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RESEARCH STATEMENT

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My area of research centers around Computability Theory, a branch of Mathematical Logic. Inside computability theory, I have worked in various different areas. I have been particularly interested in the programs of Computable Mathematics, Reverse Mathematics and Turing Degree Theory. The former one studies the computability aspects of mathematical theorems and structures. The second one analyzes the complexity of mathematical theorems in terms of the complexity of the constructions needed for their proofs. The latter, which is considered Pure Computability Theory, studies the partial ordering induced by the relation "computable from".

I have also written a papers in other areas like Effective Randomness, automata theory, the structure of linear orderings, the lattice of Π^0_1-classes, and Borel structures. However, most of my work is on the three programs described before.

In Computable and Reverse Mathematics, my research has concentrated on linear orderings, well-quasi-orderings and Boolean algebras. But I have also worked with other type of structures like torsion-free abelian groups, vector spaces, and on computable model theory where we consider general types of structures. I now have a student working on Artinian Rings. Results in Computable and Reverse Mathematics usually require a deeper understanding of the objects from classical mathematics. For instance I have obtained interesting results purely on the structure of the embeddability relation on linear orderings as explained below. I have recently written a survey paper [Mon07] about my results on linear orderings.

Various approaches have been taken to understanding the shape of the Turing Degree Structure. One is to study the algebraic properties of the structure. Once people realized the structure is a quite complicated one, methods from logic started being used to study the complexity of the structure. Another approach has been studying how algebraic properties of certain Turing degrees in this structure relate to properties about the computational power of the degree. There is a lot of interaction between these approaches and I have been interested in this program in general. I have written a survey paper [Mona] on the history of the study of the Turing Degree Structure via embeddability results where I mention my contributions to the area until 2006.

Hyperarithmetic theory appears as a tool all over my work.

In the next subsection I quickly introduce the basic concepts of Computability Theory. In the next three sections I describe my work in each of the areas of Turing Degree Theory, Computable Mathematics and Reverse Mathematics. Each of these sections starts by describing the general ideas of the subject and becomes more technical at the end. Some of my plans for future work are described along the way.

Basic concepts. The main concept in computability theory is the relation "computable from". A set $A \subseteq \mathbb{N}$ is said to be *computable from* a set $B \subseteq \mathbb{N}$, and we write $A \leq_T B$, if there is a computable procedure that can tell whether an element is in A or not using B as an *oracle*, that is, we let the procedure use the information of which elements are in B. A set A is said to be *computable* if it is computable without the use of any oracle. We chose to work with subsets of $\mathbb{N}$ because this is enough: every finite object can be encoded by a single number (using, for instance, the binary representation of the number). For instance, strings, graphs, trees, simplicial complexes, group presentations, etc., if they are finite, they can be coded by a single number.

Turing Degree Theory

Introduction. The Turing degree structure is a very natural object introduced by Kleene and Post in [KP54]. It is defined as follows. The relation $\leq_T$ is a quasi-ordering on $\mathcal{P}(\mathbb{N})$, the set of subsets of

$I\!N$. It induces an equivalence relation ($A \equiv_T B \iff A \leq_T B$ & $B \leq_T A$) and a partial ordering on the equivalence classes. The equivalence classes are called *Turing degrees*. We use $(\mathbf{D}, \leq_T)$ to denote this partial ordering. With the intention of studying the relation $\leq_T$ in abstract, one of the main goals of Computability Theory is to understand the structure of $(\mathbf{D}, \leq_T)$.

The Turing degrees form an *upper semilattice*; that is, every pair of elements $\mathbf{a}$, $\mathbf{b}$ has a least upper bound $\mathbf{a} \vee \mathbf{b}$. Intuitively, $\mathbf{a} \vee \mathbf{b}$ contains all the information that $\mathbf{a}$ and $\mathbf{b}$ have. There is another naturally defined operation called the *Turing jump* (or just *jump*). The jump of a degree $\mathbf{a}$, denoted $\mathbf{a}'$, is given by the degree of the *Halting Problem* relativized to some set in $\mathbf{a}$. (Given $A \subseteq \mathcal{P}(I\!N)$, the *Halting Problem relative to* A, denoted by A', is the set of codes for computer programs, that, when run with oracle A, halt.) It can be shown that the jump operation is strictly increasing (i.e., $\forall \mathbf{a}(\mathbf{a} <_T \mathbf{a}'))$ and monotonic. A *jump upper semilattice* is an upper semilattice together with a strictly increasing, monotonic function.

Embeddings. One approach to understanding the shape of the Turing Degree Structure has been by studying the structures that can be embedded into it. Kleene and Post, in the same paper where they introduced the Turing degree structure [KP54], proved that every finite upper semilattice can be embedded into $(\mathbf{D}, \leq_T)$. Since then, various other embeddablity results have been proved. Abraham and Shore [AS86] proved in that every upper semilattice of size at most $\aleph_1$ with the countable predecessor property can be embedded into $(\mathbf{D}, \leq_T, \vee)$, extending a previous result of Sacks [Sac61]. Hinman and Slaman [HS91] proved that every countable jump partial ordering is embeddable in $(\mathbf{D}, \leq_T, ')$. For countable structures, the most general result proved is the following:

Theorem 0.1. *(Montalbán [Mon06c]) Every countable jump upper semilattice can be embedded into the Turing Degrees* $(\mathbf{D}, \leq_T, \vee, ')$ *(of course, preserving join and jump).*

It follows from my result that the quantifier-free formulas that are always true in $(\mathbf{D}, \leq_T, \vee, ')$ are the exactly the ones that follow from the definition of jump upper semilattice. Therefore, it follows that the existential theory of $(\mathbf{D}, \leq_T, \vee, ')$ is decidable. More results about embeddings into the Turing degrees of both jump upper semilattices with $\mathbf{0}$ and uncountable jump upper semilattices can also be found in [Mon06c]. For example, I proved that the question of whether it is possible to embed every jump upper semilattice of size $\aleph_1$ satisfying the countable predecessor property into $(\mathbf{D}, \leq_T, \vee, ')$ is independent of ZFC.

Jockusch and Posner [JP78] defined the *generalized high/low hierarchy* with the intention of classifying the Turing degrees depending on how close a degree is to being computable, and on how close it is to computing the Halting Problem. In [Mon06e], I proved that every finite partial ordering labeled with the classes of the generalized high/low hierarchy can be embedded into the Turing degrees preserving labels. This result helps to understand how the degrees in the various classes of the generalized high/low hierarchy are located inside the structure of the Turing degrees. It also yields to decidability results and answers an open question posed by Lerman [Ler85].

Extensions of embeddings. Let $\mathcal{D}_{(\leq 0')}$ be the upper-semi-lattice (usl) of degrees below $0'$.

Much is know about $\mathcal{D}_{(\leq 0')}$, but we are far from having a clear understanding of what the structure looks like. We do know it is a complicated structure; its theory is one-to-one equivalent to true first order arithmetic (Shore [Sho81]). On the other hand, if we look only at existential sentences, we can decide which sentences are true (as follows from results in [KP54]).

In order to understand where the complexity of a certain structure lies, one natural question to ask is what fragments of its theory are decidable. It has always been the case that answers to this question, by exposing either decidability procedures or coding methods, have given us a good deal of information about the algebraic properties of the structure. In the figure below we show the results known so far for $\mathcal{D}_{(\leq 0')}$. These investigation have also been done for $\mathcal{D}$ (the whole structure of the Turing degrees), $\mathcal{R}$ (the computably enumerable degrees) and many other structures. We refer the reader to [Sho06] for a recent survey of know results.

Decidability results for ∃-theories and ∀∃-theories are closely related to embeddability results. For a recent survey on embeddability results see [Mona].

	∃	∀∃	∃∀∃
$(\mathbf{D}_{(\leq 0')}, \leq_T)$	decidable	decidable [LS88]	undecidable [Ler83]
$(\mathbf{D}_{(\leq 0')}, \leq_T, \vee)$	decidable [KP54]	**?**	undecidable
$(\mathbf{D}_{(\leq 0')}, \leq_T, \vee, \wedge)$	decidable[1][LL76]	undecidable[2] [MNS04]	undecidable

Definition 0.2. Let $\mathbb{E}$ be the set of pairs of usls $(\mathcal{P}, \mathbb{Q})$ such that $\mathcal{P} \subseteq \mathbb{Q}$ and such that every usl embedding of $\mathcal{P}$ into $\langle \mathbf{D}_{(\leq \mathbf{0}')}, \leq, \vee\rangle$, can be extended to an embedding of $\mathbb{Q}$ into $\langle \mathbf{D}_{(\leq \mathbf{0}')}, \leq, \vee\rangle$. We call $\mathbb{E}$ the *extensions-of-embeddings problem* for $\mathcal{D}_{(\leq 0')}$.

In order to find a decidability procedure for $\forall\exists\text{-}Th(\mathbf{D}_{(\leq 0')}, \leq, \vee)$, if it exists, we definitely have to start by solving the extensions-of-embeddings problem for this structure. Conversely, in some occasions, deciding the extensions of embeddings problem has been enough to get the decidability of ∀∃-theories. This was the case with the decidability of $\forall\exists\text{-}Th(\mathcal{D}, \leq, \vee)$ by Jockusch and Slaman [JS93] and the decidability of $\forall\exists\text{-}Th(\mathbf{D}_{(\leq 0')}, \leq_T)$ by Lerman and Shore [LS88]. However, this was not the case with the celebrated extension of embeddings problem for $(\mathcal{R}, \leq_T)$, proved decidable by Slaman and Soare [SS01].

I plan to work on the following question.

Question 1. Is the extensions-of-embeddings problem for $\mathcal{D}_{(\leq 0')}$ decidable?

Downey, Greenberg, Lewis and I [DLGM] have found a good number of necessary and sufficient conditions that they expect will eventually lead to a solution of the problem. Many of the theorems they proved for this purpose are interesting on their own right, and provide a better understanding of the structure $\mathcal{D}_{(\leq 0')}$:

(1) Simultaneous 1-genericity below c.e. sets: For every c.e. set C and every sequence $\{A_i : i \in \omega\}$ of sets uniformly computable in C, there exists a set $G \leq_T C$ that is simultaneously 1-generic relative to each A_i such that $A_i <_T C$.
(2) No-least-join theorem: Consider degrees $\mathbf{a}, \mathbf{b} \leq_T \mathbf{c}$ with $\mathbf{c}$ c.e. such that $\mathbf{a} \not\leq_T \mathbf{b}$, $\mathbf{b} \not\leq_T \mathbf{0}$. Then, $\mathbf{b}$ is not the least degree below $\mathbf{c}$ that joins $\mathbf{a}$ up to $\mathbf{a} \vee \mathbf{b}$.
(3) Join property for non-GL_2 degrees: Let $\mathbf{c}$ be a non-GL_2 degree. Then, for every degree $\mathbf{a} < \mathbf{c}$, there exists $\mathbf{x} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{x} = \mathbf{c}$.
(4) There exist c.e. sets A, B, C, D and E such that A, B, D and E are all Turing reducible to C and pairwise incomparable, and such that any Δ^0_2 set X which is computable in C and joins A above B also joins D above E.

Complexity vs structure. Another way of analyzing $(\mathbf{D}, \leq_T)$ has been by finding relations between the computational complexity of a degree $\mathbf{a}$ and the structure $\mathbf{D}(\leq_T \mathbf{a}) = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \leq_T \mathbf{a}\}$.

One of the classes in the generalized high/low hierarchy is the class of the generalized high degrees. Posner asked in [Pos81] if for every generalized high degree $\mathbf{a}$, the upper semilattice $\mathbf{D}(\leq_T \mathbf{a})$ has the complementation property. Greenberg, Shore and I answered this question affirmatively in [GMS04].

Another way of analyzing local structures of $(\mathbf{D}(\leq_T \mathbf{a}), \leq_T)$ is by studying the complexity of their theory. Shore [Sho81] proved that $Theory(\langle \mathbf{D}(\leq_T \mathbf{a}), \leq_T\rangle)$, is Turing equivalent to true 1st order arithmetic whenever $\mathbf{a}$ is arithmetic and $\geq_T \mathbf{0}'$, c.e., or high. With Greenberg [GM03], we extended this result to $\mathbf{a}$ n-CEA, 1-generic and below $\mathbf{0}'$, 2-generic and arithmetic, or arithmetically generic.

Computable Mathematics

Effective mathematics is concerned with the computable aspects of mathematical objects and constructions. I have been working on questions like the following: How computationally complicated is common mathematical practice? When can a mathematical structure be represented computably?

[1]Here ∧ is the partial binary operation that is the greatest lower bound operation.
[2]Here ∧ is any total binary operation that is the greatest lower bound operation when this exists.

Can information be encoded into an isomorphism type of a structure? How can we measure the complexity of a mathematical proof?

Equimorphism types of linear orderings. Given a mathematical structure, is there computational content in it that is intrinsic to the structure? what is the simplest way to represent the structure? These two questions of are of great importance in this area.

In [Spe55], Clifford Spector proved that every hyperarithmetic well ordering is isomorphic to a computable one. In less technical terms this says that if an ordinal has a representation of a certain complexity (hyperarithmetic, which is quite high) then it also has a very simple (computable) representation. This theorem is central in Hyperarithmetic theory. I proved the following surprising generalization to all countable linear orderings:

Theorem 0.3. [Mon05] *Every hyperarithmetic linear ordering is equimorphic with a computable one. (Two linear orderings are* equimorphic *if they can be embedded in each other.)*

The proof of this Theorem requires a deep analysis of the structure of the countable linear orderings modulo equimorphisms. Using this analysis, in [Mon06b], I define equimorphism invariants for the class of scattered linear ordering of any size. This invariants are finite trees with nodes labeled with ordinals. They are equimorphism invariants in the sense that two linear orderings are equimorphic if and only if they are assigned the same invariant. One can then study the embeddability relation on linear orderings using these invariants.

The low$_n$ conjecture. In these past two years, part of my work has been on problems surrounding the well-known *low$_n$ conjecture*:

Question 2. Does every low$_n$ Boolean algebra have a computable copy?
(A set $X \subseteq \omega$ is low$_n$ if its nth Turing jump is as low as possible, namely $X^{(n)} = 0^{(n)}$.)

This work is part of a larger ongoing program that studies the computational complexity of mathematical structures. (We will discuss this further in Section **??**.)

Kenneth Harris and I [HMb] studied the relations on a Boolean algebra that are always computably enumerable in the n-th Turing jump of the Boolean algebra. These relations are said to be *relatively intrinsically* Σ^0_{n+1}. Ash, Knight, Manasse, Slaman and independently Chisholm (see [AK00, Theorem 10.1]) showed that these are exactly the relations that can be defined by a *computably infinitary* Σ^0_{n+1} formula in the language of Boolean algebras (or whatever structure we are dealing with). (*Computably infinitary formulas* are formulas in $L_{\omega_1,\omega}$ where infinite disjunctions or conjunctions are allowed so long as they can be computably enumerated.) Harris and I defined, for each n, a finite set of Boolean algebra predicates, $P^n_1(x), ..., P^n_{k_n}(x)$, that in a sense describe the whole structural information of the Boolean algebra that can be described with n Turing jumps. For instance, Harris and I got the following quantifier elimination result. Every infinitary Σ_{n+1} formula in the language of Boolean algebras is equivalent to an infinitary Σ_1 formula that uses these finitely many predicates $P^n_1(x), ..., P^n_{k_n}(x)$. The definition of these relations came from a deep analysis of the n-back-and-forth relations on the class of Boolean algebras.

Harris and I are currently finishing a second paper [HMa] based on the tools developed in [HMb]. In this paper they prove that the level five of the low$_n$ conjecture presents some extra essential difficulties not present at the previous levels: They show that there is a low$_5$ Boolean algebra that is not $0^{(7)}$-isomorphic to any computable one. This contrasts with previous results, where for each $n = 1, 2, 3, 4$, every low$_n$ Boolean algebra is $0^{(n+2)}$-isomorphic to a computable one. The proof uses just a zero-double-jump priority argument, it does not deal with $0^{(7)}$ or anything similar. This is due to the new technique developed by Harris and I to build low$_n$ Boolean algebras.

A particular Boolean algebra relation that has been studied by computability theorist is the *atom* relation. I [Mon08] proved that for every computable Boolean algebra with infinitely many atoms, and every high$_3$ c.e. degree X, there is a computable copy of the Boolean algebra where the atom relation has degree X. For further background on this problem, see page **??**.

One of the motivations for studying the low_n conjecture is that it would give an example of a class of structures whose *degree spectra* have a very interesting and unusual property. Kach, J. Miller and I [KMM] found another class where the low_n conjecture holds, namely the class of linear ordering with finitely many descending cuts. They have also shown that there is such a linear ordering of intermediate degree with no computable copy.

Hirschfeldt, Kach and I [HKM] studied a class of degrees that is slightly larger than the class of all low_n degrees. They call the degrees in this class *low for* Δ*-Feiner*. We say that a set S is $\Delta^0_{(n\mapsto n)}(X)$ if membership of n in S is a $\Delta^0_n(X)$ question, uniformly in n. So, X' can tell if 1 is in S, X'' can decide if 2 is in S, etc.. A set X is *low for* Δ*-Feiner* if every set S that is $\Delta^0_{(n\mapsto n)}(X)$ is also $\Delta^0_{(n\mapsto n)}(\emptyset)$. Their motivation comes from a results in Kach [Kac07] that says that for a certain type of Boolean algebras (depth zero Boolean algebras), if they have an X-computable copy and X is low for Δ-Feiner, then they have a computable copy. Furthermore, this is true for X if and only X it is low for Δ-Feiner. It is not hard to see that every low_n set is low for Δ-Feiner. Hirschfeldt, Kach and I showed that the converse is not true by constructing a c.e. intermediate degree that is low for Δ-Feiner. They also studied variations of this notion, like the classes of degrees that are $\Delta^0_{(n\mapsto bn+a)}(X)$ or $\Sigma^0_{(n\mapsto bn+a)}(X)$, and the classes of sets that are low, intermediate, and high for them.

Well-quasi-orderings and linear orderings. Well-quasi-orderings have been widely studied by combinatorists, computer scientists, proof theorists, etc.. A quasi-ordering, P, is a *well-quasi-ordering* (abbreviated WQO) if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of P, there exist i and j such that $i<j$ and $x_i \leq_P x_j$, or equivalently, if P has no infinite descending sequences nor infinite anti-chains. I analyzed results on well-partial-orderings from the viewpoint of computability theory.

The length of a WQO is used to measure its well-quasi-orderedness: The *length* of a WQO is defined to be the supremum of the order types of its linearizations. Note that this definition makes sense because every linearization of a WQO is a well-ordering. Moreover, this supremum is actually reachable: De Jongh and Parikh [dJP77] showed that every well-partial-order has a linearization of maximal order type. The length can also be obtained as the well-founded-rank of the tree of finite sequences $\langle x_0,...,x_k\rangle$ of elements of P such that there no $i<j\leq k$ with $x_i \leq_P x_j$. Notice that P is WQO if and only if this tree has no infinite paths. So far, people have computed lengths of various WQOs for different applications, but always using different methods and new ideas, and it was unknown whether the length of a WQO could be found computably. Diana Schmidt posed this question in [Sch79]. I showed that computable WQOs have computable maximal linearizations. However, he showed that the process of finding such linearizations is not computably uniform, not even hyperarithmetically.

Since my Ph.D. thesis, I have been working on solving the long standing question about the proof theoretic strength of Fraïssé's conjecture (Laver's theorem [Lav71]), which states that the class of countable linear orderings is WQO under embeddability. As proposed in Montalbán's previous proposal, in the past two years, I have started a new approach to solving this problem using lengths and ordinal notations. Marcone and I proved that the length of the WQO of linear orders of finite Hausdorff rank under embeddability is $\epsilon_{\epsilon_{\cdot_{\cdot}}}$, the first fixed point of the *epsilon-ordinal-function*, $\alpha\mapsto\epsilon_\alpha$ (where ϵ_α is the αth fixed point of the ordinal function $\beta\mapsto\omega^\beta$). They then showed that Fraïssé's conjecture restricted to linear orders of finite Hausdorff rank is equivalent to "$\epsilon_{\epsilon_{\cdot_{\cdot}}}$ is well-ordered" over ACA_0^+, where ACA_0^+ is $\mathsf{RCA}_0+\ \forall X\ (X^{(\omega)}$ exists). We note that the statement "$\epsilon_{\epsilon_{\cdot_{\cdot}}}$ is well-ordered" implies the consistency of ACA_0^+ and hence is not provable in ACA_0^+.

In my Ph.D. thesis, I had already published results about the class of linear orderings unrelated to computability theory, namely a definition of invariants for linear orderings up to equimorphisms and a construction of the finitely many linear orderings that are minimal at each Hausdorff rank [Mon06b]. These results followed from the analysis required to get the computability results. Since then I published a new paper [Mon06a] also about the structure of the class of linear orderings under embeddability that is not related to computability theory. I answered positively the following question from Rosenstein's book on Linear Orderings [Ros82, page 178]: Consider countable linear orderings

$\mathcal{L}$ and $\mathcal{C}_0 \preccurlyeq \mathcal{C}_1 \preccurlyeq \cdots$, such that for every $\mathcal{C}$ with $\forall n(\mathcal{C}_n \preccurlyeq \mathcal{C})$ we have that $\mathcal{L} \preccurlyeq \mathcal{C}$ (where $\preccurlyeq$ is the embeddability relation). Then, is there some n such that $\mathcal{L} \preccurlyeq \mathcal{C}_n$?

Other work.

Elementary equivalence of BAs. With Csima and Shore [CMS06] we studied the complexity of the question: Given two computable Boolean algebras, are they elementarily equivalent? This work included a complete analysis of the complexity of index sets of Boolean algebras with a certain Tarski invariant. In joint work with Csima, Harizanov, and R. Miller [CMHM], we have studied the computational aspects of Fraïssé limits and the notion of spectral universality.

Büchi and Borel structures. Not much work has been done on the effectiveness of uncountable structures. I started to work with Hjorth, Khoussainov and Nies on the effective model theory of Büchi and Borel structures [HKMN08]. In [HKMN08], they start analyzing continuum size structures that can be presented using a Büchi automata. This interaction between automata theorists, computability theorists and a descriptive set theorist led to interesting results. For example, among their results, they solve an open question in the area of whether every Büchi structure, where equality is represented as a Büchi equivalence relation, has a presentation where equality is just the the identity equivalence relation. To solve this question they use Borel presentable structures, and the fact that the Borel equivalence relation E_0 is not smooth.

Fraïssé limits. Fraïssé studied countable structures $\mathcal{S}$ through analysis of the *age* of $\mathcal{S}$, i.e., the set of all finitely generated substructures of $\mathcal{S}$ ([Fra86], see also [Hod97]). B. Csima, V. Harizanov, R. Miller, and I [CMHM] investigated the effectiveness of his analysis, as proposed in the previous NSF project. They focus particularly on the *Fraïssé limit.* One interesting result they get is that degree spectra of relations on a sufficiently nice Fraïssé limit are always upward closed unless the relation is definable by a quantifier-free formula.

Torsion-free Abelian groups and Vector Spaces. In this last two years, I have gotten two papers accepted in the Journal of Algebra. One with Downey, Hirschfeldt, Kach, Lempp, and Mileti [DHK+07] on the computational complexity of finding subspaces of a given computable vector space. And another one with Downey [DM08] on the complexity of the isomorphism problem for Torsion free abelian groups, topic that had been studied by Descriptive Set theorists like Hjorth, Kechris and Thomas.

Reverse Mathematics

Introduction. The questions of which axioms are necessary to do mathematics is of great importance in Foundations of Mathematics and is the main question behind Friedman and Simpson's program of Reverse Mathematics. To analyze this question formally it is necessary to fix a logical system. Reverse Mathematics deals with the system of second-order arithmetic. Second-order arithmetic, though much weaker than set theory, is rich enough to be able to express an important fragment of classical mathematics. This fragment includes number theory, calculus, countable algebra, real and complex analysis, differential equations, separable metric spaces and combinatorics among others. Almost all of mathematics that can be modeled with, or coded by, countable objects can be done in second-order arithmetic.

The idea of Reverse Mathematics goes as follows. We start by fixing a basic system of axioms. The most commonly used basic system is called RCA_0, and is closely related to Computable Mathematics. Now, given a theorem of "ordinary" mathematics, the question we ask is what axioms do we need to add to the basic system to prove this theorem. It is often the case in Reverse Mathematics that we can show that a certain set of axioms is necessary to prove a theorem by showing, using the basic system, that the axioms follow from the theorem. Because of this idea, this program is called Reverse Mathematics. Many different systems of axioms have been defined and studied. But a very interesting fact is that most of the theorems, whose proof-theoretic strength has been analyzed, have been proved equivalent over RCA_0 to one of five systems, that we will call the *main five*.

I am generally interested in the program of classifying theorems by their effective content and proof-theoretic strength.

Linear orderings. My work in Reverse Mathematics started trying to find the proof-theoretic strength of Jullien's Theorem which is a classification of the countable extendible linear orderings [Jul69]. It is a theorem that seems to require more complex axioms than most of the theorems in classical mathematics. Without finding its exact proof-theoretic strength in terms of logical axioms, I ended up finding that it is equivalent to many other statements about embeddability of linear orderings. One of these statements, Fraïssé's conjecture, have been studied before. Fraïssé's conjecture (also known as Laver's Theorem [Lav71]) is the statement that says that the countable linear orderings form a wqo with respect to embeddablity. (Recall that a *well-quasi-ordering*, or *wqo* is a quasi-ordering without infinite descending sequences or infinite antichains.) It has interested logicians for many years because of the difficulty of its proof in terms of reverse mathematics. From my work, it follows that this statement has a *robustness property* in the sense that it is equivalent to many other statements talking about the same type of objects. So far, the only systems with this robustness property were the main five, but we do not know that Fraïssé's conjecture is equivalent to one of these five. It also follows that to assume Fraïssé's conjecture is sufficient and necessary to develop a reasonable theory linear orderings and the embeddability relation.

Question 3. Is Fraïssé's conjecture equivalent to Friedman's system of Arithmetic Transfinite Recursion (ATR_0).

It was conjectured by Clote [Clo90], Simpson [Sim99, Remark X.3.31] and Marcone [Mar05] that that the answer is positive. This problem is still open and plan to work on it. One possible approach is to study the length of the wqo's involved, since this usually gives proof-theoretic information. Together with Marcone and Weierman, we have recently made some progress on this approach.

Ordinal Notations. The *proof theoretic ordinal* of a theory is an extremely useful notion when trying to measure its consistency strength. Ordinal notations are the main tool to deal with these ordinals. The proof theoretic ordinal of a theory is the least ordinal that the theory cannot show is well-ordered. Usually, it is the least ordinal such that the consistency of the theory can be proved using transfinite induction along this ordinal. For instance, Gentzen's proof of the consistency of Peano Arithmetic used transfinite induction along ϵ_0, and it can be shown that Peano Arithmetic proves transfinite induction along any ordinal less than ϵ_0. So, the proof theoretic ordinal of Peano Arithmetic is ϵ_0. ACA_0 has the same proof theoretic ordinal. Ordinals are naturally defined in set theory, but to deal with ϵ_0 inside first or second order arithmetic we use ordinal notations: We use a a string of symbols to represent each ordinal below ϵ_0. In this representation we should be able to easily compare two strings and decide which one corresponds to a larger ordinal. This way we have a linear ordering that is representing ϵ_0. When we refer to the statement that says that ϵ_0 is well-ordered we actually mean the statement that this particular linear ordering formed of notations is well-ordered.

Statements about the well-orderedness of a certain ordinal are Π^1_1 and have no set-existence implications, and hence cannot imply any of the main systems after RCA_0. However, when we use *ordinal notation operations* we can get set existence implications. For instance, Hirst [Hir94] proved that the statement that says "If X is a well-ordering, then ω^X is also a well-ordering" is equivalent to ACA_0, where ω^X is a linear ordering whose elements are of the form $\omega^{x_0}\cdot n_0+...+\omega^{x_k}\cdot n_k$, with $x_0,...,x_k\in X$ and $n_0,...,n_k\in\mathbb{N}$, and the ordering on these elements is defined in the obvious way. Friedman showed that the statement "If X is a well-ordering, then $\varphi(X,0)$ is also a well-ordering" is equivalent to ATR_0, where $\varphi(\cdot,\cdot)$ is the Veblen ordinal function (Veblen 1908 [Veb08], see also [Sch77]). Marcone and I [unpublished] have recently shown that the statement "If X is a well-ordering, then ϵ_X is also a well-ordering" is equivalent to ACA_0^+, over RCA_0, where ϵ_X is, in a sense, the X'th fixed point of the function $x\mapsto\omega^x$, and ACA_0^+ is RCA_0+$\forall X$ ($X^{(\omega)}$ exists). Even though proofs with ordinal notations usually involve techniques from Proof Theory, the proof of this theorem is purely computability theoretical. They build a computable linear ordering X such that ϵ_X has a computable descending sequence, but any descending sequence in X computes $0^{(\omega)}$. Bahareh Afshari and Michael Rathjen have recently shown the same result using only proof theoretic methods [AR].

Question 4. Let α be an ordinal (in some ordinal notation system). Is the statement "If X is a well-ordering, then $\varphi(\alpha, X)$ is also a well-ordering" equivalent to the Comprehension Axiom scheme for infinitary $\Pi^0_{\omega^\alpha}$ formulas?

It is known that the proof theoretic ordinal of RCA_0+ the Comprehension Axiom scheme for infinitary $\Pi^0_{\omega^\alpha}$ formulas is $\varphi(\alpha+1,0)$, the first limit point of the function $\beta \mapsto \varphi(\alpha,\beta)$. Proving the theorem above would explain, in the language of computability theory, why it is that these ordinal operations have such interesting properties.

Theories in the higher end. With Greenberg, in [GM08] we show how for some type of structures, ATR_0 is the natural system to work with them. These structures are superatomic Boolean algebras, reduced p-groups, compact countable topological spaces and well-founded trees. This paper provides a good understanding of what ATR_0 is able to prove.

Statements of hyperarithmetical analysis. Since my Ph.D. thesis, I have been interested in the *statements of hyperarithmetical analysis*. These are statements whose minimal ω-model is the one of the hyperarithmetic sets. These theories have been studied in the seventies (see [Fri75], [Van77] and [Ste78]), but (to the author's knowledge) there was no natural example of a statement of hyperarithmetic analysis. In [Mon06d] I give the first such natural example, called the *Indecomposability Theorem* which is about linear orderings and due to Jullien [Jul69]. Other statements of hyperarithmetic analysis concerning determinacy of games are also discussed in [Mon06d] and many questions are left open. After a talk I gave in Singapore, Kazuyuki Tanaka asked me about the strength of the Π^1_1-*separation axiom scheme* pointing out that it is also a statement of hyperarithmetical analysis and that it lies in between the Δ^1_1-comprehension and Σ^1_1-choice axiom schemes over RCA_0. I [Monb] answered Tanaka's question by showing that Π^1_1-separation actually lies strictly in between these theories.

Recently Itay Neeman has proved that Jullien's Indecomposability Theorem lies strictly in between weak-Σ^1_1-choice and Δ^1_1-comprehension.

Misselaneous

Linear Orderings. Out of the analysis of linear orderings form the viewpoints of reverse mathematics and computable mathematics, I have written two papers on the structures of the embeddability relation of linear orderings which are unrelated to computability theory. [Mon06b] is mentioned in section . In [Mon06a] I study the class of *countably complementable* linear orderings.

Effective randomness. This has become a very popular area of research in Computability theory in the last years. With Csima [CM06] we have showed that there is a minimal pair of Kolmogorov degrees.

Orbits of the lattice of Π^0_1*-classes.* A Π^0_1 class P is called *thin* if, for every Π^0_1 subclass P' of P, there is a clopen C with $P' = P \cap C$. This property is preserved under automorphisms of the lattice of Π^0_1 classes under inclusion, as it is definable in this structure. Cholak, Coles, Downey and Herrmann [CCDH01] found sufficient conditions that make two thin Π^0_1 classes automorphic: They proved that if P and Q are thin Π^0_1 classes and their lattices of subclasses are isomorphic, and these lattices are a Boolean algebra with finitely many atoms, then P and Q are automorphic. Downey and I proved that this is the only case when the lattices of subclasses determine the automorphism orbit of thin Π^0_1 classes: They showed that if P is a thin Π^0_1 class and its lattice of subclasses is not a Boolean algebra with finitely many atoms, then there is another thin Π^0_1 class whose lattice of subclasses is isomorphic to the one of P, but such that P and Q are not automorphic. This was conjectured by Cholak and Downey in [CD04].

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