ASL Summer Meeting "Logic Colloquium '06".

Embeddability and Decidability in the Turing Degrees

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1. Jump upper semilattice embeddings
   - Background
   - JUSL Embeddings
   - Other Embeddability results

2. Local Structures
   - High/Low Hierarchy
   - Ordering of the classes
   - Fragments of the theory

Basic definitions

Given sets $A, B \subseteq \mathbb{N}$ we say that A is computable in B, and we write $A \leqslant_T B$, if there is a computable procedure that can tell whether an element is in A or not using B as an *oracle*.
(Note: Instead of $\mathbb{N}$ we could've chosen $2^{<\omega}$, $\omega^{<\omega}$, or $V(\omega)$,...)

This defines a quasi-ordering on $\mathcal{P}(\mathbb{N})$.

We say that A is Turing equivalent to B, and we write $A \equiv_T B$ if $A \leqslant_T B$ and $B \leqslant_T A$.

[Kleene Post 54] We let $\mathbf{D} = (\mathcal{P}(\mathcal{D})/\equiv_T)$, and $\mathcal{D} = (\mathbf{D}, \leqslant_T)$.

Question: How does $\mathcal{D}$ look like?

Some simple observations about $\mathcal{D}$

- There is a least degree $\mathbf{0}$.
  The degree of the computable sets.
- $\mathcal{D}$ has the *countable predecessor property*,
  i.e., every element has at countably many elements below it.
  Because there are countably many programs one can write.
- Each Turing degree contains countably many sets.
- So, $\mathbf{D}$ has size $2^{\aleph_0}$.
  Because $\mathcal{P}(\mathbb{N})$ has size $2^{\aleph_0}$, and each equivalence class is countable.

Operations on $\mathcal{D}$

Turing Join

Every pair of elements $\mathbf{a}, \mathbf{b}$ of $\mathcal{D}$ has a least upper bound (or *join*), that we denote by $\mathbf{a} \cup \mathbf{b}$. So, $\mathcal{D}$ is an upper semilattice.

Given $A, B \subseteq \mathbb{N}$, we let $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$.
Clearly $A \leqslant_T A \oplus B$ and $B \leqslant_T A \oplus B$,
and if both $A \leqslant_T C$ and $B \leqslant_T C$ then $A \oplus B \leqslant_T C$.

Turing Jump

Given $A \subseteq \mathbb{N}$, we let A' be the *Turing jump of A*, that is,

$A' =${programs that HALT, when run with oracle A }.

For $\mathbf{a} \in \mathbf{D}$, let $\mathbf{a}'$ be the degree of the Turing jump of any set in $\mathbf{a}$

- $\mathbf{a} <_T \mathbf{a}'$
- If $\mathbf{a} \leqslant_T \mathbf{b}$ then $\mathbf{a}' \leqslant_T \mathbf{b}'$.

Operations on $\mathcal{D}$.

Definition

A jump upper semilattice (JUSL) is structure $(A, \leqslant, \vee, \mathrm{j})$ such that

- $(A, \leqslant)$ is a partial ordering.
- For every $x, y \in A$, $x \vee y$ is the l.u.b. of x and y,
- $x < \mathrm{j}(x)$, and
- if $x \leqslant y$, then $\mathrm{j}(x) \leqslant \mathrm{j}(y)$.

$\mathcal{D} = (\mathbf{D}, \leqslant_T, \vee, ')$ is a JUSL.

Questions one may ask

- Are there incomparable degrees? YES
- Are there infinitely many degrees such that non of them can be computed from all the other ones toghether? YES
- What about $\aleph_1$ many? YES
- Is there a descending sequence of degrees $\mathbf{a}_0, \geqslant_T \mathbf{a}_1 \geqslant_T \dots$? YES
- Could we also get such a sequence with $\mathbf{a}'_{n+1} = \mathbf{a}_n$? YES

A more general question:

Which structures can be embedded into $\mathcal{D}$?

Embedding structures into $\mathcal{D}$

Theorem: The following structures can be embedded into the Turing degrees.

- Every countable upper semilattice. [Kleene, Post '54]
- Every partial ordering of size $\aleph_1$ with the countable predecessor property (c.p.p.). [Sacks '61]
  (It's open whether this is true for size $2^{\aleph_0}$.)
- Every upper semilattice of size $\aleph_1$ with the c.p.p. Moreover, the embedding can be onto an initial segment. [Abraham, Shore '86]
- Every ctble. jump partial ordering $(A, \leqslant, ')$.[Hinman, Slaman '91]

Theorem (M.)

Every ctble. jump upper semilattice $(A, \leqslant, \vee, ')$ is embeddable in $\mathcal{D}$.

Idea of the proof

Definition: A JUSL $\mathcal{J}$ is *h-embeddable* if there is a map $H\colon J \to \mathcal{P}(\mathbb{N})$ s.t., for all $x, y \in P$,

- if $x <_{\mathcal{J}} y$ then $H(x)' \leqslant_T H(y)$.
- uniformity condition : $\mathcal{J} \leqslant_T H(y)$, and $\bigoplus_{x \leqslant_{\mathcal{J}} y} H(x) \leqslant_T H(y)$;

Obs: Every well-founded JUSL is h-embeddable, by taking $x \mapsto 0^{\mathrm{rk}(x)}$.

Theorem

Every ctble JUSL which is h-embeddable, is embeddable into $\mathcal{D}$.

Proof: Forcing Construction. □

Lemma

Every ctble JUSL embeds into one which is h-embeddable.

Proof: Uses Fraïssé limits and non-standard ordinals. □

Corollary

Emedabbility results are usually related to the decidability of existential theories.

Corollary

$\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \vee, ')$ *is decidable.*

Note: $\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \vee, ')$ is the set of existential forumulas, in the language of JUSL, true about $\mathcal{D}$

Proof: An $\exists$-formula about $(\mathbf{D}, \leqslant_T, \vee, ')$ is true iff is does not contradict the axioms of jump upper semilattice. □

History of Decidability Results.

- $\mathrm{Th}(\mathbf{D}, \leqslant_T)$ is undecidable. [Lachlan '68]
- $\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T)$ is decidable. [Kleene, Post '54]

Question: Which fragments of $\mathrm{Th}(\mathbf{D}, \leqslant_T, \vee, ')$ are decidable?

- $\exists\forall\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T)$ is undecidable. [Shmerl]
- $\forall\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \vee)$ is decidable. [Jockusch, Slaman '93]
- $\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, ')$ is decidable. [Hinman, Slaman '91]
- $\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \vee, ')$ is decidable. [M. 03]
- $\forall\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \vee, ')$ is undecidable. [Slaman, Shore '05].
- $\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \vee, ', 0)$ is decidable. [Lerman, in preparation]

Question: Is $\forall\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, ')$ decidable?

Other Embeddability results.

Definition: A *jump upper semilattice with 0* (JUSL w/0) is a structure $\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \cup, \mathrm{j}, 0\rangle$

such that
- $\langle J, \leqslant_{\mathcal{J}}, \cup, \mathrm{j}\rangle$ is a JUSL, and
- 0 is the least element of $\langle J, \leqslant_{\mathcal{J}}\rangle$.

Q: Which JUSL w/0 can be embedded into $\mathcal{D}$?

Q: What about among the ones which have only finitely many generators?

other results.

Theorem (M. 03)

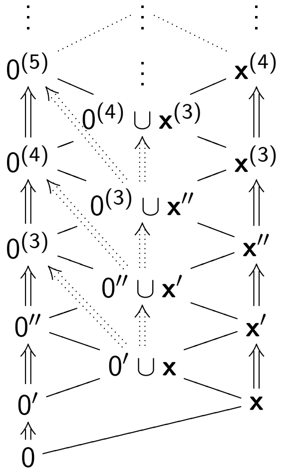
Not every JUSLw/0 even with one generator is embeddable in $\mathcal{D}$.

Proof: There are $2^{\aleph_0}$ JUSLw/0 with a generator x satisfying $x \leqslant 0''$. □

Theorem (Hinman, Slaman 91; M.03)

Every JPOw/0 with one genrator is realized in $\mathcal{D}$.

Question: What about JPOw/0 and with two generators?



Other results.

Let κ be a cardinal, $\aleph_0 < \kappa \leqslant 2^{\aleph_0}$.
Q: Is every JUSL with the c.p.p. and size κ embeddable in $\mathcal{D}$?

Proposition

If $\kappa = 2^{\aleph_0}$, then the answer is **NO**.

Proposition

If Martin's axiom holds at κ, the answer is **YES**.

Corollary

For $\kappa = \aleph_1$, it is independent of ZFC.

Proof: It is FALSE under CH, but TRUE under MA($\aleph_1$). □

$\mathbf{D}(\leqslant 0')$

Limit lemma: Let $\mathcal{A} \subseteq \mathbb{N}$. The following are equivalent.

- $A \leqslant_T 0'$,
- A is $\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$,
- there is a computable func. $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $\forall n$
  $n \in A \;\Leftrightarrow\; \lim_{s\to\infty} f(n, s) = 1 \;\Leftrightarrow\; (\exists m)(\forall s > m)\ f(n, s) = 1$
  $n \notin A \;\Leftrightarrow\; \lim_{s\to\infty} f(n, s) = 0 \;\Leftrightarrow\; (\exists m)(\forall s > m)\ f(n, s) = 0$

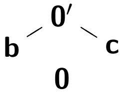
Notation: $\mathbf{D}(\leqslant \mathbf{0'}) = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \leqslant_\mathbf{T} \mathbf{0'}\}$.

Order-theoretic Properties of $\mathbf{0}'$

There is a history of results showing that $\mathbf{D}(\leqslant \mathbf{0}')$ has special properties. To cite a few:

- Every ctble poset can be embedded below $\mathbf{0}'$ [Kleene-Post '54].
- There are minimal degrees below $\mathbf{0}'$ [Sacks 61].
- Every degree below $\mathbf{0}'$ joins up to $\mathbf{0}'$ [Robison, Posner 72, 81]
- There are 1-generic degrees below $\mathbf{0}'$

$$(\forall \mathbf{b} \leqslant_T \mathbf{0}')(\exists \mathbf{c} <_T \mathbf{0}')\ \mathbf{0}' = \mathbf{b} \vee \mathbf{c}$$



What is the relation between the computabtional complexity of a

High/Low Hierarchy.

Definition: A Turing degree $\mathbf{a} \leqslant_T \mathbf{0}'$ is

- low if $\mathbf{a}' = \mathbf{0}'$.
- high if $\mathbf{a}' = \mathbf{0}''$.

Definition[Soare '74][Cooper '74] A Turing degree $\mathbf{a} \leqslant_T \mathbf{0}'$ is

- low$_n$ (L_n) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$.
- high$_n$ (H_n) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$.
- intermediate (I) if $\forall n\ (\mathbf{0}^{(n)} <_T \mathbf{a}^{(n)} <_T \mathbf{0}^{(n+1)})$.

Properties of $\mathbf{D}(\leqslant \mathbf{a})$

- Any ctble poset embeds below any $\mathbf{a} \notin L_2$. [Jockusch-Posner 78]
- There are minimal degrees below $\mathbf{a} \in H_1$. [Cooper 73]
- Every degree below $\mathbf{a} \in H_1$ joins up to $\mathbf{a}$. [Posner 77]
- There are 1-generic degrees below $\mathbf{a} \notin L_2$. [Jockusch-Posner 78]

Generalized High/Low Hierarchy

Definition: [Jockusch, Posner '78] A Turing degree $\mathbf{a}$

- is generalized low$_n$ (GL_n) if $\mathbf{a}^{(n)} = (\mathbf{a} \cup \mathbf{0}')^{(n-1)}$.
- is generalized high$_n$ (GH_n) if $\mathbf{a}^{(n)} = (\mathbf{a} \cup \mathbf{0}')^{(n)}$.
- is generalized intermediate (GI) if
$$\forall n \; ((\mathbf{a} \cup \mathbf{0}')^{(n-1)} <_T \mathbf{a}^{(n)} <_T (\mathbf{a} \cup \mathbf{0}')^{(n)}).$$

This hierarchy coincides with the High/Low one below $\mathbf{0}'$.

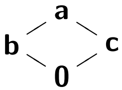
Question: Does it actually classify the degrees in terms of their complexity?

Properties of $\mathbf{D}(\leqslant \mathbf{a})$

- Any ctble poset embeds below any non-GL_2. [JP 78]
- There are minimal degrees below any $\mathbf{a} \in GH_1$. [Jockusch 77]
- Every degree below $\mathbf{a} \in GH_1$ joins up to $\mathbf{a}$. [Posner 77]
- There are 1-generic deg. below any $\mathbf{a} \notin GL_2$. [JP 78]

Complementation

Definition: We say that a degree $\mathbf{a}$ has the *complementation property* if
$(\forall \mathbf{b} \leqslant_T \mathbf{a})(\exists \mathbf{c} \leqslant_T \mathbf{a})\ \mathbf{b} \vee \mathbf{c} = \mathbf{a} \ \ \& \ \ \mathbf{b} \wedge \mathbf{c} = \mathbf{0}$.



Theorem: $\mathbf{0}'$ has the complementation property. History:

- Every $\mathbf{b} \in L_2$ has a complement below $\mathbf{0}'$. [Robinson 72]
- Every $\mathbf{b} \in H_1$ has a complement below $\mathbf{0}'$. [Posner 77]
- Every c.e. degree $\mathbf{0}$ has a complement below $\mathbf{0}'$. [Epstein 75]
- Every $\mathbf{b} \notin L_2$ has a complement below $\mathbf{0}'$. [Posner 81]
- The complement can be found uniformly, and can be choosen to be a 1-generic degree. [Slaman-Steel 89]
- The complement can be choosen a minimal degree. [Lewis 03]

Q: Does every GH_1 have the complementation property? [Posner 81]

- Yes, it does. [Greenberg-M.-Shore 04]

Q: Can the complement be found uniformly?

Ordering of the High/Low Hierarchy

Definition:

- $L_n^* = L_n \setminus L_{n-1}$ and $H_n^* = H_n \setminus H_{n-1}$.
- $L_1^* = L_1$, $H_1^* = H_1$, $I^* = I$,

This induces a partition of $\mathcal{D}(\leqslant_T \mathbf{0}')$:

$$\mathcal{C}^* = \{L_1^*, L_2^*, ...\} \cup \{I^*\} \cup \{H_1^*, H_2^*, ...\}.$$

On $\mathcal{C}^*$ we define a linear ordering:

$$L_1^* \prec L_2^* \prec \cdots \prec I^* \prec \cdots \prec H_2^* \prec H_1^*.$$

Observation: For all $\mathbf{x} \in X \in \mathcal{C}^*$ and $\mathbf{y} \in Y \in \mathcal{C}^*$

$$\mathbf{x} \leqslant_T \mathbf{y} \Rightarrow X \preceq Y. \qquad (*)$$

Theorem:[Lerman '85] Every finite partial ordering labeled with elements of $\mathcal{C}^*$ satisfying $(*)$ can be embedded into $\mathcal{D}(\leqslant_T \mathbf{0}')$ (of course, preserving labels).

Corollary:[Lerman '85]

$\exists - \mathrm{Th}(\mathbf{D}(\leqslant_T \mathbf{0}'), \leqslant_T, \mathbf{0}, \mathbf{0}', L_1, L_2, ..., I, ..., H_1)$ is decidable.

Non-ordering of the Generalized High/Low Hierarchy.

Question:[Lerman '85]

Can this be proved for the generalized high/low hierarchy?

The generalized high/low hierarchy induces a partition of $\mathcal{D}$:

$$\mathcal{G}^* = \{\mathrm{GL}_1^*, \mathrm{GL}_2^*, ...\} \cup \{\mathrm{GI}^*\} \cup \{\mathrm{GH}_1^*, \mathrm{GH}_2^*, ...\}.$$

Theorem (M.)

Every finite partial ordering labeled with elements of $\mathcal{G}^*$ can be embedded into $\mathcal{D}$.

Note that there is no restriction at all on the labels.

Corollary

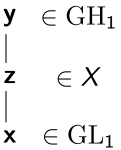
$\exists - \mathrm{Th}(\mathbf{D}, \leqslant_T, \mathbf{0}, \mathrm{GL}_1, \mathrm{GL}_2, ..., \mathrm{GI}, ..., \mathrm{GH}_1)$ *is decidable.*

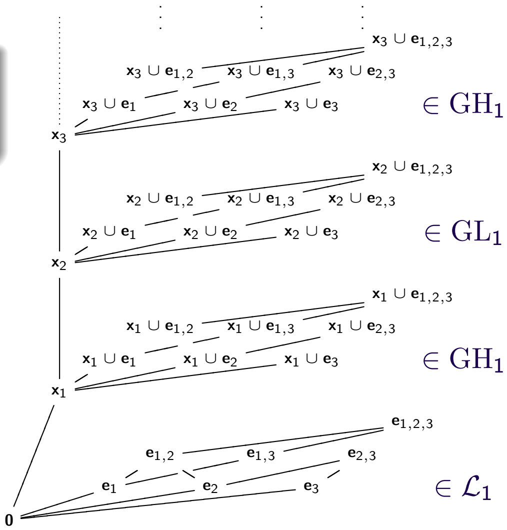
Idea of the proof

Lemma (M.)

There exists sets $\mathbf{e}_i$ *and* $\mathbf{x}_i$ *as in the picture.*

Lerman's bounding lemma: Given $\mathbf{x} \leqslant_T \mathbf{y}$, $\mathbf{x} \in \mathrm{GL}_1$, $\mathbf{y} \in \mathrm{GH}_1$, and $X \in \mathcal{G}^*$, there exists $\mathbf{z} \in X$ with $\mathbf{x} \leqslant_T \mathbf{z} \leqslant_T \mathbf{y}$.





Complexity of $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}'), \leqslant)$.

Question:

How does the complexity of **a** relates

to the complexity of $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}'), \leqslant)$?

Complexities of the Theories

Obs: $\mathrm{Th}(\mathbf{D}, \leqslant_T) \ \leqslant_1 \ \mathrm{Th}^2(\mathbb{N}, +, \times)$.

Theorem: [Simpson 77]
$\mathrm{Th}(\mathbf{D}, \leqslant_T) \ \equiv_1 \ \mathrm{Th}^2(\mathbb{N}, +, \times)$.

Obs: $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{0}'), \leqslant_T) \ \leqslant_1 \ \mathrm{Th}(\mathbb{N}, +, \times) \ \equiv_1 \ 0^{(\omega)}$.

Theorem: [Shore 81]
$\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{0}'), \leqslant_T) \ \equiv_1 \ \mathrm{Th}(\mathbb{N}, +, \times) \ \equiv_1 \ 0^{(\omega)}$.

Theorem: [Harrington, Slaman, Woodin]
$\mathrm{Th}(\mathcal{R}, \leqslant_T) \ \equiv_1 \ \mathrm{Th}(\mathbb{N}, +, \times) \ \equiv_1 \ 0^{(\omega)}$.

Upper bound of $Th(\mathbf{D}(\leqslant \mathbf{a}'), \leqslant_T)$

- $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}), \leqslant_T) \ \leqslant_1 \ \mathbf{a}^{(\omega)}$.
- $(\mathbf{D}(\leqslant \mathbf{a}), \leqslant)$ has a presentation $\Sigma^0_3(\mathbf{a})$

Theorem: [Lachlan - Lerman - Abraham,Shore]
Every countable upper semilattice can be embedded as an initial segment of $\mathcal{D}$.

- there are degrees $\mathbf{a}$ such that $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}), \leqslant_T)$ is decidable. (Lerman's method only produces L_2 such degrees.)
- there are degrees $\mathbf{a}$ such that $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}), \leqslant_T) \ \geqslant_1 \ \mathbf{0}^{(\omega)}$

Local Theories

Theorem: [Shore 81] $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}), \leqslant_T) \geqslant_1 0^{(\omega)}$ whenever **a** is either

- $\geqslant \mathbf{0}'$,
- computable enumerable,
- or high.

Proof:

- Find a way of defining models of arithmetic embedded in $\mathbf{D}(\leqslant \mathbf{a})$ using only finitely many parameters.
- Find a way to recognize when the finitely many parameters are coding the stardard model of arithmetic.
- Translate formulas..

□

Local theory below a 1-generic

Theorem

[Greenberg, M.] $\mathrm{Th}(\mathbf{D}(\leqslant \mathbf{a}), \leqslant_T) \geqslant_1 0^{(\omega)}$ *whenever* $\mathbf{a}$ *is either*

- *1-generic and* $\leqslant \mathbf{0}'$,
- *2-generic,*
- *n-REA*

Recall that a set $G \in 2^{\mathbb{N}}$ is 1-generic if

for every Σ^0_1 formula φ, $\exists p \subset G (p \Vdash \varphi) \vee (p \Vdash \neg\varphi)$.

Slaman-Woodin coding

Let $\mathcal{J}$ be an antichain of Turing degrees. There are degrees $\mathbf{c}$, $\mathbf{g}_0$, and $\mathbf{g}_1$ such that
the elements of $\mathcal{J}$ are the minimal solutions below $\mathbf{c}$ of the following inequality in $\mathbf{x}$:

$$(\mathbf{g}_0 \vee \mathbf{x}) \cap (\mathbf{g}_1 \vee \mathbf{x}) \neq \mathbf{x}.$$

Moreover, this degrees $\mathbf{c}$, $\mathbf{g}_0$, and $\mathbf{g}_1$ can be found below any 2-generic over $\mathcal{J}$.
[Odifreddi, Shore 91] The can also be found below $\mathbf{0}'$ if $\mathcal{J} \subseteq \mathcal{D}(\leqslant \mathbf{0}')$.

Lemma (Greenberg, M.)

1-genericity is enough to find the parameters $\mathbf{c}$, $\mathbf{g}_0$, *and* $\mathbf{g}_1$.