq 0$ *there exist* $x, y \in \mathcal{M}_g$ *such that* $d_T(x,y) > M$ *and* $|d_{Sc}(x,y) - r| < \varepsilon$.

In particular, I deduce:

Corollary 4.2. *There is no function* $f : (0,\infty) \to (0,\infty)$ *which is a distortion function for* d_T *and* d_{Sc}.

Theorem 4.1 shows that the obstruction to finding a distortion function is particularly severe in that such a function cannot be defined on any subset of $(0,\infty)$. Despite this obstruction, I do wish to describe the distortion between the two metrics. In order to do so I consider only certain subsets of $\mathcal{M}_g$.

The first result in this vein is a description of d_T-geodesic rays in $\mathcal{M}_g$ along which a distortion function can be found. A geodesic ray is locally distance minimizing for points on the ray, but not necessarily globally distance minimizing. Following [28], say that a ray $r(t) : [0,\infty) \to \mathcal{M}_g$ is *eventually distance minimizing* (or EDM) if there exists $t_0 \geq 0$ such that $\forall t > t_0$:

$$d_T(r(t_0, r(t))) = |t - t_0|.$$

Say that a ray is *almost distance minimizing* (or ADM) if here exists $C, t_0 \geq 0$ such that $\forall t > t_0$:

$$d_T(r(t_0, r(t))) \geq |t - t_0| - C.$$

A quadratic differential q on the surface Y is called Strebel (resp. mixed Strebel) if all (resp. some) of its vertical trajectories are closed. A ray in $\mathcal{M}_g$ is called Strebel if it is the projection to $\mathcal{M}_g$ of a ray corresponding to a surface equipped with a Strebel (resp. mixed Strebel) differential. In [16], Farb and Masur prove that a ray r in $\mathcal{M}_g$ is EDM (resp. ADM) if and only if r is Strebel (resp. mixed Strebel). I use this description to prove the following result.

Theorem 4.3. *Let r be an ADM ray in $\mathcal{M}_g$. Then there is a function on r which is a distortion function for d_T and d_{Sc} if and only if the quadratic differential associated to r has no vertical trajectories which are separating curves.*

As a corollary, I deduce the following.

Corollary 4.4. *Let r be an ADM ray in $\mathcal{M}_g$. Then the ray r, considered as a map $r : [0,\infty) \to \mathcal{M}_g$, is proper if and only if the quadratic differential associated to r has no vertical trajectories which are separating curves.*

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Dept. of Mathematics, University of Chicago
5734 University Ave. Chicago, IL 60637
E-mail: asaf@math.uchicago.edu